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A Note on High Discount Rates and Depletion of Primary Forests

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Conventional wisdom implies that high discount rates accelerate depletion of tropical forests. As shown in this article, this result does not necessarily hold in a two-state variable model that distinguishes between primary and secondary forest stocks. In the context of a fixed concession period and imperfect government control, logging of primary forests may be both accelerated and depressed as discount rates increase.

Key words: conservation of primary and secondary forests, discount rates, timber concessions, tropical deforestation

Introduction

Conventional wisdom implies that high discount rates discourage sustainable forestry. High discount rates typically accelerate harvesting, depress investments in sustainable resource management, and reduce the weight attached to the needs and desires of distant future generations. In the economic literature, the process of discounting has been labelled ethically indefensible (Ramsey), a polite expression for rapacity (Harrod), and is believed by some to advance doomsday (Koopmans). At first sight, the higher the discount rate the more pronounced the negative impact on intergenerational equity and sustainability. Since it has been argued by many that current deforestation is excessive and socially wasteful (e.g., Barbier et al.; Brown and Pearce), and since intertemporal harvesting paths are for an important part driven by discount rates (Hotelling; Dasgupta and Heal), a logical conclusion would be that the timber industry applies a discount rate that is too high.[1]

However, the adverse effects of high discount rates on intergenerational equity and sustainability is not generally valid in resource economics. Farzin demonstrated that if the backstop price of a resource is a function of the interest rate or if production is capital intensive, the depletion period of a mine may increase or decrease as the interest rate shifts. With respect to forestry, Price argued that if discount rates are lowered, the opportunity costs of investment funds are also reduced, thereby making forest exploitation more profitable. Hence, lower discount rates may enhance the case for short-term exploitation of forests.

In this study we obtain similar results for tropical forestry without making any assumptions about the capital requirements of substitutes and without exploring the con-

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[1] To evaluate whether harvesting is "excessive" or not, one needs to relate it to a benchmark. It is often unclear what this may be; invariably, subjective opinions will play a role.

sequences of changes in the discount rate on the opportunity costs of investment funds. Instead, we analyze the influence of the discount rate on the way in which tropical forests are converted into agricultural land. It is explicitly recognized that primary forests are converted into secondary forests by the timber industry and we model the interlinkages in a simple two-state variable model. An important assumption in this study is that the government (as the owner of the forest) has imperfect control over the harvesting decisions of the logging firm that is granted a concession for harvesting, such that the firm has a certain freedom to allocate its intertemporal supply. The model is solved numerically to highlight the impact of discount rates on the depletion period of mature primary forest.

A Two-State Variable Model

Many governments in developing countries consider their forest base to be suboptimally large (Myers) and develop forest zonage plans in which part of the forest area is designated as conversion forests. Conversion forests are destined to be cleared for alternative uses, such as agricultural cultivation (see, for example, the case of Cameroon, as described in Côté). Land can be cleared from its forest cover by clearcutting the entire area as quickly as possible, or it can be transformed gradually such that the net present value of land use is maximized. Of course there are situations in which instantaneous clearcutting maximizes net present value of land, but this is certainly not generally true. During a gradual conversion phase two types of forests can be discerned. The first type is untouched forests, generally referred to as primary forests. Trees have matured fully and the timber extracted from these forests can be sold at high prices in international markets. Primary forests have no net growth: they are in ecological equilibrium where growth equals decay. By definition, primary forests are turned into secondary forests, the second type, if they are selectively logged (for example, Kummar and Sham). In contrast to primary forests, secondary forests display net growth because half-grown trees are exposed to more sunlight and face less competition for other scarce resources.[2]

Suppose that the government owns an area of primary forest $x_1(0)$. If the benefits accruing to society from sustainable forestry are less than the benefits associated with alternative land use (for instance, agriculture), then eventually the forest cover will be cleared. The government will solve the following problem:

$$\max U = \int_0^T B(t)e^{-\delta t}\,dt + e^{-\delta T}\int_T^\infty A(\tau)e^{-\delta(\tau-T)}\,d\tau, \tag{1}$$

subject to appropriate state equations, nonnegativity constraints, and the constraint that cumulative extraction from 0 to T equals the total primary forest stock at $t = 0$, which is $x_1(0)$. In (1), $B(t)$ are (social) benefits derived from timber exploitation and $A(t)$ are (social) benefits from the alternative land use option.[3] Finally, δ is the opportunity cost of capital, used by the government as discount rate. Solving the government's problem

[2] Additional differences between primary and secondary forests, not elaborated further upon in this article, are differences in accessibility for shifting cultivators (Myers) and differences in preservation values.

[3] We can assume that the government solves for a harvest plan that is socially optimal. Alternatively and without loss, we can assume that the government maximizes, for example, a political preference function (e.g., Becker).

yields (*a*) an optimal intertemporal extraction path for the forest stock and (*b*) the optimal period T in which all forest cover is cleared.[4]

Tropical commercial logging typically involves a contract between the government, as the owner of the resource, and a private firm. The firm is granted the right to harvest and manage a certain parcel of forest for a specific period of time. We assume that the government aims to achieve optimal deforestation as determined by the optimality conditions that follow from the government's problem by setting the terms of a concession contract with a logging firm. However, limited ability of governments in tropical countries to enforce concession contracts is well documented (for example, Grut, Gray, and Egli). An alternative interpretation is that the transaction or enforcement costs associated with enforcing full compliance exceed the benefits of compliance. For that reason we assume that the government is able to set the optimal "depletion time" of the forest stock T, but due to limited capability of the government to monitor the firm's logging activities or output, the harvesting decisions of the firm cannot be fully controlled.[5] The consequence is that the firm has a certain freedom to allocate intertemporal supply, which will result in an intertemporal harvesting path that is optimal for the firm (conditional on the predetermined depletion time T) but not necessarily for the government. Discrepancies arise if, for instance, harvesting involves external effects or if the firm applies a different discount rate than the government. In the remaining part of this article we focus exclusively on the latter.

In a simple model, the discount rate of a private firm may be based on the opportunity cost of capital (δ) and a possible risk premium (σ). In the case of the timber industry in tropical countries, the latter may be a function of the security of its tenure rights (Deacon; Mendelsohn). We assume that the timber industry perceives a constant probability κ of losing its tenure rights due to hostile government policy. When $\sigma = -\ln(1 - \kappa)$, then $e^{-\sigma t}$ is the probability of having control over the stock at time t. We define $\sigma + \delta = r$. Due to this risk premium the government and firm will prefer different extraction paths, even if they have the same opportunity cost of capital.

Now the firm's optimization problem can be sketched. The firm has agreed to deforest an area of mature, nongrowing primary forest ($x_1(0)$) in T years. Suppose that it is optimal for the firm to log the primary forest selectively such that trees with small diameters are allowed to grow and reach commercially (more) profitable stem sizes. Selective logging turns the primary forest into a forest with net growth. The firm's problem consists of two linked subproblems: (*a*) with respect to the primary stock, an optimal extraction path and *depletion* time (denoted by T_1) must be solved for; and (*b*) with respect to the stock of secondary forest, an optimal extraction path and *starting* time (denoted by T_2) must be solved for. Due to the set-up of the model, and more specifically the transversality condition that every hectare must be cleared from its (secondary) forest cover at T, the firm's subproblem is concerned with finding the starting time that maximizes net present value of exploitation, rather than the optimal depletion time. Of course it is possible to harvest primary forest and secondary forest at the same time (though not on

[4] User cost and its development over time is an important determinant of the optimal timing of switching to alternative land use. We refer to McConnell, Daberkow, and Hardie for a model that determines (approximately) optimal harvesting when timber production eventually ends. Since we are interested in optimal management of the firm (which faces a given T) rather than the government, we do not deal explicitly with the complexities of solving the government's problem.

[5] Hence the timber industry is not allowed to harvest after T and is not allowed to stop harvesting before T. A reason for the latter may be that the government wants to get a steady stream of revenues from forest exploitation. Because of imperfect monitoring, the government can only enforce that a positive harvest takes place in each period.

the same hectare, obviously), as T_1 can be arbitrarily near T and T_2 can be arbitrarily near zero.

Formally, the objective function of the firm is specified as follows:

$$\max_{y_1, y_2} P = \int_0^T [P_1(t)y_1(t) + P_2(t)y_2(t)]e^{-\delta t}e^{-\sigma t}\, dt, \tag{2}$$

where Π indicates the present value of the profit stream, T is the concession period as determined by the government, $y_i(t)$, $i = 1, 2$ indicates harvesting in primary and secondary forest in period t, respectively, and $P_i(t)$ represents the (net) price of wood from forest type i in period t. Because of the differences between primary and secondary forests (in tree quality and in growth), the rate of harvesting in primary forest ($y_1(t)$) and the rate of harvesting in secondary forests ($y_2(t)$) are expressed in different dimensions. Harvesting in primary forests is measured in hectares logged, whereas harvesting of the growing secondary forest is expressed in cubic meters of wood. As will become clear below, this makes more sense in the context of the equation of motion for the stock of secondary forest.

Most probably (that is, unless tenure rights are defined for an extremely long period and the discount rate applied by the firm is extremely low or negative), there will be a difference in stem size between wood harvested in primary and secondary forests. In our model, this difference in quality of wood extracted is reflected by the fact that the harvests from primary and secondary forests are sold for different prices at different markets. Furthermore, prices are assumed to be net of extraction costs, and revenues are net revenues. If marginal extraction costs are constant, this assumption is harmless, but even with marginal extraction costs that are not constant, the qualitative results are generally not affected. However, there is one aspect of ignoring extraction costs that is potentially restrictive: if there are significant economies of scale in harvesting, it may be more attractive for firms to clearcut the entire tree cover and save on exploitation costs than to harvest selectively and benefit from forest growth for future harvesting. In the remainder of this article we assume that possible economies of scale in exploitation are outweighed by the benefits from harvesting additional forest growth after selective logging, such that selective logging is optimal for the firm. If scale economies dominate growth benefits and we maintain the condition that output should be strictly positive from 0 to T, interior solutions for the firm's problem may be infeasible.[6]

In order to make this an interesting problem, we assume that the firm faces a downward sloping, inverse demand function for wood: $\partial P_i/\partial y_i < 0$. More specifically, in the numerical solution we will assume that the inverse demand function is linear: $P_i(t) = \bar{P}_i - \alpha_i\, y_i(t)$. In the absence of extraction costs and with constant prices, the logger's optimal decision when to remove all commercially interesting trees would simply be to deplete the mature stock in the first period and benefit from the growth potential of secondary forests in all periods that follow.[7]

[6] Suppose that the terminal point of the optimal extraction path ($y_1(T)$) and the starting time ($t = 0$) are specified. Then, at most 1 extraction path will satisfy the conditions that (*a*) ${}_0\int^T y_1(t)dt = X_1(0)$; (*b*) $y(t) > 0$, $\forall\, t \in [0, T]$; and (*c*) the necessary nonarbitrage condition that describes the development of the costate variable over time [see equation (7)]. This path may be optimal, depending on the discount rate applied by the firm, but most probably it won't be. Changing the firm's discount rate and evaluating the effect on deforestation makes no sense in this context.

[7] Note that, without loss, a so-called bang-bang solution can also be avoided when we model harvesting costs explicitly and assume $\partial C_i/\partial y_i > 0$.

The equations of motion of the model will be explained next. With respect to mature forest, the model is an extension of the standard mining model (Hotelling; Dasgupta and Heal):

$$\dot{x}_1(t) = -y_1(t), \tag{3}$$

where $x_1(t)$ is the stock of primary forest in period t, measured in hectares, and $y_1(t)$ is the number of hectares of primary forest that is selectively logged in period t. The dot over a variable indicates a change in time; hence, $\dot{x}$ represents a change in the size of the stock.

In order to derive the specification of the second equation of motion, we need to make the translation from the area of logged-over primary forest to the quantity of commercially valuable trees in secondary forest. Selectively harvesting the stock of primary forest implies accumulating a stock of secondary forest. For this purpose we multiply the area harvested in primary forests by a conversion factor γ. The constant is derived as follows. Assume that the timber volume per hectare of undisturbed forests equals ψ_1 units of trees, and that it is optimal for the firm to restrict harvesting to ψ_2 trees. Now, γ is given by $(\psi_1 - \psi_2)$. If the different age classes of trees are homogenously distributed over the total area, ψ_1 and γ will be constant. Furthermore, the number of cubic meters of commercially valuable trees in secondary forests falls over time because of harvesting and increases over time because of net growth. Hence, the equation of motion of secondary forests is

$$\dot{x}_2(t) = \gamma y_1(t) - y_2(t) + g(x_2(t)). \tag{4}$$

In this equation $x_2(t)$ is the stock of timber in cubic meters available in secondary forest, $y_2(t)$ is the number of cubic meters of timber harvested, and $g(x_2)$ describes secondary forest growth. Invoking the maximum principle and assuming an interior solution gives the following necessary conditions for an optimal solution:

$$P_1(t) + \gamma\mu(t) = \lambda(t), \tag{5}$$

$$P_2(t) = \mu(t), \tag{6}$$

$$\frac{\dot{\lambda}(t)}{\lambda(t)} = r, \quad \text{and} \tag{7}$$

$$\frac{\dot{\mu}(t)}{\mu(t)} = r - g'. \tag{8}$$

In these equations, $\lambda(t)$ and $\mu(t)$ are shadow prices (costate variables) associated with the state variables: they basically reflect how much the logging firm would be willing to pay for an extra unit of primary forest land and secondary forest land, respectively. The interpretation of (5) is that the marginal benefits of harvesting a unit of primary forest, measured as the sum of direct revenues and future harvesting of secondary forest, are equal to the foregone future timber benefits from primary forest. Equation (6) states that marginal timber benefits from secondary forest should equal the marginal cost of foregone future timber benefits. Equations (7) and (8) are nonarbitrage conditions: (7) is simply the Hotelling rule, and (8) is an extended version of this rule that accommodates the growth of the resource. After substituting the solutions of (7) and (8) into (5) and (6), we find

$$P_1(t) - (\bar{P}_1 + \gamma\bar{P}_2 e^{(r-g')(T_1-T)})e^{r(t-T_1)} + \gamma P_2(t) = 0, \tag{9}$$

and

$$P_2(t) - \bar{P}_2 e^{(r-g')(t-T)} = 0, \tag{10}$$

where $\bar{P}_i$, $i = 1,2$ is the backstop price for wood extracted from forest type i, which will be reached at $t = T_1$ for primary forest and $t = T$ for secondary forest, because by definition at T_1 and T, respectively, primary and secondary forest stocks must be depleted. Harvesting the stock of secondary forest starts at time T_2. By integrating (3) and (4) we find

$$x_1(0) - \int_0^{T_1} y_1(t)\, dt = 0, \tag{11}$$

and

$$e^{g'T} \int_0^T (\gamma y_1(t) - y_2(t))e^{-g't}\, dt = 0. \tag{12}$$

Solving this model we obtain the optimal depletion time of the primary forest T_1 and the starting period of secondary forest extraction T_2 [See appendix. We have chosen a linear growth function ($g(x_2) = \rho x_2$) because it facilitates the mathematics considerably without affecting the qualitative results.][8] Next, the optimal depletion paths for primary and secondary forests can be derived. The response of T_1 to changes in r is the subject of analysis in the next section. The discount rate r can change, for instance, because the firm perceives its tenure security differently as time passes. There are numerous reasons why (the perception of) tenure security may change over time. For example, the firm may fear that the present political elite loses its dominant position, through elections or a coup d'etat, and that the new government will renege on the contract (for a discussion and empirical study of tenure security, see Deacon). Alternatively, political preferences of the government may be subject to change over time. This latter will especially be important if government decision making is in accordance with public choice theories (e.g., Mueller) and not so much aimed at achieving a socially optimal solution.

If the government does not want to renegotiate the concession contract because the opportunity cost of capital has not changed (hence, T is constant), the firm will alter the intertemporal allocation of supply and redefine its optimal T_1 using the procedure outlined above.[9]

Results

Given the mathematical results as presented in the appendix, the effects of an increase in the rate of discount r on the optimal depletion time of primary forests T_1 and on the

[8] With this specification, even though $g(x)$ is a function of γ, it is easily verified that $g'(x)$ and γ are independent terms. This is clearly a violation of standard biological relationships central to selective tree harvesting. However, since the focus is on the transformation of nongrowing primary forest into growing secondary forest, this simplification is harmless. It is recognised that the present model specification is not suitable to analyse, for instance, optimal thinning in the secondary forest.

[9] In fact, the model we use does not investigate the impact of a change in r after a few years of harvesting but, instead, determines optimal exploitation paths from period 0 onwards.

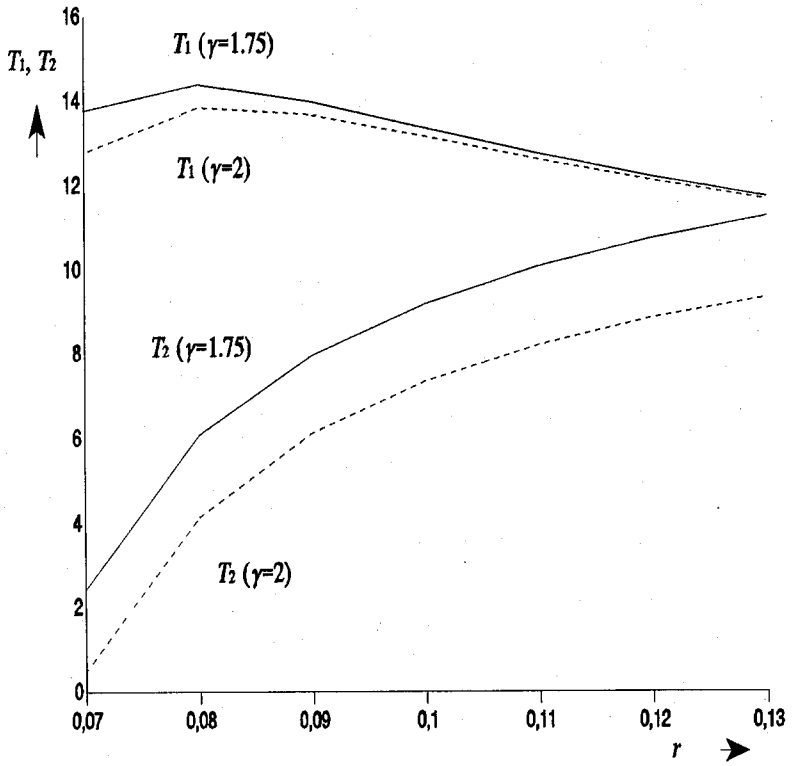

Figure 1. Optimal depletion times of primary forests and optimal starting times of logging in secondary forests, for γ = 1.75 and γ = 2.

Note: Additional parameter values: ρ = 0.05, $x_1(0)$ = 550, T = 50, P_1 = 30, P_2 = 15, α_1 = 0.3, α_2 = 0.15.

optimal time at which the firm starts logging secondary forests T_2 can be derived. The model is complicated and analytically solving it in order to illustrate the relation between T_1 and r proves to be extremely cumbersome. Therefore, we resort to a numerical solution. We have arbitrarily selected values for the parameters of the inverse demand functions and for $x_1(0)$ and T, as reported in figure 1. Representative results are presented in figure 1.

As is clear from this figure, the higher the rate of discount, the more logging in secondary forests is postponed while the effect on the depletion period of primary forests is less clearcut; the results are presented for two different values of γ but are robust for other parameter values (as long as the nonnegativity constraints are not violated). The fact that a higher γ leads to lower optimal values of T_1 and T_2 can easily be explained by analyzing equation (5). Ceteris paribus, an increase in γ (which may correspond with a high initial stocking density) will raise the marginal benefits of primary forest exploitation because the investment aspect of harvesting, hence the role of converted primary forest as an input in the secondary forest production process, gains weight. In order to benefit more from the stock of secondary forest, the concessionaire wants to accelerate access to this stock. Hence, the higher the conversion factor γ the shorter the optimal rotation of the primary forest, and logging in secondary forests can start at an earlier date.

Second, and more important, the curve that relates optimal depletion to the discount

rate is not monotonically declining but has an inverted U-shape. This implies that there exists a range of r-values for which raising the discount rate actually postpones depletion of the primary forest. The reason is that a high discount rate "tilts" the price paths and optimal exploitation paths of both the primary and the secondary forest stock. With respect to the primary forest, on the one hand, this results is an incentive to harvest more in early periods, which is the standard Hotelling result. On the other hand, the shift in the price path of the stock of secondary wood has a countervailing effect on optimal exploitation of the virgin stock. The reason is as follows. Raising r while keeping T and $\bar{P}_2$ fixed implies that the new price path for secondary wood ($P'_2(t)$) will be steeper and necessarily located entirely below the original price path. If the growth rate of $P_2(t)$ increases and the terminal point is identical, then automatically the starting point of the price path must be lower. Hence,

$$\dot{\mu}'(t) > \dot{\mu}(t) \wedge \mu'(t) < \mu(t), \qquad \forall t \in [0, T]. \tag{13}$$

The implications are as follows. From (5) it is clear that the low level of $\mu(t)$ reduces the marginal benefits of converting primary forest into a productive asset. This slows down the optimal extraction rate of the stock of virgin forest, which explains the inverted U-shaped T_1 path as shown in figure 1. Perhaps this explains why empirical support for the hypothesis that high risk premiums should accelerate deforestation, provided by Deacon (tables 4 and 5, p. 424), is weak.

From the inverse demand function we know that low realizations of $\mu(t)$ [which equals $P_2(t)$] correspond with a relatively high supply of secondary wood. To satisfy condition (12), this means that T_2 must be shifted to the future, as the increase in supply per period must necessarily be compensated for by a reduction in the number of periods in which timber is actually supplied. Thus supply is increased and prices are depressed, but harvesting the secondary stock starts later.

Conclusions

It is well documented that high discount rates are detrimental for natural resource conservation. If supply is restricted, for instance because of a tropical timber concession contract, this general conclusion no longer holds. If the concession period is exogenously determined and fixed and we recognize that depleting a stock of primary forest implies building a stock of secondary forest, then the effect of high discount rates on the stock of primary forest is ambiguous. There is a range of r-values over which an increase in the discount rate actually postpones depletion. In addition, the effect of higher discount rates on the stock of secondary forest is that the first period of exploitation is shifted to the future for all r. Whether this phenomenon is likely to occur in reality depends on the strength of the government to enforce concession contracts.

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Appendix

In order to determine T_1 and T_2, equations (11) and (12) must be solved. First, we have to determine the optimal paths of harvesting in primary and secondary forest areas. These paths can be derived by inserting the inverse demand functions into equations (9) and (10), and then solving them for $y_1(t)$ and $y_2(t)$:

$$\text{(A1)} \qquad y_1(t) = \frac{1}{\alpha_1}[\bar{P}_1 - [\bar{P}_1 + \gamma\bar{P}_2 e^{(r-\rho)(T_1-T)}]e^{r(t-T_1)} + \gamma\bar{P}_2 e^{(r-\rho)(t-T)}],$$

and

$$\text{(A2)} \qquad y_2(t) = \frac{1}{\alpha_2}[\bar{P}_2 - \bar{P}_2 e^{(r-\rho)(t-T)}].$$

Now the integrals (11) and (12) can be derived using the optimal depletion paths (A1) and (A2). The result for equation (11) is

$$\text{(A3)} \quad \alpha_1 x_1(0) = \bar{P}_1 T_1 - \frac{1}{r}[\bar{P}_1 + \gamma \bar{P}_2 e^{(r-\rho)(T_1 - T)}][1 - e^{-rT_1}] + \frac{\gamma \bar{P}_2 e^{-(r-\rho)T}}{(r-\rho)}[e^{(r-\rho)T_1} - e^{(r-\rho)T_2}].$$

The result for equation (12) is

$$\text{(A4)} \quad \frac{\alpha_1 \bar{P}_2}{\alpha_2 \gamma}\left[\frac{1}{\rho}(e^{-\rho T_2} - e^{-\rho T}) - \frac{e^{-(r-\rho)T}}{r - 2\rho}(e^{(r-2\rho)T} - e^{(r-2\rho)T_2})\right]$$

$$= \frac{\bar{P}_1}{\rho}(1 - e^{-\rho T_1}) + \frac{\gamma \bar{P}_2 e^{-(r-\rho)T}}{r - 2\rho}[e^{(r-2\rho)T_1} - e^{(r-2\rho)T_2}]$$

$$- [\bar{P}_1 + \gamma \bar{P}_2 e^{(r-\rho)(T_1 - T)}]\frac{e^{-rT_1}}{r - \rho}[e^{(r-\rho)T_1} - 1].$$

T_1 and T_2 are determined simultaneously by these two equations.