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STATIONARY EQUILIBRIUM IN A MARKET FOR DURABLE ASSETS

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ABSTRACT

This paper presents a dynamic model of consumer trading on the primary, secondary, and scrap markets for a stochastically deteriorating durable good in a stationary economy with perfect information and no transaction costs. We explicitly model the trading process by tracking each durable from its "birth" in the primary market, through its sequence of owners in the secondary market, until its "death" in the scrap market. We prove that a stationary equilibrium exists, characterize the distribution of consumer holdings of durables, and show that equilibrium asset prices are shadow prices to a particular regenerative optimal stopping problem. We show that each heterogeneous agent equilibrium is observationally equivalent to a homogeneous agent equilibrium. We derive a differential equation for equilibrium rental rates, and a functional equation which links rental rates to asset prices. These equations show precisely how the structure of durable prices and rental rates embody the functional form and population distribution of preferences, and the technological characteristics of durable goods.

1. Introduction

The defining characteristic of a durable good is that it yields consumption or productive services over multiple time periods. Since most durables deteriorate with use and eventually wear out, it is often the case that one wishes to trade a durable well before the end of its economic lifespan. When transactions costs and information asymmetries are minimal, or when the potential gains from trade are large enough to overcome these barriers, a secondary market comes into existence. Secondary markets commonly exist for relatively portable, standardized durables such as automobiles, ships, trucks, aircraft, railroad cars, and farm equipment, as well as for some non-portable, non-standardized durables such as housing. In either case, the essential benefit of a secondary market is to create dynamic trading opportunities similar to a securities market: the consumer can hold the current durable for any desired length of time and has the opportunity of trading it for a new asset or choosing from an array of used assets of different ages and physical conditions. In certain cases, rental markets for durable goods also come into existence. A rental market helps to "complete" the secondary market; it offers consumers the option to rent a durable for a given period of time at a predetermined price rather than own the asset over the same period and face the risk of capital loss upon resale.

The dominant framework for analyzing durable goods markets is the "user cost" model developed by Wicksell [36] in the 1930s, and applied in several recent studies such as Bulow [10], Chow [11], Parks [26], Sieper and Swan [30], and Stokey [31]. The essential features of this model are the assumptions that 1) the lifetime distribution of a durable is inalterable once fixed at date of manufacture, 2) new and used durables are perfect substitutes, and 3) a complete, competitive rental market exists.[1] The latter assumption implies that the equilibrium price of a durable equals the expected discounted value of the stream of rentals over the remaining life of the asset. While the Wicksellian model has proven useful in many contexts, the strong assumption that new and used assets are perfect substitutes limits its usefulness as a model of a secondary market for durable goods.

Recent work by White [35], Swan [32], [33], Sweeney [34], Wykoff [37], Miller [24], Bond [9], Berkovec [3], and Manski [21], can be viewed as attempts to relax the strong assumptions of the Wicksellian model in order to produce more realistic models of durable goods markets. Of these studies, Manski [21], [22] has provided the first true equilibrium analysis of durables markets by deriving durable prices from underlying assumptions about preferences of consumers and the technological characteristics of the asset. Given the importance of durable assets in economic activity, it is somewhat surprising that these more realistic equilibrium models of durables markets have arisen only recently. This can probably be explained by several difficulties involved in constructing a model of a secondary market for durables. First, due to expectations and the carry-over of current durable stocks into future periods, one obtains a strong intertemporal linkage of prices: holdings of today's stocks depend on expectations of tomorrow's prices, but prices of tomorrow's stocks depend on carry-overs of today's stocks (which depend on today's prices). Effectively, one must solve for the entire equilibrium price path in order to determine prices at any particular date. Furthermore, consumer expectations must be self-fulfilling along an equilibrium price path in any viable long-run equilibrium. A second problem stems from the logistical difficulties of "tracking" the stock of durables over time as they are traded among consumers. Since durables are typically subject to gradual deterioration with use, and since durables in different conditions are effectively different goods, one obtains a market with many distinct, but closely substitutable goods. Unless the economy is in a stationary state, the composition of the stock of durables is constantly changing as new durables are supplied to the economy by producers in the primary market and obsolete durables are removed from the economy in the scrap market. Consequently, it is necessary to follow the life history of each durable through its sequence of owners in order to determine its current physical condition and whether or not the current owner will supply it to secondary market, keep it, or scrap it. One quickly obtains a market with an unmanageable number of goods.

The Manski model (and a similar model by Berkovec [3]) is a computable general equilibrium model which deals head-on with most of the difficulties outlined above. The result is a highly realistic model of a secondary market. However, because the model is designed for numerical computation, it is difficult to characterize the basic properties of equilibrium in this framework. The present study is an attempt to follow the spirit of Manski's approach using a different analytical apparatus which enables us to deduce general properties of the equilibrium and derive closed-form solutions for specific examples.

Section 2 describes the durables market and the objectives of three types of agents in the model: producers, scrappers, and consumers. We derive the consumer's optimal trading strategy when secondary markets exist, and the optimal replacement policy when secondary markets don't exist. It turns out that the latter problem, formulated as a regenerative optimal stopping problem, provides the key to understanding how equilibrium prices and quantities are determined in secondary markets.

Section 3 defines our concept of a stationary equilibrium in a secondary market. Generalizing the Wicksellian framework, we allow the distribution of asset lifetimes to be determined endogenously in equilibrium, and allow new and used assets to be imperfect substitutes. Further, we do not require the existence of a rental market for durables: prices are determined in neoclassical fashion as values which equate supply and demand.

In Section 4 we prove the existence and uniqueness of stationary equilibrium and characterize its basic properties. We show that the equilibrium distribution of asset lifetimes is simply the first passage time distribution to the optimal stopping barrier of the optimal replacement problem of Section 2. Further, equilibrium prices are the shadow prices associated with this optimal stopping problem. The objective function of the optimal stopping problem turns out to be an equally weighted average of the utility functions of consumers who purchase used assets; oddly enough, consumers who purchase new assets are given zero weight. Interpreting this optimal stopping problem as defining a "modified utilitarian" social welfare function, it follows that stationary equilibria are efficient. Finally, this "dual" optimal stopping problem allows us to show that each heterogeneous agent stationary equilibrium is observationally equivalent to a particular homogeneous agent stationary equilibrium; i.e., we exhibit a homogeneous consumer economy with identical prices <u>and</u> quantities.

Section 5 derives the basic properties of stationary equilibrium under the additional assumption that a competitive rental market exists. We show that rental companies can be viewed as insurance intermediaries pooling independently evolving durable assets in order to provide insurance against capital losses at actuarially fair rates. This implies that if consumers are risk averse, all consumers would prefer to rent rather than own. This fact allows us to prove the existence of stationary equilibria for very general specifications of consumer preferences and guarantees that our partial-equilibrium characterization of the properties of stationary equilibria are in fact properties which hold in any general equilibrium system in which our durables market is imbedded.

2. Description of the Market for Durables

The durables market consists of three types of agents: scrappers, producers, and consumers, who trade in three corresponding markets: scrap, primary, and secondary.

Scrappers. Scrappers offer an infinitely elastic demand for durables at a fixed price $\underline{P}$. $\underline{P}$ represents the scrap value of material contained in the durable and is assumed to have the same value regardless of the physical condition of the asset.[2]

Producers. There is a single producer of durable goods. A durable can be represented quite generally as a discrete parameter markov process whose state at time t is represented by a nonnegative real number x_t. x_t can be interpreted as an index of the physical condition of the asset. Thus, a newly produced durable is in state 0 and larger x represents increased physical deterioration. Durables can be regarded as providing decreasing levels of service and/or requiring increasing operating and maintenance expenditures as the asset's condition deteriorates. A new durable consists of the triple $\{\overline{P}, m(\cdot;\tau), \Phi(\cdot,\cdot)\}$ where τ is a parameter indexing a particular consumer, $\overline{P}$ is the sales price of a new durable set by the producer, $m(x;\tau)$ is the conditional expectation of one period operating and maintenance costs for consumer τ as a function of the asset's state x, and $\Phi(x,y)$ is the conditional probability distribution of next period's condition y as a function of current condition x. Since this paper focuses on equilibrium in secondary markets, we assume that the producer's choice of price $\overline{P}$ and durability Φ are exogenously specified.[3]

Consumers. There are a continuum of consumers distinguished by a characteristic τ where τ is a real number in the interval $[\underline{\tau},\overline{\tau}]$. The population distribution of consumers is given by a probability distribution H on the interval $[\underline{\tau},\overline{\tau}]$, where $H(\tau)$ is the fraction of population whose preference parameter is less than τ. We assume that the per period utility of having an asset in condition x and income I is given by $U(I,x;\tau) = a(\tau)I+q(x;\tau)$.[4] Each consumer holds at most one durable per period over an infinite horizon and chooses an optimal asset selection and replacement policy to maximize the expected utility of owning an infinite sequence of assets. Consider consumer behavior in the presence of secondary markets. Under the assumption of stationarity, the cost-structure consumer τ faces is given by

$m(x;\tau)$	conditional expectation of 1 period maintenance costs for asset x
$P(z)$	purchase price of an asset in condition z
$T(x)$	trade-in value of an asset in condition x.

At the beginning of each time period the consumer has the option of (a) continue to operate the currently held asset, or (b) trade in the currently held asset for a replacement in condition z. If option (a) is chosen an expected cost $m(x;\tau)$ is incurred, the consumer obtains a one period expected utility of $a(\tau)[I-m(x;\tau)]+q(x;\tau)$ and the asset makes a transition to a new state in the next time period according to $\Phi(x,\cdot)$. If option (b) is chosen a cost of $P(z)-T(x)+m(z;\tau)$ is incurred, the consumer obtains a one period utility of $a(\tau)[I-P(z)+T(x)-m(z;\tau)]+q(z;\tau)$, and the decision process repeats itself in the next time period. The functions P, T and M and the transition probability Φ generate expectations of future operating costs and resale values on the basis of which the consumer decides to keep or sell the currently held asset. Formally, consumer τ who initially owns a durable in condition $x_0=x$, seeks an infinite sequence of decision-rules u_t which attain the infimum $j_\tau(x)$ given by

$$(2.3) \qquad j_\tau(x) = \sup_\Pi E \left\{ \sum_{t=0}^{\infty} \beta^t g(x_t,u_t) \,\middle|\, x \right\}$$

where:

$0<\beta<1$

$\Pi=(u_0,u_1,\ldots)$

and where:

$$g(x_t,u_t) = \begin{cases} a(\tau)[I_t-m(x_t;\tau)]+q(x_t;\tau) & \text{if } u_t(x_t)=x_t \\ a(\tau)[I_t-P(u_t)+T(x_t)-m(u_t;\tau)]+q(u_t;\tau) & \text{otherwise} \end{cases}$$

In order to simplify notation it is useful to observe that (2.3) can be reformulated as an equivalent cost minimization problem. Define an "operating cost" function $M(\cdot;\tau)$ by $M(x;\tau)=m(x;\tau)-q(x;\tau)/a(\tau)$. Then a solution Π to (2.3) will exist if Π is a solution to

$$(2.4) \qquad J_\tau(x)=\inf_\Pi E\{\sum_{t=0}^{\infty} \beta^t G(x_t,u_t) \mid x\}$$

where:

$$G(x_t,u_t) = \begin{cases} M(x_t;\tau) & \text{if } u_t(x_t)=x_t \\ P(u_t)-T(x_t)+M(u_t;\tau) & \text{otherwise} \end{cases}$$

Throughout the remainder of the paper we will work with the equivalent cost minimization form of the agent's objective function.

When no secondary market exists, the consumer has the option of (a) keeping the currently held asset in condition x at a cost of $M(x;\tau)$, or, (b) trading the currently held asset for a new asset at a cost of $\overline{P}-\underline{P}+M(0;\tau)$. The objective function remains the same as (2.4) with the modification

$$(2.5) \qquad G(x_t,u_t) = \begin{cases} M(x_t;\tau) & \text{if } u_t(x_t)=x_t \\ \overline{P}-\underline{P}+M(0;\tau) & \text{if } u_t(x_t)=0,\ x_t > 0 \end{cases}$$

The consumer's decision problem under the objective function (2.5) defines a (regenerative) optimal stopping problem: how long should the current asset be held until it is optimal to incur the transactions costs and purchase a new asset? As we shall see in Section 4, there is an intimate connection between the solution to this problem and the solution for equilibrium in secondary markets. We complete the description of the durables market with the following assumptions:

(A1) There are no taxes or transactions costs and all trading occurs at equilibrium prices; there is no individual bargaining over price.

(A2) Each consumer can costlessly verify the physical condition of each asset, and knows the structure of secondary market prices P, and the technological characteristics of durables, Φ.

(A3) For each $x \varepsilon R^+$, $\Phi(x,\cdot)$ is an absolutely continuous distribution function which satisfies $\Phi(x,0)=0$.

(A4) For each $y \varepsilon R^+$, $\Phi(\cdot,y)$ is a continuous, nonincreasing function.

(A5) Φ satisfies Doeblin's condition (ref. Doob [15], Futia, [17]).

(A6) For any constant $\gamma > 0$ the mean first passage time from 0 to γ under Φ is finite.

(A7) $M(x;\tau)$ is twice continuously differentiable in each argument and satisfies

$\partial M/\partial x > \varepsilon > 0$, $\partial M/\partial \tau > 0$, $\partial^2 M/\partial x \partial \tau > 0$.

(A8) H has density h which is nonzero and continuous on $[\underline{\tau},\overline{\tau}]$.

Note that the transition probability Φ defines a continuous linear transformation E on the Banach space of bounded Borel measureable functions defined by

$Ef(x) = \int_0^\infty f(y)\Phi(x,dy)$. E is the conditional expectation operator generated by Φ.

To solve the consumer's problem (2.4) we must solve for the value function J_τ which is the unique solution to Bellman's equation given by

(2.6) $$J_\tau(x)=\min[M(x;\tau)+\beta EJ_\tau(x),\ \inf_z\{P(z)-P(x)+M(z;\tau)+\beta EJ_\tau(z)\}]$$

Notice that in (2.6) the assumption of no transaction costs, (A1), implies that T=P.

Theorem 2.1 Under assumptions (A1),...,(A8) the consumer's optimal holding strategy is given by

(2.7) $$u(x_t,\tau) = z^*(\tau)$$

where $z^*(\tau)$ is a solution to:

(2.8) $$\min_z M(z;\tau)+P(z)-\beta EP(z)$$

and where $J_\tau(x)$ is given by:

(2.9) $$J_\tau(x)=[1/(1-\beta)][P(z^*(\tau))+M(z^*(\tau);\tau)-\beta EP(z^*(\tau))]-P(x)$$

For proof, see Appendix. According to Theorem 2.1, with zero transactions costs the optimal policy entails trading each period for an optimal asset $z^*(\tau)$. The formula for the consumer's value function J_τ given by (2.9) has an intuitive explanation: the present value of holding an infinite sequence of assets in condition z is equal to the initial cost of the asset P(z) plus the expected discounted value of operating and maintenance costs $M(z;\tau)/(1-\beta)$ plus the expected value of trading costs $[\beta/(1-\beta)][P(z)-EP(z)]$ less the trade-in value of the currently owned asset, $-P(x)$. Obviously we want to choose z to minimize the sum of these terms, so $z^*(\tau)$ must minimize (2.8). Formula (2.8) serves as a formalization of the intuitive notion that consumers' choice of durables involves a trade-off between capital and operating costs.

Now consider the consumer's problem (2.5) when no secondary market exists. The consumer's value function is the unique solution to Bellman's equation defined by

$$J_\tau(x) = \min[M(x;\tau)+\beta EJ_\tau(x), \overline{P}-\underline{P}+M(0;\tau)+\beta EJ_\tau(0)] \tag{2.10}$$

<u>Theorem 2.2</u> Under assumptions (A1),...,(A8) the consumer's optimal holding strategy is given by

$$u(x_t,\tau) = \begin{cases} x_t & \text{if } 0 \leq x_t \leq \gamma^* \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

where γ^* is the smallest solution to:

$$\overline{P}-\underline{P}+J_\tau(0) = M(\gamma^*;\tau) + \beta EJ_\tau(\gamma^*) = J_\tau(\gamma^*) \tag{2.12}$$

and where J_τ is the unique solution to (2.10).

The proof of Theorem 2.2 is given in Bertsekas [5], pp. 232-234.

To conclude this section we briefly indicate how the model implicitly allows for maintenance and usage decisions. These decisions can be made explicit by adding a variable λ in $\Phi(\cdot,\cdot;\lambda)$ as a control variable governing the mean rate of deterioration and in $M(\cdot;\lambda)$ to represent increasing consumer utility for higher rates of utilization (or lower levels of maintenance). Bellman's equation then becomes

$$(2.13) \quad J(x)=\min[\inf_{\lambda}\{M(x;\lambda)+\beta EJ(x;\lambda)\},\ \inf_{z}\inf_{\lambda}\{M(z;\lambda)+P(z)-T(x)+\beta EJ(z;\lambda)\}]$$

where $EJ(x;\lambda) = \int_0^{\infty} J(y)\Phi(x,dy;\lambda)$. Suppose that $\lambda(x)$ attains the infimum in (2.13) for each x. $\lambda(x)$ represents the consumer's optimal choice of utilization of the durable when it is in condition x. The residual uncertainty regarding the end of period condition of the asset is given by the probability distribution $\Lambda(x,y)=\Phi(x,y;\lambda(x))$. results of Propositions 2.1 and 2.2 are valid for general transition probabilities which satisfy (A5),...,(A8), it follows that if Φ is continuous in λ, and if λ is continuous in x, then Propositions 2.1 and 2.2 will also be valid for the transition probability Λ which incorporates the optimal usage decision. Although maintenance and usage decisions are not explicitly analyzed in this paper, the existence of such decisions provides an important reason why rental markets for durable goods may not exist: rental companies may not be able to observe the level of usage or maintenance λ creating an incentive for renters to over-use or inadequately maintain rented durables. This moral hazard problem will be discussed in Section 5.

3. Stationary Equilibrium in Secondary Markets

The durables market consists of a continuum of goods indexed by x and a continuum of consumers indexed by τ. We describe the price of all durables by a price function P and the quantities of durables by a holdings distribution F. P(x) represents the price of asset x and F(x) represents the fraction of durables whose condition is in the interval [0,x]. P and F should be interpreted as the continuous analogs of price and quantity vectors. Define the scrap point γ as the smallest constant for which $P(x)=\underline{P}$ if $x>\gamma$. If such a γ exists, scrappage behavior in the durables market will take the form of a threshold rule similar to a single consumer's optimal scrappage behavior given in Theorem 2.2: keep the asset if $x\in[0,\gamma]$ and scrap it otherwise. We now show how a stationary price function, holdings distribution, and associated scrap point are determined. Unfortunately, the traditional equilibrium condition, supply equals demand, is not well-defined in a market with a continuum of goods and consumers. The natural generalization involves equating the population distribution of supply to the population distribution of demand. This motivates the following definition.

<u>Definition</u>: A <u>stationary equilibrium</u> is given by $\{F(\cdot;\gamma^*), P(\cdot), \gamma^*\}$ where F and P are functions, and γ^* is a constant which satisfy:

1. $P(0)=\overline{P}$
2. $P(x)=\underline{P}$ of $x>\gamma^*$
3. $F(x;\gamma^*)$ a stationary holdings distribution corresponding to Φ and γ^*, and is the unique solution to the functional equation

$$F(x;\gamma^*) = \int_0^\infty [1-\Phi(y,\gamma^*)+\Phi(y,x)]F(dy;\gamma^*) \tag{3.1}$$

4. Each consumer $\tau\varepsilon[\underline{\tau},\overline{\tau}]$ chooses a holding policy $Z^*(\tau)$ which minimizes the expected cost (maximizes expected utility) of owning an infinite sequence of durables. From Theorem 2.1 we know that the optimal trading policy for consumer τ is to trade each period for $Z^*(\tau)$ given by

$$Z^*(\tau) = \underset{0<x<\gamma^*}{\operatorname{argmin}}[M(x;\tau)+P(x)-\beta EP(x)] \tag{3.2}$$

5. There exists a complete,[5] measurable selection $z^*(\cdot)$ from the correspondence $Z^*(\cdot)$ which satisfies for each x

$$(3.3) \qquad \int_{\underline{\tau}}^{\overline{\tau}} I\{\tau \mid z^*(\tau) \leq x\} H(d\tau) = F(x;\gamma^*)$$

where $I\{\tau \mid z^*(\tau) \leq x\} = 1$ if $z^*(\tau) \leq x$, 0 otherwise.

Condition 5 is the continuous equivalent of supply=demand. Condition 4 requires that consumers employ self-confirming expectations of secondary market prices to determine their optimal asset holdings. Condition 3 defines the stationary holdings distribution F as an ergodic or self-reproducing distribution corresponding to the transition probability Ψ given by

$$(3.4) \qquad \Psi(x,y) = \begin{cases} 1-\Phi(x,\gamma)+\Phi(x,y) & y \in [0,\gamma] \\ 1 & y \in (\gamma,\infty) \end{cases}$$

According to (3.4) an asset is allowed to age normally if $x \in [0,\gamma]$, but is replaced with a new asset once x exceeds the threshold γ. Thus, Ψ provides us with a mathematical formalization of the concept of flow equilibrium: each period the volume of new production equals the volume of scrappage.

The easiest way to understand why formula (3.1) should define the relevant notion of a "quantity vector" is to consider the sequence of holdings distributions F_t which evolve over time according to the functional difference equation

$$(3.5) \qquad \begin{aligned} \Lambda_t(x) &= \int_0^{\gamma} \Phi(y,x) F_t(dy) \\ F_{t+1}(x) &= [1 - \Lambda_t(\gamma)] + \Lambda_t(x) \end{aligned}$$

Formula (3.5) is simply a formalization of the basic accounting identity: current period stock = {last period stock (age adjusted) + new production - scrappage}.

If flow equilibrium is maintained each period, Doeblin's condition (A5) implies that given any initial distribution F_0 the sequence F_t defined by (3.5) converges at a geometric rate to an equilibrium distribution F which is the unique solution to (3.1). F represents the unique self-reproducing structure of durable holdings for a market which is in stationary flow equilibrium. Rust [28] shows that if $\Phi(x,x)=0$, F(x) equals the ratio of the durable's mean first passage time to the set (x,∞) divided by the mean first passage time to the set (γ,∞), i.e., F(x) is a ratio of stopping times.

To show how the model represents the dynamic trading process in the durables market, we will "track" a single durable from the beginning of its life as a newly produced asset in the primary market, through its sequence of owners in the secondary market, until its ultimate disposition in the scrap market. Consumers locate in the interval $[0,\gamma^*]$ according to their optimal asset choice. Let $Z^*(\tau)$ denote the optimal durable for consumer τ. In general $Z^*(\cdot)$ is a correspondence, but if the market is in equilibrium we can choose a measurable selection $z^*(\cdot)$ from $Z^*(\cdot)$ which assigns a unique asset to consumer τ. Assume that z^* is chosen so that its range equals the interval $[0,\gamma^*]$. This is simply the completeness condition which guarantees that there is a demand for every possible asset in the interval $[0,\gamma]$. We can now describe the history of an asset from its 'birth' until its 'death.' Let the asset begin at time 0 as a newly produced asset. It is initially bought by some consumer τ_1 who sells the asset next period to consumer τ_2 whose optimal policy is to always hold an asset in condition y_2. Each period a similar trade is made between the successive owners of the asset until the first time the asset's condition exceeds γ^*, at which time the asset is sold for scrap value $\underline{P}$. In the language of stochastic processes, the life history of a durable is a realization of a point process with an absorbing barrier at γ^*.

4. Characterization of Stationary Equilibrium in Secondary Markets

It is convenient to begin with the case of a homogeneous consumers population

$$(4.1) \qquad H(\tau) = \begin{cases} 1 & \text{if } \tau \geq \tau^* \\ 0 & \text{if } \tau < \tau^* \end{cases}$$

The results in this case are transparent, and turn out generalize directly to general population distributions H. With homogeneous consumers it is clear that in any equilibrium all assets must be priced to yield the same level of utility

$$(4.2) \qquad [1/(1-\beta)][M(x;\tau^*)+P(x)-\beta EP(x)]=[1/(1-\beta)][M(0;\tau^*)+\overline{P}-\beta EP(0)] \quad 0\leq x\leq\gamma^*$$

Any price function P(x) which satisfies (4.2) together with the corresponding holdings distribution $F(x;\gamma^*)$ constitute a stationary equilibrium provided P(x) = $\underline{P}$, $x\geq\gamma^*$. Although consumers are indifferent to the choice of durable, the equilibrium price function causes consumers to "locate" along the interval $[0,\gamma^*]$ according to $F(x;\gamma^*)$ so that stock equilibrium is fulfilled. If consumers were not located this way, there would be a tendency for prices to rise or fall in locations where there are too many or too few consumers relative to supply. This would entail instantaneous relocations of consumers until all assets are priced to yield the same utility. Once the location distribution coincides with $F(x;\gamma^*)$ and prices are given by P(x) there will be no changes in prices or holdings and no incentive to relocate.

Since $P(x)\geq\underline{P}$ we can rewrite (4.2) as

$$(4.3) \qquad P(x) = \max[\underline{P},\overline{P}-\beta EP(0)+M(0;\tau^*)-M(x;\tau^*)+\beta EP(x)]$$

Thus, a stationary equilibrium will by any solution to the functional equation (4.3) which satisfies $P(x)=\underline{P}$, $x>\gamma^*$.

Theorem 4.1 Under assumptions (A1),...,(A8) if H is given by (4.1), then a unique stationary equilibrium $\{P,F,\gamma^*\}$ exists and is given by

$$P(x) = \overline{P}-[J_\tau^*(x)-J_\tau^*(0)] \tag{4.4}$$

where γ^* and J_τ^* are the optimal stopping barrier and value function from the solution to the optimal stopping problem (2.12), and where $F(x;\gamma^*)$ is the unique solution to the functional equation (3.1).

Proof. By Theorem 2.2 the optimal value function J_τ^* exists and is the unique solution to Bellman's equation (2.10). Assumptions (A4) and (A7) imply the monotonicity of $M + EJ_\tau^*$ which implies that the optimal stopping barrier γ^* is the unique solution to equation (2.12). Consider the candidate price function given by (4.4). Clearly $P(0)=\overline{P}$. Equation (2.12) implies that $P(x)=\underline{P}$ for $x > \gamma^*$. It only remains to show that P(x) is the unique solution to (4.3). It is easy to verify that (4.3) defines a fixed point to a contraction mapping, so a unique solution exists. Substituting (4.4) into (4.3) and using the fact that J_τ^* is the unique solution to the functional equation (2.10), it follows that $P(x)=\overline{P}-[J_\tau^*(x)-J_\tau^*(0)]$ is the required solution to (4.3) Q.E.D.

Theorem 4.1 shows that a stationary equilibrium arises from the solution to the optimal stopping problem (2.10). Equilibrium quantities are determined from the optimal stopping barrier γ^* and the stationary holdings distribution $F(x;\gamma^*)$. Equilibrium prices are simply "shadow prices" from the optimal stopping problem. Formula (4.4) shows that the price of asset x equals the price of a new asset $\overline{P}$, less a discount $[J_\tau^*(x)-J_\tau^*(0)]$ to compensate for the increased operating costs of holding asset x. Since the optimal value function J_τ^* can be computed by successive approximations, it follows that equilibrium prices can be computed for any cost function M and transition probability Φ. Given J_τ^*, one can solve for γ^* and compute $F(x;\gamma^*)$ by standard numerical methods, see Futia [17]. For more specific parametric forms we can obtain explicit closed-form solutions for equilibrium.

Theorem 4.2 Under assumptions (A1),...,(A8) if $\Phi(x,y) = 1-e^{-\lambda(y-x)}$, then the unique stationary equilibrium is given by

$$F(x;\gamma^*)=(1+\lambda x)/(1+\lambda\gamma^*) \tag{4.5}$$

$$P(x) = \max\left[\underline{P},\ \underline{P}+[1/(1-\beta)]\int_x^{\gamma^*} \partial M(y;\tau^*)/\partial y[1-\beta e^{-\lambda(1-\beta)(y-x)}]dy\right] \tag{4.6}$$

where γ^* is the unique solution to

$$(\overline{P}-\underline{P})(1-\beta) = \int_0^{\gamma^*} \partial M(y;\tau^*)/\partial y[1-\beta e^{-\lambda(1-\beta)y}]dy \tag{4.7}$$

The important point to notice about Theorem 4.2 is that if M is concave, then the equilibrium price function is convex. This yields a pattern of rapid early depreciation which we observe for many durable assets such as automobiles. Note that such a depreciation pattern is generated without assuming the existence of informational asymmetries which create the "lemon's problem" which is commonly believed to be responsible for rapid early depreciation. Theorem 4.2 suggests that observed price structures might be equally well explained in a perfect information framework with natural specifications of consumer preferences and aging process for durables. The convexity result extends to general transition probabilities Φ which are convexity-preserving. We say that a linear operator E is convexity-preserving if Ef is convex whenever f is. We say that the transition probability Φ convexity-preserving if its associated conditional expectation operator E is convexity-preserving. The following theorem, whose proof is given in the appendix, shows that this concept provides a general sufficient condition for a convex price structure.

<u>Theorem 4.3</u> Under assumptions (A1),...,(A8) if Φ is convexity preserving and M is concave, then P is convex.

The homogeneous consumer model is objectionable because it implies an absence of gains from trade: consumers are no better off with a secondary market than under autarky. It follows that a homogeneous consumer model cannot provide a positive theory for the existence of secondary markets. To remedy this defect we now consider heterogeneous consumer economies with general population distributions H. Gains from trade due to heterogeneity imply that consumers are unambiguously better off than under autarky. Our analysis of the homogeneous consumer case is not superfluous, however, since we will show that each heterogeneous agent stationary equilibrium is observationally equivalent to a homogeneous consumer stationary equilibrium.

With a continuum of distinct consumer types and only one equilibrium price function, it follows that consumers will no longer be indifferent between various conditions of durables. For a given price function, consumer τ will "locate" at the condition $Z^*(\tau)$ which maximizes expected discounted utility. Consumers sort themselves by type; those with strong preference for newness buy new durables and sell their used assets to consumers with less intense preferences for newness. The reduced price of used durables compensates these consumers for holding older durable goods. The "shape" of the price function must be chosen to cause the distribution of consumer demand to equal the distribution of supply. The added complication is that since scrappage is determined endogenously, the range of available conditions must be determined simultaneously with equilibrium prices. The following theorem shows how this is done.[6]

<u>Theorem 4.4</u> Under assumptions (A1),...,(A8) a unique stationary equilibrium exists and is given by the unique solution to the functional equation

$$P(x)=\max[\underline{P},\overline{P}-\beta EP(0) - \int_0^x \partial M(y;H^{-1}\{1-F(y;\gamma^*)\})/\partial y dy + \beta EP(x)] \tag{4.8}$$

where γ^* is the smallest solution to

$$\overline{P} - \beta EP(0) = \underline{P} - \beta EP(\gamma^*) + \int_0^{\gamma^*} \partial M(y;H^{-1}\{1-F(y;\gamma^*)\})/\partial y dy \tag{4.9}$$

and where $F(x;\gamma^*)$ is the unique solution to (3.1).

<u>Proof</u>. By assumption (A8) we conjecture that the optimal asset $z^*(\tau)$ for consumer τ is unique, and that the function z^* is monotonically decreasing on the interval $[\underline{\tau},\hat{\tau})$, where $\hat{\tau}$ is the smallest solution to $z^*(\hat{\tau})=0$. Using the equilibrium condition (3.3) and letting $\tau(x)$ denote the inverse of $z^*(\tau)$, $\tau(x)=z^{*-1}(x)$, we have

$$\int_{\underline{\tau}}^{\overline{\tau}} I\{\tau \mid z^*(\tau)<x\}H(d\tau) = 1-H(\tau(x)) = F(x;\gamma^*) \tag{4.10}$$

which implies

$$\tau(x) = H^{-1}\{1-F(x;\gamma^*)\} \qquad x \in (0,\gamma^*] \tag{4.11}$$

From the first order condition of the consumer's cost minimization problem we have;

$$0=\partial M(z^*(\tau);\tau)/\partial z + P'(z^*(\tau)) - \beta EP'(z^*(\tau)) \quad \tau \in [\underline{\tau},\hat{\tau}) \tag{4.12}$$

Taking inverses we obtain

(4.13) $0 = \partial M(x;\tau(x))/\partial x + P'(x) - \beta EP'(x) \quad x \in (0,\gamma^*]$

Integrating (4.13) and using the fact that $P(x) \geq \underline{P}$ we obtain Equation (4.8). Given a solution γ^* to (4.9), it follows that (4.8) defines a fixed point to a contraction mapping so a unique solution P exists. By construction $\{P,F,\gamma^*\}$ satisfies conditions 1, 2, 3 and 5 of the definition of a stationary equilibrium. To complete the proof we must verify that given P, it is indeed optimal for a consumer τ to choose asset $z^*(\tau)$. The details of this argument are presented in the appendix. Q.E.D.

The striking feature of Theorem 4.4 is that the functional equation which determines equilibrium prices (4.8), is identical to the corresponding equation for the homogeneous consumer economy. If we define the function $M(x;\gamma^*)$ by

(4.14) $M(x;\gamma^*) = \int_0^x \partial M(y;H^{-1}\{1-F(y;\gamma^*)\})/\partial y dy$

then the equilibrium price function P from the solution to (4.8) could equally well be regarded as the equilibrium price function for a homogeneous consumer economy when consumers have preferences $M(x;\gamma^*)$ given by (4.14). In this sense a heterogeneous consumer economy is observationally equivalent to a homogeneous consumer economy: given any heterogeneous consumer stationary equilibrium we can find a homogeneous consumer stationary equilibrium with the same price function and holdings distribution. In one sense this result is disheartening; given that we observe only equilibrium prices and quantities we cannot separately identify the functional form of consumer utility functions $M(x;\tau)$ from the population distribution of preferences $H(\tau)$. On the other hand, the result shows that for empirical purposes we can restrict our attention to the simpler homogeneous consumer framework without loss of generality. Even though the actual economy has heterogeneous consumers, we can test the implications of the stationary equilibrium model using in equivalent homogeneous consumer model, estimating the "composite" utility function $M(x;\gamma^*)$ given in (4.14).

Theorem 4.4 gives us an explicit characterization of consumer holdings of durables. Inverting $\tau(x)$ to obtain $z^*(\tau)$ we obtain

$$(4.15) \qquad z^*(\tau) = \begin{cases} F^{-1}\{(1-H(\tau);\gamma^*\} & \tau \in [\underline{\tau},\hat{\tau}) \\ 0 & \tau \in [\hat{\tau},\overline{\tau}] \end{cases}$$

where $\hat{\tau} = H^{-1}\{1-F(0;\gamma^*)\}$. This is a "location function" which identifies the type of durable chosen by each consumer τ. Notice that the price structure influences consumer location decisions only through the "sufficient statistic," γ^*. Location decisions are primarily governed by technological considerations as embodied by F^{-1}; this function allocates consumers in the right way so that consumer demand exactly matches the available stock. Consumer preferences do not enter except insofar as they determine equilibrium prices and equilibrium scrappage. From (4.14) we can see directly how the holdings distribution F enters the "composite utility function" $M(x;\gamma^*)$ which in turn determines prices. The following Lemma provides a useful alternative interpretation of the composite utility function $M(x;\gamma^*)$.

<u>Lemma 4.1</u> Under assumptions (A1),...,(A8) we have

$$(4.16) \qquad M(x;\gamma^*) = \int_0^x \partial M(y;H^{-1}\{1-F(y;\gamma^*)\})/\partial y dy = \int_{H^{-1}\{1-F(x;\gamma^*)\}}^{\hat{\tau}} \partial M(F^{-1}\{1-H(\tau);\gamma^*\};\tau)/\partial x d\tau$$

The proof is a simple exercise in a change of variables. According to Lemma 4.1 the composite utility function $M(x;\gamma^*)$ is simply an equally weighted average of marginal utilities for all consumers buying assets in the interval (0,x], with the exception that consumers $\tau > \hat{\tau}$ who buy new assets are not given any weight at all. This is due to the fact that in this model, once a consumer has bought a new asset x=0, there is nothing more they can do to express the intensity of their preferences for newness. Only consumers at the marginal level $\hat{\tau}$ and below can affect allocation decisions and indeed it is only they who are included in the social welfare function. All included consumers are given equal weight, and for this reason we call $M(x;\gamma^*)$ a "modified utilitarian" social welfare function. The next result shows that equilibrium scrappage is the solution to an optimal stopping problem using the modified utilitarian welfare function as the objective function.

Theorem 4.5 Under assumptions (A1),...,(A8) the stationary equilibrium $\{P,F,\gamma^*\}$ is given by

$$P(x)=\overline{P}-[J_{mu}(x)-J_{mu}(0)] \tag{4.17}$$

where γ^* and J_{mu} are the optimal stopping barrier and value function from the solution to the optimal stopping problem (2.10) with the modified utilitarian objective function $M(x;\gamma^*)$, and where F is given by the unique solution to (3.1).

The proof of Theorem 4.5 follows directly from Theorem 4.1. It follows that stationary equilibrium with heterogeneous consumers is Pareto efficient, and equilibrium prices are simply "shadow prices" from the solution to the social optimal stopping problem. Thus, stationary equilibrium with heterogeneous consumers can be calculated in exactly the same way as with homogeneous consumers, but with one catch: the social planner must know the optimal stopping barrier γ^* ahead of time in order to determine the correct social welfare function $M(x;\gamma^*)$. If the social planner chooses an arbitrary value γ, then computes the optimal stopping barrier $f(\gamma)$ corresponding to the social welfare function $M(x;\gamma)$, it may turn at that $\gamma^{*} \neq f(\gamma)$. An equilibrium is located when $\gamma^{*}=f(\gamma^{*})$. To conclude this section, we show that for specific functional forms for Φ we can derive closed-form solutions for equilibrium.

Theorem 4.6 Under assumptions (A1),...,(A8) if $\Phi(x,y)=1-e^{-\lambda(y-x)}$, then the unique stationary equilibrium is given by

$$F(x;\gamma^{*}) = (1+\lambda x)/(1+\lambda\gamma^{*}) \tag{4.18}$$

$$P(x)=\max[\underline{P},\underline{P}+[1/(1-\beta)]\int_{x}^{\gamma^{*}}\partial M(y;H^{-1}\{\lambda(\gamma^{*}-y)/(1+\lambda\gamma^{*})\})/\partial y[1-\beta e^{-\lambda(1-\beta)(y-x)}]dy] \tag{4.19}$$

where γ^{*} is the unique solution to

$$(\overline{P}-\underline{P})(1-\beta) = \int_{0}^{\gamma^{*}}\partial M(y;H^{-1}\{\lambda(\gamma^{*}-y)/(1+\lambda\gamma^{*})\})/\partial y[1-\beta e^{-\lambda(1-\beta)y}]dy \tag{4.20}$$

The equilibrium price function (4.19) shows explicitly how the structure of durable prices depends on the technological characteristics of the asset, and the form and distribution of consumer preferences. Notice the striking similarity of this solution and the solution to the homogeneous case given in Theorem 4.2. The result underscores the observational equivalence of heterogeneous and homogeneous stationary equilibria.

5. Stationary Equilibrium in Rental Markets

Our formulation of consumer preferences for durable goods given in Section 2 is restrictive in two respects: a) we have assumed that consumers are risk neutral, and b) we have implicitly assumed that consumer preferences for consumption of durables is strongly separable from consumption of other goods. The latter assumption allowed us to isolate the durable purchase decision from the consumption decision for other goods. In this section we show that if a competitive rental market for durable assets exists, stationary equilibria will exist for very general non-separable specifications of consumer preferences, and equilibrium prices and quantities will continue to satisfy the same functional equations derived in Section 4. A further consequence is that all the properties of stationary equilibrium derived in our partial equilibrium framework continue to hold in any general equilibrium system in which our durables market is imbedded.

To simplify the presentation, we initially assume that no secondary market exists and that each durable x must be rented on a one period contract from a risk neutral rental intermediary at a nonstochastic rate R(x). The rental intermediary buys new assets from the producer at price $\overline{P}$ and rents the asset each period of its economic lifespan which is determined from the solution to an optimal rental policy. Let V(x) be the present discounted value of a rental intermediary which owns a single asset in condition x, and let β be the risk-free discount factor. Then V is the unique solution to the functional equation

$$(5.1) \qquad V(x) = \max[\underline{P}, R(x)+\beta EV(x)]$$

The optimal rental policy corresponding to (5.1) is simply a stopping rule S($\cdot$) of the form

$$(5.2) \qquad S(x) = \begin{cases} 1 \text{ (rent)} & \text{If } x \in [0,\gamma] \\ 0 \text{ (scrap)} & \text{If } x \in (\gamma,\infty) \end{cases}$$

where γ is the smallest solution to $V(\gamma)=\underline{P}$.

By duality theory, a general representation of preferences of consumer τ for alternative durable assets x is given by a conditional indirect utility function

$U(x,I-R(x),\tau)$ defined by

$$(5.3) \qquad U(x,I-R(x),\tau) = \max_{(q_1,\ldots,q_n)} u(q_1,\ldots,q_n,x,\tau)$$

subject to

$$(5.4) \qquad \sum_{i=1}^{n} p_i q_i + R(x) \leq I$$

where we leave implicit the dependence of U on the price vector $(p_1,\ldots,p_n)$. We need the following assumptions.

(A9) The rental market is perfectly competitive and all rental intermediaries have access to a perfect capital market for borrowing and lending a risk-free discount factor β.

(A10) Rental intermediaries can costlessly verify the level of maintenance, usage, or any other action undertaken by the renter which affects the rate of deterioration of the durable.

(A11) U is twice continuously differentiable in all of its arguments and satisfies for some ε and δ

$$\partial U/\partial x \leq \varepsilon < 0 \qquad \delta \geq \partial U/\partial I > 0 \qquad \partial^2 U/\partial x \partial\tau < 0 \qquad \partial^2 U/\partial I \partial\tau \geq 0$$

Assumptions (A9) and (A10) enable a competitive rental market to exist. Assumption (A11) allows us to rank consumer preferences by consumer type τ: higher τ consumers have stronger preferences for newness and possibly a higher marginal utility of income.

<u>Definition</u>: A <u>stationary equilibrium with rental markets</u> is given by $\{R,V,F,\gamma^*\}$ where R, V, and F are functions and γ^* is a constant which satisfies

1. $V(0)=\overline{P}$
2. $V(x)=\underline{P}\ \ x > \gamma^*$
3. $F(\cdot;\gamma^*)$ is a stationary holdings distribution corresponding to Φ and γ^*; the unique solution to the functional equation (3.1).
4. Each consumer $\tau \in [\underline{\tau},\overline{\tau}]$ chooses a rental asset in the set $Z^*(\tau)$ given by[7]

$$Z^*(\tau) = \underset{0<x<\gamma^*}{\text{argmax}}\ U(x,I-R(x),\tau) \tag{5.5}$$

5. There exists a complete, measurable selection z^* from Z^* which satisfies for all x

$$\int_{\underline{\tau}}^{\overline{\tau}} I\{\tau \mid z^*(\tau)<x\}H(d\tau) = F(x;\gamma^*) \tag{5.6}$$

6. V is the unique solution to the functional equation

$$V(x) = \max\ [\underline{P},R(x)+\beta EV(x)] \tag{5.7}$$

Most of the above conditions should be self-explanatory. Condition 1 is an equilibrium condition for rental intermediary purchases in the primary market. If $V(0)<\overline{P}$, it would be unprofitable to buy new assets so demand would fall to zero. If $V(0)>\overline{P}$ competition and new entry would drive down rental rates until $V(0)=\overline{P}$. In condition 4, stationarity and the one period length of rental contracts implies that the consumer's intertemporal decision problem reduces to a sequence of one period optimization problems (5.5).

From the first order condition for an optimum, consumer τ's choice of rental asset $z^*(\tau)\varepsilon Z^*(\tau)$ satisfies

$$(5.8)\qquad 0=\partial U(x,I-R(x),\tau)/\partial x-[\partial U(x,I-R(x),\tau)/\partial I]\partial R(x)/\partial x\Big|_{x=z^*(\tau)}$$

conjecturing that Z^* is single-valued and monotonic it follows that z^* has inverse $z^{*-1}(x) \equiv \tau(x) = H^{-1}\{1-F(x;\gamma)\}$, $x \varepsilon [0,\gamma]$. Substituting $\tau(x)$ for τ in (5.8) and rearranging we obtain the following differential equation for rental prices

$$(5.9)\qquad \partial R(x)/\partial x = \frac{\partial U(x,I-R(x),H^{-1}\{1-F(x;\gamma)]\})/\partial x}{\partial U(x,I-R(x),H^{-1}\{1-F(x;\gamma)\})/\partial I}$$

By assumption (A11) and the contraction mapping theorem (Hoffman, [19], pp. 267-268), it follows that for each γ (5.9) has a unique solution R_γ.

Theorem 5.1 Under assumptions (A1),...,(A11) a unique stationary equilibrium in rental markets $\{R,F,\gamma^*\}$ exists where R is the unique solution to (5.9) on $[0,\gamma^*]$.

The proof of Theorem 5.1 is given in the appendix. The significance of Theorem 5.1 is that if shares of rental intermediaries are traded in a stock market, then V(x) is precisely the purchase price of asset x, V(x)=P(x). It then follows immediately that these prices are the solution to the same functional equation derived for prices in Section 4 under more restrictive assumptions about consumer preferences (compare (5.1) with (4.3) and (4.6)).[8] Clearly, the functional equations determining rental prices (5.9), and asset prices (5.7), take the same form regardless of the prices of other goods and services. Thus, even though prices of other goods enter consumer's indirect utility function U and affect the shape of the individual solutions to (5.7) and (5.9), the general properties of equilibrium are determined by the form of these functional equations and hence are independent of the particular general equilibrium system which our durable market is imbedded.

Using (5.1) and the fact that V=P, we have for all x

$$(5.10) \quad R(x) = P(x)-\beta EP(x)$$

Competitive rental rates equal expected depreciation: in effect, rental intermediaries provide insurance against capital losses at actuarially fair rates. When rental markets do not exist (5.10) defines the "shadow rental price" of the asset. Solving for $P(x)-\beta EP(x)$ from equation (4.1) we obtain

$$(5.11) \quad R(x) = \overline{P}-\beta EP(0)-[M(x)-M(0)]$$

The shadow rental rate for asset x equals the expected depreciation on a new asset less a discount for the excess operating costs of the used asset.

Theorem 5.1 appears to depend on two restrictive conditions, a) rental intermediaries are risk neutral, and b) consumers do not have the option of owning in addition to renting. In fact, neither of these conditions involve any loss of generality, as the following arguments demonstrate.[9] The law of large numbers implies that assumption a) is actually a consequence of the assumptions of zero transactions costs and infinitely many consumers. The rental intermediary is simply an insurance cooperative which allows consumers to pool and subdivide their risk of capital loss. Thus, suppose a rental intermediary owned by N shareholders purchases N durable assets in condition x. The resale value of these assets on a per share basis is $\sum_{i=1}^{N} P(\tilde{y}_i)\frac{1}{N}$, where $\tilde{y}_i$ is the random end of period condition of asset i. As

$N \to \infty$ this resale value converges to its expectation $EP(x)$. Since resale risk is completely diversifiable, it follows that the relevant discount rate is the risk free rate β and risk averse shareholders will unanimously agree that the rental intermediary should maximize expected discounted profits.

Jensen's inequality implies that condition b) involves no loss of generality; risk averse consumers always prefer to rent at actuarially fair rates rather than own. For example, comparing the expected utility of owning an asset over one period versus renting for one period we have

$$(5.12) \quad E\{U(x,I-[P(x)-\beta P(\tilde{y})],\tau)\} \leq U(x,I-R(x),\tau)$$

since $R(x) = P(x)-\beta EP(x)$ by (5.10).

Theorem 5.1 provides our final interpretation of equilibrium asset prices: $P(x)$ is the expected discounted value of the future rental stream of asset x under an optimal rental policy. This interpretation of equilibrium prices is the starting point of the Wicksellian analysis of durable goods markets: given a rental schedule R, equilibrium prices P are given by the solution to the functional equation (5.1). The contribution of our analysis is to generalize the Wicksellian framework to allow new and used assets to be imperfect substitutes, and to show how the equilibrium rental schedule is determined from the technology Φ, and the form and distribution of consumer prefrences, U,H.

We conclude by emphasizing that when rental markets do not exist, the general approach outlined in this section is inapplicable. In this case asset prices rather than rental prices must bear the burden of equating supply and demand. The nonexistence of rental markets is often the result of a moral hazard problem: the rental intermediary cannot observe whether or not a renter abuses the equipment.

Recalling our discussion in Section 2, this problem can be modelled by allowing the deterioration of the asset Φ to depend on a consumer action λ. Even though the asset's condition x is observable by both parties, only the renter observes the level of care λ. If an asset is returned in poor physical condition x, the rental intermediary cannot determine whether x is due to improper maintenance λ_I, or due to proper maintenance λ_p and an unlucky draw from Φ. The problems of nonexistence of equilibria in a rental market are exactly the same as those present in other insurance markets with moral hazard: in order to cover the losses due to inadequate maintenance, rental rates are driven up. This creates more incentives for dishonest renters to abuse the equipment leading to a spiral of increasing rental rates as honest renters decide to own their own equipment rather than pay the price of others' negligence. In the limit, this adverse selection process may drive rental markets out of existence. In these cases private ownership may be a relatively efficient incentive-compatible allocation mechanism. This may be one reason why we sometimes observe active secondary markets for durable goods, but weak or nonexistent rental markets. A complete analysis of this issue might be an interesting topic for future research.

Footnotes

[1] A common alternative formulation of the Wicksellian model is to treat the durable good as perfectly divisible with service flow proportional to quantity available, regardless of the vintage of this total quantity. If the initial quantity of the durable is normalized to 1 and Q(t) denotes the quantity remaining at time t, it is easy to see that an equivalent reinterpretation is to treat the durable as an indivisible unit, and interpret Q(t) as the complement of the cdf of the asset's lifetime distribution. For details, see Parks [26].

[2] Our model can be easily generalized to allow scrap prices $\underline{P}$ to be determined endogenously by equating supply and demand in the scrap market.

[3] Rust [29] studies equilibrium in the primary market where $\overline{P}$ and Φ are chosen to maximize the expected discounted profits of a monopolist.

[4] In the following analysis we assume that the consumer's per period endowment of income is sufficient to finance any desired holding strategy, or alternatively, that the consumer has access to a perfect capital market so that period budget constraints can be ignored.

[5] We say that z^* is <u>complete</u> if its range equals the interval $[0,\gamma^*]$.

[6] Theorem 4.4 may be of independent interest to researchers in monopolistic competition theory. Durable choice can be interpreted as choice of a continuously differentiated commodity. Our equilibrium price function matches supply and demand in each point in the commodity space, with the range of available products determined endogenously. In this sense, Theorem 4.4 provides a solution for a class of monopolistically competitive equilibria very similar to a class first proposed by Rosen [27].

[7] We have assumed that income I is the same for all consumers only to simplify notation: all our results go through if I is a continuously differentiable function of τ with $\partial I/\partial \tau > 0$.

[8] Note that although the results of Section 4 require more restrictions on consumer preferences, they do not require existence of a rental market.

[9] In the interest of space we substitute hueristic arguments in place of formal proofs.

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Appendix: Proofs of Theorems.

Theorem 2.1 Since the set of available assets is the compact interval $[0,\gamma^*]$, there exists an optimal replacement asset $z^*(\tau)$; an asset which attains the infimum in (2.6). Suppose for $x \neq z^*(\tau)$ it is optimal to keep x, then (2.6) implies

$$(1) \qquad P(z^*(\tau)) + M(z^*(\tau);\tau) + \beta EJ_\tau(z^*(\tau)) > P(x) + M(x;\tau) + \beta EJ_\tau(x)$$

This contradicts the definition of $z^*(\tau)$. Thus, $u_t(x) = z^*(\tau)$ for all x. Using this fact (2.6) reduces to

$$(2) \qquad J_\tau(x) = P(z^*(\tau)) - P(x) + M(z^*(\tau);\tau) + \beta EJ_\tau(z^*(\tau))$$

Taking expectations on both sides of (2) and solving for $EJ_\tau(z^*(\tau))$ yields the solution for J_τ given in (2.9). Substituting this solution for J_τ into (2.6), it follows $z^*(\tau)$ is a solution to (2.8). Q.E.D.

Theorem 4.3 Since equilibrium prices P are the fixed point of a contraction mapping, we have

$$(3) \qquad P = \lim_{j\to\infty} T^j(J_o)$$

where $J_o = 0$, $T(J_o)(x) = \max[\underline{P}, R(x) + \beta EJ_o(x)]$, and $R(x) = \overline{P} - \beta EP(0) + M(0;\tau^*) - M(x;\tau^*)$. Since M is concave and Φ is convexity preserving, it follows that for each j, $T^j(J_o)$ is convex. Since P is the uniform limit of convex functions, it is also convex. Q.E.D.

The following lemma will be used in the proof of Theorem 4.4.

Lemma The stationary distribution $F(\cdot;\gamma)$ given by the unique solution to (3.1) is a weakly continuous function of γ.

Proof Assumptions (A3) and (A4) imply that for each x the transition probability Ψ_γ given by (3.4) is a weakly continuous function of γ, i.e., $\Psi_{\gamma_n}(x,y) \to \Psi_\gamma(x,y)$ for every continuity point y of Ψ_γ and for each sequence $\{\gamma_n\}$ converging by γ. The Lesbesgue dominated convergence theorem implies that for any probability measure μ, the probability distribution $E^*_\gamma\mu(x) = \int_0^\infty \Psi_\gamma(y,x)\mu(dy)$ is weakly continuous in γ. By Theorem 2.10 of Futia [17], the sequence $\frac{1}{N}\sum_{i=0}^{N-1}(E^*_\gamma)^i(\mu)$ converges weakly to $F(\cdot;\gamma)$ for any initial probability μ. By induction $(E^*_\gamma)^i(\mu)$ is weakly continuous in γ for each i, therefore it follows immediately that $F(\cdot;\gamma)$ is weakly continuous in γ.

Q.E.D.

Theorem 4.4 To complete the proof given in the text we must verify that 1) a solution γ^* to (4.9) exists, and 2) given the conjectured solution (4.8) for P, $z^*(\tau) = F^{-1}\{1-H(\tau);\gamma^*\}$ is in fact the optimal asset choice for consumer τ. To establish the first result, note that Theorem 4.1 implies that for each γ, the solution for P in (4.8) is simply the dual or shadow price solution to the optimal stopping problem (2.10) with the cost function $M(x;\gamma) = \int_0^x \partial M(y;H^{-1}\{1-F(y;\gamma)\})/\partial y dy$. Let the optimal stopping barrier corresponding to this cost function be denoted by $f(\gamma)$. It is easy to verify that the required γ^* which satisfies (4.9) is simply the smallest fixed point of f. Since $f(0) > 0$ and $\lim_{\gamma\to\infty} f(\gamma) < +\infty$, it follows that such a fixed point $f(\gamma^*) = \gamma^*$ will exist provided f is a continuous function of γ. By assumption (A3), $F(\cdot;\gamma)$ is absolutely continuous so that by the Lemma proven above $F(x;\gamma)$ is a continuous function of γ for each x. Since H is absolutely continuous by (A8) it follows that for each y, $\partial M(y;H^{-1}\{1-F(y;\gamma)\})/\partial y$ continuous function of γ. By the Lesbesgue dominated convergence theorem, it follows that $M(x;\gamma)$ is a continuous function of γ for each x. Let J_γ by the unique solution to (2.10) with cost function $M(x;\gamma)$. Theorem 3 of Kantorovich and Aikilov [20] (p. 476) implies that J_γ is a continuous in γ, uniformly in x. Since $f(\gamma)$ is the unique solution to (2.12) it follows that $f(\gamma) = [M + \beta EJ_\gamma]^{-1}(\overline{P} - \underline{P} + J_\gamma(0))$ is a continuous function of γ. We conclude that a smallest fixed point $\gamma^* = f(\gamma^*)$ which solves (4.9) exists.

To complete the proof we verify that $z^*(\tau) = F^{-1}\{1-H(\tau);\gamma^*\}$ is in fact the optimal asset choice for consumer τ. There are actually two cases to verify:

Case I - interior minimum: if $\tau \in [\underline{\tau}, H^{-1}\{1-F(0;\gamma^*)\})$, then $z^*(\tau) = F^{-1}\{1-H(\tau);\gamma^*\}$ minimizes $M(x;\tau) + P(x) - \beta EP(x)$.

Case II - boundary minimum: if $\tau \in [H^{-1}\{1-F(0;\gamma^*)\}, \overline{\tau}]$, then $z^*(\tau) = 0$ minimizes $M(x;\tau) + P(x) - \beta EP(x)$.

Consider case I. By construction it is easy to see that $z^*(\tau)$ is a critical point of $M(x;\tau) + P(x) - \beta EP(x)$; see (4.12) and (4.13). To show this is a local minimum we compute the second derivative

$$(4) \qquad (1-\beta)J''(z^*(\tau)) = \partial^2 M(z^*(\tau);\tau)/\partial x \partial x + P''(z^*(\tau)) - \beta EP''(z^*(\tau))$$

Differentiating the identity (4.13), solving for $P''(x)$, evaluating this at $z^*(\tau)$, and substituting into (5), we obtain

$$(5) \qquad (1-\beta)J''(z^*(\tau)) = \frac{[\partial^2 M(z^*(\tau);\tau)/\partial x \partial \tau][\partial F(z^*(\tau);\gamma^*)/\partial x]}{h(1-F(z^*(\tau);\gamma^*))}$$

Assumption (A3) implies $\partial F(x;\gamma^*)/\partial x$ exists and is positive, so that by assumptions (A7) and (A8) $J''(z^*(\tau)) > 0$, so $z^*(\tau)$ is a local minimum. To show this is the unique global minimum, suppose there exists a y such that $J_\tau(y) < J_\tau(z^*(\tau))$. Suppose that $y < z^*(\tau)$. Choose $w \in (y, z^*(\tau))$ so that $J_\tau'(w) < 0$ and $J_\tau(z^*(\tau)) < J_\tau(w)$ (the existence of such a w follows from the fact that $z^*(\tau)$ is a local minimum). By the mean value theorem, there exists a $v \in (y,w)$ such that

$J_\tau'(v) = [J_\tau(w)-J_\tau(y)]/(w-y) > 0$. By the intermediate value theorem it follows that there exists a $u \in (v,w)$ such that $J_\tau'(u) = 0$. We then have

(6) $$0 = \partial M(u;\tau)/\partial x + P'(u) - \beta EP'(u)$$

(7) $$0 = \partial M(u;\tau(u))/\partial x + P'(u) - \beta EP'(u)$$

Since $u < z^*(\tau)$ it follows that $\tau(u) > \tau$. By assumption (A7), $\partial^2 M(x;\tau)/\partial x\partial\tau > 0$ so that (7) and (8) imply a contradiction. A symmetric argument shows that $y > z^*(\tau)$ implies $J_\tau(y) > J_\tau(z^*(\tau))$. Thus, $z^*(\tau)$ is the unique global minimum for case I. To verify case II, let $\hat{\tau} = H^{-1}\{1-F(0;\gamma^*)\}$. Formula (4.12) implies that $J_{\hat{\tau}}'(0) = 0$. Since $\partial^2 M(x;\tau)/\partial x\partial\tau > 0$, it follows that $J_\tau'(0) > 0$ for $\tau > \hat{\tau}$, so that 0 is a local minimum of $J_\tau(0)$ for $\tau \in (\hat{\tau}\ \bar{\tau}]$. Suppose for some $\tau \in (\hat{\tau},\bar{\tau}]$ there exists an x such $J_\tau(x) < J_\tau(0)$. By the mean value theorem there exists a $v \in (0,x)$ such that $J_\tau'(v) = [J_\tau'(x) - J_\tau'(0)]/x < 0$. By the intermediate value theorem there exists a $w \in (0,v)$ such that $J_\tau'(w) = 0$. This implies that the following equations hold simultaneously

(9) $$0 = \partial M(w;\tau)/\partial x + P'(w) - \beta EP'(w)$$

(10) $$0 = \partial M(w;\tau(w))/\partial x + P'(w) - \beta EP'(w)$$

which is a contradiction since $\partial^2 M(x;\tau)/\partial x\partial\tau > 0$ and $\tau(w) < \tau$. Q.E.D.

<u>Theorem 5.1</u> Assumption (A11) implies that $\partial R/\partial x < \varepsilon/\delta < 0$. The contraction mapping theorem guarantees that given any constant γ and any initial condition the differential equation (5.9) has a unique solution $R_{k,\gamma}$ on the compact set $[0,-\delta k/\varepsilon]$ satisfying $R_{k,\gamma}(0) = k$. Let $V_{k,\gamma}$ be the unique solution to (5.7) with rental function $R_{k,\gamma}$ (extended to R+ by defining $R_{k,\gamma}(x) = R_{k,\gamma}(-\delta k/\varepsilon)$ for $x > -\delta k/\varepsilon$). Since

$R_{k,\gamma}$ and $V_{k,\gamma}$ are fixed points to contraction mappings which depend continuously on γ and k, Theorem 4 of Kantorovich and Aikilov [20], (p. 476) guarantees that $R_{k,\gamma}$ and $V_{k,\gamma}$ are continuous functions of γ and k, uniformly in x. Since $V_{\underline{P},\gamma}(0) = \underline{P}$ and $\lim_{k\to\infty} V_{k,\gamma}(0) = +\infty$, it follows that for each γ there exists a smallest constant $k(\gamma)$ such that $V_{k(\gamma),\gamma}(0) = \overline{P}$. To simplify notation, let $R_\gamma \equiv R_{k(\gamma),\gamma}$ and $V_\gamma \equiv V_{k(\gamma),\gamma}$. Let $f(\gamma)$ be the optimal stopping barrier for V_γ, i.e., the smallest solution to $\underline{P} = R_\gamma(f(\gamma)) + \beta EV_\gamma(f(\gamma))$. Since $R_\gamma(x)$ is negative for $x > -\delta k(\gamma)/\varepsilon$, it follows that $f(\gamma) < -\delta k(\gamma)/\varepsilon$. Since $f(0) > 0$, a fixed point $f(\gamma^*) = \gamma^*$ will exist provided f is continuous. Continuity of f in γ is established exactly as in Theorem 4.4, using the fact that both R_γ and V_γ are continuous in γ, uniformly for x in $[0,-\delta k(\gamma)/\varepsilon]$. It follows that a fixed point γ^* exists such that $V_{\gamma^*}(0) = \overline{P}$ and $V_{\gamma^*}(x) = \underline{P}$ for $x > \gamma^*$. To conclude the proof we must verify that given the rental function $R = R_{\gamma^*}$, $z^*(\tau) = F^{-1}\{1-H(\tau);\gamma^*\}$ is in fact the optimal asset choice for consumer τ. This may be verified using an argument identical to the one used to establish this result in Theorem 4.4.

Q.E.D.