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ECONOMETRIC INSTITUTE

SCHEDULING SUBJECT TO RESOURCE  
CONSTRAINTS: CLASSIFICATION  
AND COMPLEXITY

J. BLAZEWICZ J.K. LENSTRA A.H.G. RINNOOY KAN

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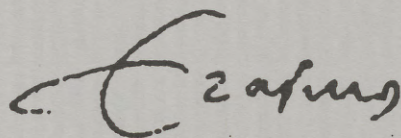
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REPORT 8018/0

SCHEDULING SUBJECT TO RESOURCE CONSTRAINTS:  
CLASSIFICATION AND COMPLEXITY

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ABSTRACT

In deterministic sequencing and scheduling problems, jobs are to be processed on machines of limited capacity. We consider an extension of this class of problems, in which the jobs require the use of additional scarce resources during their execution. A classification scheme for resource constraints is proposed and the computational complexity of the extended problem class is investigated in terms of this classification. Models involving parallel machines, unit-time jobs and the maximum completion time criterion are studied in detail; other models are briefly discussed.

KEY WORDS & PHRASES: *scheduling, jobs, machines, resource constraints, classification, polynomial algorithm, NP-hardness.*

NOTE: This report will be submitted for publication in a journal.

## 1. INTRODUCTION

In the traditional class of deterministic sequencing and scheduling problems [Conway et al. 1967; Graham et al. 1979], jobs  $J_1, \dots, J_n$  consisting of one or more operations are to be processed on machines  $M_1, \dots, M_m$ . Each machine can handle at most one job at a time and each job can be executed by at most one machine at a time. Thus, at any time, the execution of a job is restricted by the presence of a single scarce resource. We shall consider an extension of this class by allowing for the presence of more than one scarce resource. Each operation of a job requires the use of a given fraction of each of the resources, and the problem is to find an optimal schedule subject to these additional resource constraints. Such models occur for example in the context of computer operating systems and project scheduling.

Various assumptions can be made about the number of resources, about the amounts in which they are available, and about the amounts which are required by the operations. Section 2 introduces a simple classification scheme for resource constraints that captures many variations of the model. It expands the classification scheme for scheduling problems given in [Graham et al. 1979], the relevant part of which is included as an Appendix.

In general, the addition of resource constraints to a scheduling problem may affect its computational complexity. In particular, certain well-solved problems, for which polynomial-time algorithms exist, may be transformed into NP-hard ones, for which the existence of such algorithms is very unlikely [Karp 1975; Garey & Johnson 1979]. The obvious research program would be to determine the borderline between easy and hard resource constrained scheduling problems, much in the same vein as has been done for the traditional class, and possibly through the use of an extension of the computer aided complexity classification developed for that purpose [Lageweg et al. 1980]. Rather than attempting such a complete and probably somewhat tedious analysis, we will concentrate on single operation models with unit processing times and the maximum completion time criterion. Section 3 presents our results for these models. Section 4 deals briefly with some other models, viz. extensions to other optimality criteria, preemptive scheduling, and multi-operation models. Section 5 contains some concluding remarks.



## 2. CLASSIFICATION OF RESOURCE CONSTRAINTS

The classification scheme for resource constrained scheduling problems introduced in [Graham et al. 1979] will be used in this paper as well. Briefly, a problem type corresponds to a three-field notation  $\alpha|\beta|\gamma$ , where  $\alpha$  specifies the *machine environment*,  $\beta$  indicates certain *job characteristics*, and  $\gamma$  denotes the *optimality criterion*. Readers not familiar with this notation are referred to the Appendix, where all the relevant definitions can be found.

We shall expand this classification scheme by allowing the jobs to require the use of additional scarce resources. Suppose that there are  $\ell$  resources  $R_1, \dots, R_\ell$ . For each resource  $R_h$ , there is a positive integer size  $s_h$  which is the total amount of  $R_h$  available at any given time. In single-operation models, there is for each resource  $R_h$  and job  $J_j$  a nonnegative integer requirement  $r_{hj}$  which is the amount of  $R_h$  required by  $J_j$  at all times during its execution. A schedule is *feasible* with respect to the resource constraints if at any time  $t$  the index set  $S_t$  of jobs being executed at  $t$  satisfies  $\sum_{j \in S_t} r_{hj} \leq s_h$  ( $h = 1, \dots, \ell$ ). In multi-operation models, there is for each resource  $R_h$  and operation  $O_{ij}$  a nonnegative integer requirement  $r_{hij}$ , with a similar condition for the feasibility of a schedule.

The presence of scarce resources will be indicated in the second field of our classification scheme by

$$res\lambda\sigma\rho$$

where  $\lambda$ ,  $\sigma$  and  $\rho$  are characterized as follows.

- If  $\lambda$  is a positive integer, then the *number of resources*  $\ell$  is constant and equal to  $\lambda$ ; if  $\lambda = \cdot$ , then  $\ell$  is part of the input.
- If  $\sigma$  is a positive integer, then all *resource sizes*  $s_h$  are constant and equal to  $\sigma$ ; if  $\sigma = \cdot$ , then all  $s_h$  are part of the input.
- If  $\rho$  is a positive integer, then all *resource requirements*  $r_{hj}$  ( $r_{hij}$ ) have a constant upper bound equal to  $\rho$ ; if  $\rho = \cdot$ , then no such bounds are specified.

Many types of resource constraints are not represented by this classification, but in a sense more than enough detail is included already. In fact, we shall

assume that  $\lambda$ ,  $\sigma$  and  $\rho$  are either equal to 1 or to  $\cdot$ ; this restriction still generates most of the relevant and previously studied problem types.

Remembering that  $\sigma = 1$  excludes  $\rho = \cdot$ , we obtain six types of resource constraints, some of which are obvious generalizations of others. Figure 1 illustrates these six types and the simple transformations between them; an arc from type (a) to type (b) indicates that (a) is a special case of (b).

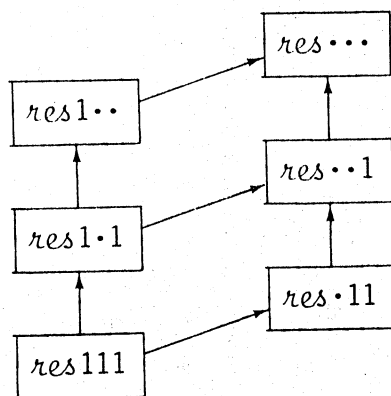


Figure 1. Reductions between six types of resource constraints.

We can draw an additional arc from  $res\cdots$  to  $res1\cdots$  under the restriction that the machines and resources are all *saturated* in each feasible schedule, i.e.,  $|S_t| = m$  and  $\sum_{j \in S_t} r_{hj} = s_h$  ( $h = 1, \dots, \ell$ ) at any time  $t$  until a given deadline. In this case the  $\ell$  requirements  $r_{1j}, \dots, r_{\ell j}$  can be encoded into a single mixed radix number  $r'_{1j}$  [Garey & Johnson 1975].

### 3. SINGLE-OPERATION MODELS WITH UNIT PROCESSING TIMES AND THE $C_{\max}$ CRITERION

We will now investigate the computational complexity of models involving parallel *identical* or *uniform* machines, unit-time jobs, (possibly empty) precedence constraints and the maximum completion time criterion. Theorems 1 to 7 determine the complexity of all such problems; the complete picture is given in Figure 2.

Our starting point is the observation that a polynomial algorithm exists for the case of two identical machines, even under the most general type of resource constraints.

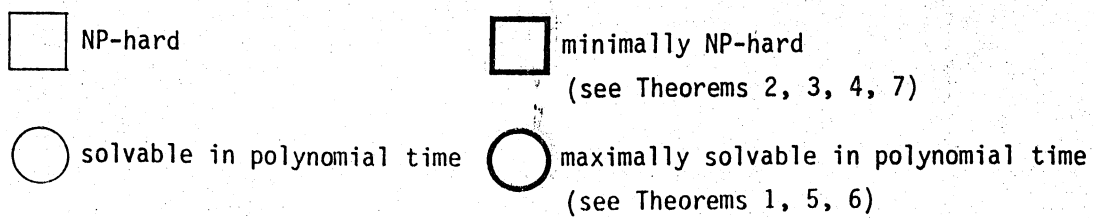
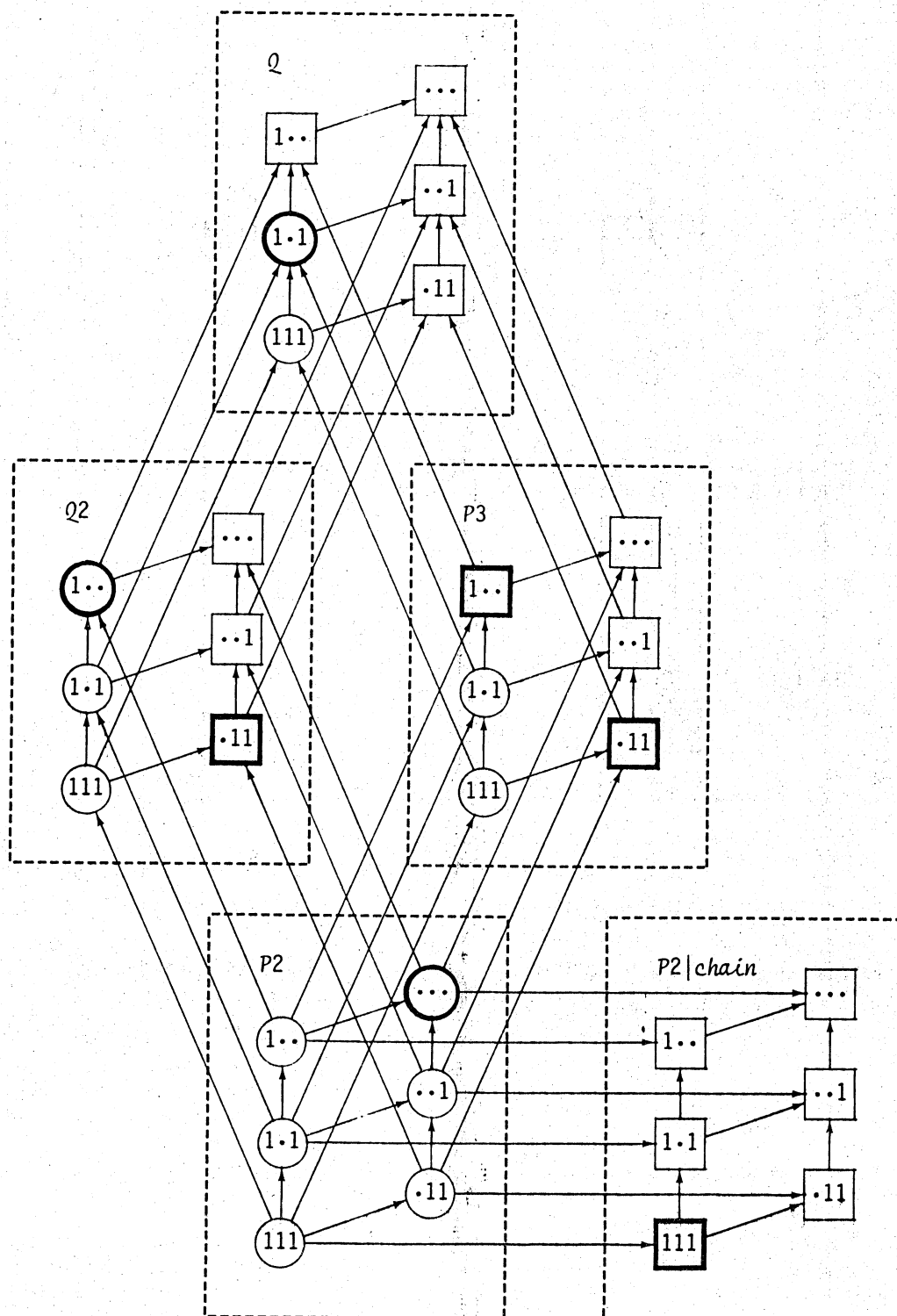


Figure 2. Complexity of  $\alpha|reslo p, B_3, p_j=1|C_{\max}$  problems.



THEOREM 1 [Garey & Johnson 1975].  $P2|res \dots, p_j=1|C_{\max}$  is solvable in  $O(\ln^2 + n^{2\frac{1}{2}})$  time.

*Proof.* Given any instance of  $P2|res \dots, p_j=1|C_{\max}$ , construct a graph  $G$  with vertices  $1, \dots, n$  and edges  $\{j, k\}$  whenever  $r_{hj} + r_{hk} \leq s_h$  ( $h = 1, \dots, \ell$ ). Thus, the vertices correspond to the jobs and the edges to pairs of jobs that can be executed simultaneously. Next, obtain a matching  $S$  (i.e., a set of vertex-disjoint edges) in  $G$  of maximum cardinality. Obviously, the minimum value of  $C_{\max}$  is equal to  $n - |S|$ . Construction of  $G$  requires  $O(\ln^2)$  time, and the algorithm from [Even & Kariv 1975] finds  $S$  in  $O(n^{2\frac{1}{2}})$  time. This proves the polynomial time bound.  $\square$

The correspondence between resource feasible sets of jobs and certain subsets of vertices in a graph can be turned around to obtain NP-hardness results for problems with three identical or two uniform machines. Given any graph  $G$  with vertex set  $V$  and edge set  $E$ , jobs and resource constraints of type  $res \cdot 11$  can be defined in the following way:

- for each vertex  $j \in V$ , introduce a job  $J_j$ ;
- for each vertex pair  $\{j, k\} \notin E$ , introduce a resource  $R_{\{j, k\}}$  of size  $s_{\{j, k\}} = 1$  with requirements  $r_{\{j, k\}, j} = r_{\{j, k\}, k} = 1$ ,  $r_{\{j, k\}, i} = 0$  otherwise.

Thus, two jobs can be executed simultaneously if and only if the corresponding vertices are adjacent.

THEOREM 2.  $P3|res \cdot 11, p_j=1|C_{\max}$  is NP-hard in the strong sense.

*Proof.* We present a straightforward transformation from the following NP-complete problem [Garey & Johnson 1979]:

PARTITION INTO TRIANGLES: Given a graph  $G = (V, E)$  with  $|V| = 3t$ , can  $V$  be partitioned into  $t$  disjoint subsets, each containing three pairwise adjacent vertices?

Given any instance of this problem, we construct an instance of  $P3|res \cdot 11, p_j=1|C_{\max}$  in the way indicated above. Clearly, PARTITION INTO TRIANGLES has a solution if and only if there exists a feasible schedule with value  $C_{\max} \leq t$ .  $\square$

THEOREM 3.  $Q2|res \cdot 11, p_j=1|C_{\max}$  is NP-hard in the strong sense.

*Proof.* In this case, we start from the following NP-complete problem [Garey & Johnson 1979]:

PARTITION INTO PATHS OF LENGTH 2: Given a graph  $G = (V, E)$  with  $|V| = 3t$ , can  $V$  be partitioned into  $t$  disjoint subsets, each containing three vertices, at most two of which are nonadjacent?

Given any instance of this problem, we construct an instance of  $Q2|res \cdot 11, p_j=1|C_{\max}$  in the way indicated above, with machine speeds  $q_1 = 2, q_2 = 1$ . It is easily seen that PARTITION INTO PATHS OF LENGTH 2 has a solution if and only if there exists a feasible schedule with value  $C_{\max} \leq t$ .  $\square$

Theorems 1, 2 and 3 indicate that, when there are no precedence constraints, we can restrict our attention to the case of a single resource. First, we recall a classical NP-hardness result.

THEOREM 4 [Garey & Johnson 1975].  $P3|res 1 \dots, p_j=1|C_{\max}$  is NP-hard in the strong sense.

*Proof.* When the machines and resources are all saturated,  $P3|res 1 \dots, p_j=1|C_{\max}$  is equivalent to the following problem:

3-PARTITION: Given a set  $S = \{1, \dots, 3t\}$  and positive integers  $a_1, \dots, a_{3t}, b$  with  $\sum_{j \in S} a_j = tb$ , can  $S$  be partitioned into  $t$  disjoint 3-element subsets  $S_i$  such that  $\sum_{j \in S_i} a_j = b$  ( $i = 1, \dots, t$ )?

This celebrated problem was the first number problem proved to be NP-complete in the strong sense.  $\square$

It turns out that polynomial algorithms exist for all special cases of  $Q|res \dots, p_j=1|C_{\max}$  whose complexity status has not been settled so far. The solution methods are presented in Theorems 5 and 6.

THEOREM 5.  $Q2|res 1 \dots, p_j=1|C_{\max}$  is solvable in  $O(n \log n)$  time.

*Proof.* Given any instance of  $Q2|res 1 \dots, p_j=1|C_{\max}$ , an optimal schedule can be obtained in the following way. Suppose that  $q_1 \geq q_2$ . First, schedule all

jobs on  $M_1$  in order of nonincreasing resource requirement. Next, successively remove the last job from  $M_1$  and schedule it as early as possible on  $M_2$ , as long as this reduces the value of  $C_{\max}$ .

This  $O(n \log n)$  algorithm clearly generates the best schedule among those satisfying the following properties:

- (a) the jobs  $J_j$  on  $M_1$  are executed in order of nonincreasing  $r_{1j}$  without machine idle time;
- (b) the jobs  $J_k$  on  $M_2$  are executed in order of nondecreasing  $r_{1k}$ ;
- (c)  $r_{1j} \geq r_{1k}$  for all  $J_j$  on  $M_1$  and all  $J_k$  on  $M_2$ .

The correctness of the algorithm will now be proved by showing that any feasible schedule can be transformed into a schedule that is at least as good and satisfies properties (a), (b) and (c).

To avoid the introduction of some cumbersome notation, the transformation is presented in an informal way. Starting from a feasible initial schedule, one proceeds as follows (cf. Figure 3).

Step 1. Move the jobs that are executed on  $M_2$  while  $M_1$  is idle to  $M_1$ . Interchange parts of the schedule simultaneously on both machines such that the jobs or fractions of jobs that are executed on  $M_1$  while  $M_2$  is idle are in the first positions on  $M_1$ .

Step 2. Interchange parts of the schedule simultaneously on both machines such that all jobs  $J_k$  on  $M_2$  are in order of nondecreasing  $r_{1k}$ .

Step 3. Rearrange the (fractional) jobs  $J_j$  that are executed on  $M_1$  while  $M_2$  is busy in such a way that they are in order of nonincreasing  $r_{1j}$  and the preemptions created by Step 2 are eliminated. (This does not lead to resource infeasibility.)

Step 4. Insert the (fractional) jobs that are executed on  $M_1$  while  $M_2$  is idle in positions on  $M_1$  chosen in such a way that all jobs  $J_j$  on  $M_1$  are in order of nonincreasing  $r_{1j}$  and the preemptions created by Step 1 are eliminated; it may be necessary to introduce periods of idle time on  $M_2$ . Left-justify the resulting schedule.

Step 5. Let  $J_j$  be the last job on  $M_1$  and  $J_k$  the last job on  $M_2$ . If  $r_{1j} \geq r_{1k}$ , the transformation terminates. Otherwise, schedule  $J_j$  in the position of  $J_k$  on  $M_2$ , schedule  $J_k$  as early as possible on  $M_1$ , left-justify the schedule, and return to Step 1.

None of these steps increases the value of  $C_{\max}$ . After each application

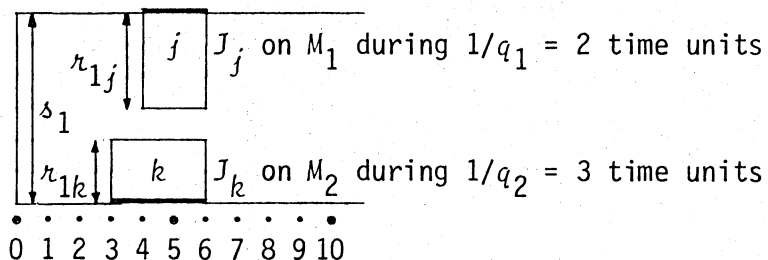


of Steps 1 to 4, properties (a) and (b) are satisfied, and after a finite number of applications of Step 5, property (c) holds as well. This validates the algorithm given above.  $\square$

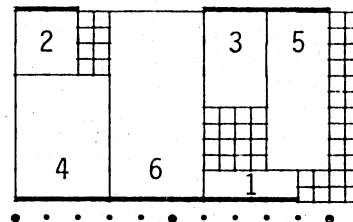
Instance of  $Q2|res1 \dots, p_j=1|C_{\max}$ :

$n = 6$ ;  $q_1 = 1/2$ ,  $q_2 = 1/3$ ;  $s_1 = 6$ ,  $r_{1j} = j$  ( $j = 1, \dots, 6$ ).

Notation:

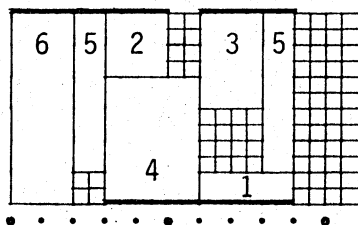


Initial schedule

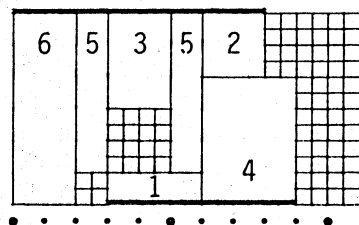


Iteration 1

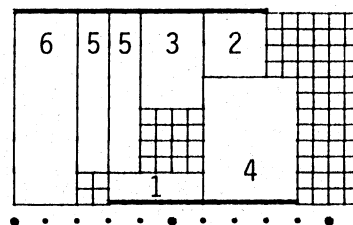
Step 1



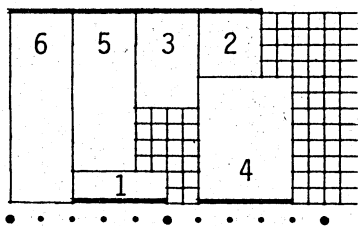
Step 2



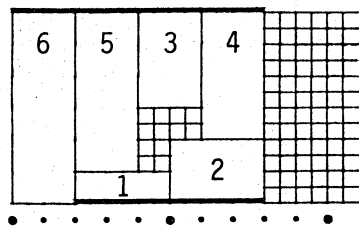
Step 3



Step 4



Step 5



Iteration 2

Optimal schedule

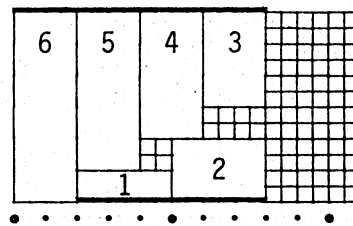


Figure 3. Illustration of transformation of a  $Q2|res1 \dots, p_j=1|C_{\max}$  schedule.

**THEOREM 6.**  $Q|res1 \dots, p_j=1|C_{\max}$  is solvable in  $O(n^3)$  time.

*Proof.* Given any instance of  $Q|res1 \dots, p_j=1|C_{\max}$ , construct a transportation network with  $n$  sources  $j$  ( $j = 1, \dots, n$ ) and  $mn$  sinks  $(i, k)$  ( $i = 1, \dots, m$ ;  $k = 1, \dots, n$ ). Each arc  $(j, (i, k))$  has a cost  $c_{ijk}$ , to be defined below. The arc

flow  $x_{ijk}$  is to have the following interpretation:

$$x_{ijk} = \begin{cases} 1 & \text{if } J_j \text{ is executed on } M_i \text{ in the } k\text{-th position,} \\ 0 & \text{otherwise.} \end{cases}$$

The number of resource requiring jobs executed simultaneously must never be allowed to exceed the resource size. This can be effectuated by requiring that these jobs are assigned only to the fastest  $s_1$  machines. Thus, assume that  $q_h \geq q_i$  for all  $h = 1, \dots, s_1$  and all  $i = s_1+1, \dots, m$ , and define

$$c_{ijk} = \begin{cases} \infty & \text{if } i \geq s_1+1 \text{ and } r_{1j} = 1, \\ k/q_i & \text{otherwise.} \end{cases}$$

Then the problem is to minimize

$$\max_{i,j,k} \{c_{ijk} x_{ijk}\}$$

subject to

$$\begin{aligned} \sum_{i=1}^m \sum_{k=1}^n x_{ijk} &= 1 \quad (j = 1, \dots, n), \\ \sum_{j=1}^n x_{ijk} &\leq 1 \quad (i = 1, \dots, m; k = 1, \dots, n), \\ x_{ijk} &\geq 0 \quad (i = 1, \dots, m; j = 1, \dots, n; k = 1, \dots, n). \end{aligned}$$

This bottleneck transportation problem can be formulated and solved in  $O(n^3)$  time.  $\square$

*Note.* Similar transportation network models provide efficient solution methods for  $Q|res1 \cdot 1, p_j=1|\gamma$ , where  $\gamma \in \{\max_j \{f_j(C_j)\}, \sum_j f_j(C_j)\}$  for arbitrary nondecreasing cost functions  $f_j$  ( $j = 1, \dots, n$ ).

When the presence of precedence constraints between the jobs is allowed, NP-hardness in the strong sense has been established for  $P2|res1 \cdot \cdot, tree, p_j=1|C_{\max}$  [Garey & Johnson 1975] and  $P2|res111, prec, p_j=1|C_{\max}$  [Ullman 1976]. These results are both dominated by Theorem 7.

**THEOREM 7.**  $P2|res111, chain, p_j=1|C_{\max}$  is NP-hard in the strong sense.

*Proof.* We prove this result by means of a transformation from 3-PARTITION (see Theorem 4), where we assume without loss of generality that  $\frac{1}{4}b < a_j < \frac{1}{2}b$  for all  $j \in S$ . Given any instance of this problem, we construct an instance of  $P2|res111, chain, p_j=1|C_{max}$  in the following way:

- there is a single chain  $L$  of  $2tb$  jobs:

$$L = J'_1 \rightarrow J'_2 \rightarrow \dots \rightarrow J'_b \rightarrow J_1 \rightarrow J_2 \rightarrow \dots \rightarrow J_b \rightarrow \\ \rightarrow J'_{b+1} \rightarrow J'_{b+2} \rightarrow \dots \rightarrow J'_{2b} \rightarrow J_{b+1} \rightarrow J_{b+2} \rightarrow \dots \rightarrow J_{2b} \rightarrow \\ \dots \\ \rightarrow J'_{(t-1)b+1} \rightarrow J'_{(t-1)b+2} \rightarrow \dots \rightarrow J'_{tb} \rightarrow J_{(t-1)b+1} \rightarrow J_{(t-1)b+2} \rightarrow \dots \rightarrow J_{tb};$$

- for each  $j \in S$ , there are two chains  $K_j$  and  $K'_j$ , each of  $a_j$  jobs:

$$K_j = J_{j1} \rightarrow J_{j2} \rightarrow \dots \rightarrow J_{ja_j},$$

$$K'_j = J'_{j1} \rightarrow J'_{j2} \rightarrow \dots \rightarrow J'_{ja_j};$$

moreover, it is required that  $K_j$  precedes  $K'_j$ , i.e.,  $J_{ja_j} \rightarrow J'_{j1}$ ;

- the primed jobs do require the resource, the unprimed jobs do not.

We claim that 3-PARTITION has a solution if and only if there exists a feasible schedule with value  $C_{max} \leq 2tb$ .

Suppose that 3-PARTITION has a solution  $\{S_1, \dots, S_t\}$ . A feasible schedule with value  $C_{max} = 2tb$  is then obtained as follows (cf. Figure 4). First, the chain  $L$  is scheduled on machine  $M_1$  in the interval  $[0, 2tb]$ ; note that this leaves the resource available only in the intervals  $[2(i-1)b, (2i-1)b]$  ( $i = 1, \dots, t$ ). For each  $i \in \{1, \dots, t\}$ , it is now possible to schedule the three chains  $K_j$  ( $j \in S_i$ ) on machine  $M_2$  in the interval  $[2(i-1)b, (2i-1)b]$  and the chains  $K'_j$  ( $j \in S_i$ ) on  $M_2$  in  $[(2i-1)b, 2ib]$ . The resulting schedule is feasible with respect to resource and precedence constraints and has total length  $2tb$ .

Part of feasible instance for 3-PARTITION:

$$b = 15, a_1 = 4, a_2 = 5, a_3 = 6; S_i = \{1, 2, 3\}.$$

Part of feasible schedule for  $P2|res111, chain, p_j=1|C_{max}$ :

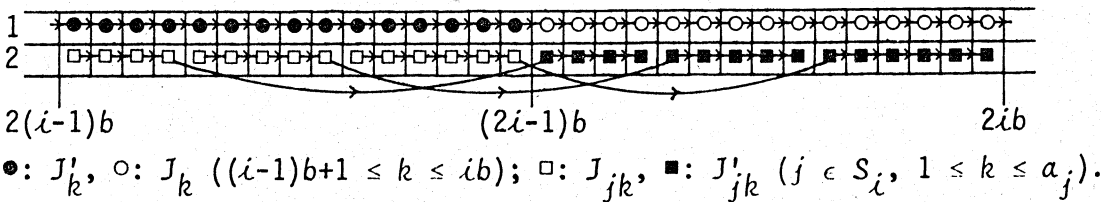


Figure 4. Illustration of transformation from 3-PARTITION to

$P2|res111, chain, p_j=1|C_{max}$ .



Conversely, suppose that there exists a feasible schedule with value  $C_{\max} \leq 2tb$ . It is clear that in this schedule both machines and the resource are saturated until time  $2tb$ . Moreover, the chains  $K_j$  ( $j \in S$ ) are executed in the intervals  $[2(i-1)b, (2i-1)b]$  ( $i = 1, \dots, t$ ) and the chains  $K'_j$  ( $j \in S$ ) in the remaining intervals. Let  $S_i$  be the index set of chains  $K_j$  completed in the interval  $[2(i-1)b, (2i-1)b]$ , for  $i = 1, \dots, t$ . Consider the set  $S_1$ . It is impossible that  $\sum_{j \in S_1} a_j > b$ , due to the definition of  $S_1$ ; the case  $\sum_{j \in S_1} a_j < b$  cannot occur either, since this would lead to machine idle time in  $[b, 2b]$ . It follows that  $\sum_{j \in S_1} a_j = b$ , and our assumption about the size of  $a_j$  ( $j \in S$ ) implies that  $|S_1| = 3$ . This argument is easily extended to an inductive proof that  $\{S_1, \dots, S_t\}$  constitutes a solution to 3-PARTITION.  $\square$

#### 4. OTHER MODELS

We will next comment on the computational complexity of three variations of the models considered in the previous section, viz.

- (1) extensions to other optimality criteria,
- (2) preemptive scheduling, and
- (3) multi-operation models.

##### 4.1. Other optimality criteria

If the  $C_{\max}$  criterion is replaced by other optimality criteria such as the total completion time  $\sum C_j$  or the maximum lateness  $L_{\max}$ , most results can be extended in a straightforward way.

In fact, all the NP-hardness results of Theorems 2, 3, 4 and 7 carry over immediately to both  $\sum C_j$  and  $L_{\max}$ . For  $\sum C_j$ , we use the fact that the machines are saturated in each of the transformations; e.g., in Theorems 2 and 4 we have  $C_{\max} \leq t$  if and only if  $\sum C_j \leq \frac{3}{2}t(t+1)$ . For  $L_{\max}$ , we define due dates  $d_j = 0$  for all jobs, so that  $L_{\max} = C_{\max}$ .

It has been noted already that the transportation network model of Theorem 6 provides polynomial algorithms for  $Q|res1 \cdot 1, p_j=1|\gamma$ , where  $\gamma = \sum C_j$  or  $\gamma = L_{\max}$ . The matching approach of Theorem 1 is easily adapted to solve

$P2|res\cdots, p_j=1|\sum C_j$  as well: simply schedule the paired jobs before the remaining ones. It seems a safe conjecture that the algorithm of Theorem 5 can be modified to solve  $Q2|res1\cdots, p_j=1|\sum C_j$ ; we leave this as a challenge to the reader. However,  $P2|res\cdots, p_j=1|L_{\max}$  and  $Q2|res1\cdots, p_j=1|L_{\max}$  remain open problems. We mention that  $P|res1\cdots, r_j, p_j=1|L_{\max}$ , where the  $r_j$  denote integer release dates at which the jobs become available, is solvable in polynomial time [Blazewicz 1979].

#### 4.2. Preemptive scheduling

If the processing times are arbitrary and preemption is allowed, the nature of the models changes considerably. It now becomes of interest to consider the general case of parallel *unrelated* machines.

The problem  $R|pmtn, res\cdots|C_{\max}$  can be formulated as a linear program in the following way (cf. [Weglarz et al. 1977; Slowinski 1980]). First, introduce a dummy job  $J_0$  with  $r_{h0} = 0$  for  $h = 1, \dots, \ell$ , representing machine idle time. Define  $S$  as the set of all resource feasible  $m$ -tuples  $k = (k_1, \dots, k_m)$  of job indices; each  $k$  is characterized by:

- $k_i \in \{0, 1, \dots, n\}$  for  $i = 1, \dots, m$ ;
- each  $j \in \{1, \dots, n\}$  occurs at most once;
- $\sum_{i=1}^m r_{hk_i} \leq s_h$  for  $h = 1, \dots, \ell$ .

To each  $k \in S$ , associate a variable  $x_k$ , representing the time during which  $J_{k_1}, \dots, J_{k_m}$  are simultaneously executed on  $M_1, \dots, M_m$  respectively. Then the problem is to minimize

$$\sum_{k \in S} x_k$$

subject to

$$\sum_{i=1}^m \left( \sum_{k \in S, k_i=j} x_k \right) / p_{ij} = 1 \quad (j = 1, \dots, n),$$

$$x_k \geq 0 \quad (k \in S).$$

This linear programming problem has  $O(n^m)$  variables. For a fixed number of machines, its size is bounded by a polynomial in the size of the scheduling problem. The existence of a polynomial algorithm for linear programming [Khachian 1979] therefore implies that  $Rm|pmtn, res\cdots|C_{\max}$  is solvable in

polynomial time.

For a variable number of machines,  $Q|pmtn, res111|C_{\max}$  can be solved as follows. Replace the resource requiring jobs by a single job with execution requirement  $\sum_{r_{1j}=1} p_j$ ; this eliminates the resource constraints. Next, apply the  $O(m \log m + n)$  algorithm for  $Q|pmtn|C_{\max}$  from [Gonzalez & Sahni 1978] to solve the resulting problem.

#### 4.3. Multi-operation models

Multi-operation models, in which each operation has its own specific resource requirements, give rise to various interesting results and to many open problems. By way of example, we consider open shops, flow shops and job shops with two machines, nonpreemptable operations and the  $C_{\max}$  criterion.

In the case of an *open shop*,  $O2|res\cdots, p_{ij}=1|C_{\max}$  is solvable by a matching approach similar to the one used in Theorem 1. If the processing times are arbitrary, even  $O2|res111|C_{\max}$  remains unresolved.

*Flow shop* problems seem to be more difficult.  $F2|res111, p_{ij}=1|C_{\max}$  is solvable in linear time by appropriately grouping jobs together according to their overall resource requirements. Little can be said about the immediate extensions of this model with unit processing times, but  $F2|res111|C_{\max}$  is NP-hard in the strong sense by virtue of a simple transformation from 3-PARTITION.

The simplest *job shop* model in this context,  $J2|res111, p_{ij}=1|C_{\max}$ , is already NP-hard in the strong sense; the transformation from 3-PARTITION is nontrivial.

#### 5. CONCLUDING REMARKS

We have proposed a classification scheme for resource constrained scheduling problems and outlined a range of initial results on their computational complexity. Presumably, many of the remaining open problems can be resolved along similar lines. We hope to have stimulated others to continue the investigation of this interesting research area.



## APPENDIX. CLASSIFICATION OF SCHEDULING PROBLEMS

Suppose that  $n$  jobs  $J_1, \dots, J_n$  have to be processed on  $m$  machines  $M_1, \dots, M_m$ . Each machine can handle at most one job at a time and each job can be executed by at most one machine at a time. Various job, machine and scheduling characteristics are reflected by a three-field problem classification  $\alpha|\beta|\gamma$  [Graham et al. 1979]. Let  $\circ$  denote the empty symbol.

Machine environment

The first field  $\alpha = \alpha_1 \alpha_2$  specifies the machine environment.

If  $\alpha_1 \in \{P, Q, R\}$ , each  $J_j$  consists of a single operation that can be processed on any  $M_i$ ; the processing time of  $J_j$  on  $M_i$  is  $p_{ij}$  ( $i = 1, \dots, m$ ;  $j = 1, \dots, n$ ). The three values are characterized as follows.

- $\alpha_1 = P$  (parallel identical machines):  $p_{ij} = p_j$  for a given execution requirement  $p_j$  of  $J_j$ .
- $\alpha_1 = Q$  (parallel uniform machines):  $p_{ij} = p_j/q_i$  for a given execution requirement  $p_j$  of  $J_j$  and a given speed  $q_i$  of  $M_i$ .
- $\alpha_1 = R$  (parallel unrelated machines):  $p_{ij}$  is arbitrary.

If  $\alpha_1 \in \{O, F, J\}$ , each  $J_j$  consists of a set of  $m_j$  operations  $O_{ij}$ ;  $O_{ij}$  has to be processed on a given machine  $\mu_{ij}$  during  $p_{ij}$  time units ( $i = 1, \dots, m_j$ ;  $j = 1, \dots, n$ ). The three values are characterized as follows.

- $\alpha_1 = O$  (open shop):  $m_j = m$ ,  $\mu_{ij} = M_i$ .
- $\alpha_1 = F$  (flow shop):  $m_j = m$ ,  $\mu_{ij} = M_i$ ;  $O_{i-1,j}$  has to be completed before  $O_{ij}$  can start ( $i = 2, \dots, m$ ).
- $\alpha_1 = J$  (job shop):  $m_j$  and  $\mu_{ij}$  are arbitrary;  $\mu_{i-1,j} \neq \mu_{ij}$  and  $O_{i-1,j}$  has to be completed before  $O_{ij}$  can start ( $i = 2, \dots, m_j$ ).

If  $\alpha_2$  is a positive integer, then  $m$  is constant and equal to  $\alpha_2$ ; if  $\alpha_2$  is  $\circ$ , then  $m$  is part of the input.

Job characteristics

The second field  $\beta \in \{\beta_1, \beta_2, \beta_3, \beta_4\}$  indicates a number of job characteristics, which are defined as follows.

- (1)  $\beta_1 \in \{pmtn, \circ\}$ .

- $\beta_1 = \text{pmtn}$ : Preemption (job splitting) is allowed; the processing of any job may arbitrarily often be interrupted and resumed at the same time on a different machine or at a later time on any machine.
- $\beta_1 = \circ$ : No preemption is allowed.
- (2)  $\beta_2$  specifies the resource constraints; see Section 2.
- (3)  $\beta_3 \in \{\text{prec}, \text{tree}, \text{chain}, \circ\}$ .
- $\beta_3 = \text{prec}$  (arbitrary precedence constraints): A directed acyclic graph  $H$  with vertices  $1, \dots, n$  is given; if  $H$  contains a directed path from  $j$  to  $k$ , we write  $J_j \rightarrow J_k$  and require that  $J_j$  is completed before  $J_k$  can start.
- $\beta_3 = \text{tree}$  (tree-like precedence constraints):  $H$  has outdegree at most one for each vertex or indegree at most one for each vertex.
- $\beta_3 = \text{chain}$  (chain-like precedence constraints):  $H$  has both outdegree and indegree at most one for each vertex.
- $\beta_3 = \circ$  (no precedence constraints):  $H$  has no arcs.
- (4)  $\beta_4 \in \{p_{ij}=1, \circ\}$ .
- $\beta_4 = p_{ij}=1$ : Each operation has unit processing time.
- $\beta_4 = \circ$ : The processing times are arbitrary nonnegative integers.  
(If  $\alpha_1 \in \{P, Q\}$ , then  $p_{ij}$  is replaced by  $p_j$ ; if  $\alpha_1 = R$ , then  $\beta_4 = \circ$ .)

### Optimality criteria

The third field  $\gamma$  denotes the optimality criterion chosen. Any feasible schedule defines for each  $J_j$  a completion time  $C_j$  and, given an integer due date  $d_j$ , a lateness  $L_j = C_j - d_j$  ( $j = 1, \dots, n$ ). Some common optimality criteria involve the minimization of

- $C_{\max} = \max\{C_1, \dots, C_n\}$  (maximum completion time);
- $\sum C_j = C_1 + \dots + C_n$  (total completion time);
- $L_{\max} = \max\{L_1, \dots, L_n\}$  (maximum lateness).

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