QUANTIFYING LONG RUN AGRICULTURAL RISKS AND EVALUATING FARMER RESPONSES TO RISK

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CHAOS, ECONOMICS, AND RISK

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Chaos, Economics, and Risk*

ABSTRACT

In recent years, research in both mathematics and the applied sciences has produced a revolution in the understanding of nonlinear dynamical systems. Used widely in economics and other disciplines to model change over time, these systems are now known to be vulnerable to a kind of "chaotic," unpredictable behavior. This paper places this revolution in historical context, explains several of the important mathematical ideas on which it is based, and discusses some of its implications for a probabilistic interpretation of longrun economic phenomena.

Keywords: Limits to predictability; nonlinear and chaotic dynamical systems; fractals; stochastic processes.

* This paper is adapted from "Nonlinear and Chaotic Dynamics: An Economist's Guide," to appear in *The Journal of Agricultural Economics Research* (1991). The forthcoming paper will include extensive discussions of structural stability and symbolic dynamics; the present one incorporates an expanded treatment of the probabilistic character of chaotic dynamics. The author thanks John McClelland and other participants in the ERS Chaos Theory Seminar for many stimulating discussions on chaotic dynamics. Helpful review comments were received from Carlos Arnade and Richard Heifner.

INTRODUCTION

In the last two decades, the world of science has come to a fundamentally new understanding of the dynamics of phenomena that vary over time. Grounded in mathematical discovery, yet given empirical substance by evidence from a variety of disciplines, this new perspective has led to nothing less than a re-examination of the concept of the predictability of dynamic behavior. Our implicit confidence in the orderliness of dynamical systems—specifically, of *nonlinear* dynamical systems—has not, it turns out, been entirely justified. Such systems are capable of behaving in ways that are far more erratic and unpredictable than once believed. Fittingly, the new ideas are said to concern *chaotic dynamics*, or, simply, *chaos*.

Economics is not immune from the implications of this new understanding. After all, our subject is replete with dynamical phenomena ranging from cattle cycles to stock market catastrophes to the back-and-forth interplay of advertising and product sales. Ideas related to the notion of chaotic behavior are now part of the basic mathematical toolkit needed for insightful dynamical modeling. Agricultural economists need to gain an understanding of these ideas just as they would any other significant mathematical contribution to their field. This paper is intended to assist in this educational process.

What exactly has chaos theory revealed? In order to address this question, let us consider an economy, subject to change over time, whose state at time t can be described by a vector, v_t, of (say) 14 numbers (money supply at time t, inflation rate at time t, etc.). Formally, this vector is a point in the 14-dimensional *state space* $\mathbb{R}^{14}$ (where $\mathbb{R}$ is the real number line). Suppose that the economy evolves deterministically in such a way that its state at any time uniquely determines its state at all later times. Then, if the initial position of the economy in $\mathbb{R}^{14}$ at time 0 is v_0, the evolution of the economy through time will be represented by a path in $\mathbb{R}^{14}$ starting at v_0 and traced out by v_t as time, t, moves forward. This path—called the *orbit* generated by the initial position v_0—represents a "future history" of the system. Figure 1 shows an orbit of a system whose state space is the plane, $\mathbb{R}^2$. Questions about the behavior of the economy over time are really questions about its orbits. We are often interested not so much in the *near-term* behavior of orbits as in their *eventual* behavior, as when we engage in long-range forecasting or study an economy's response to a new government policy or an unexpected "shock" after the initial period of adjustment has passed and the economy has "settled down."

Fractals, Sensitive Dependence, and Chaos

Scientists long have known that it is possible for a system's state space to contain an isolated, "unstable" point p such that different initial points near p can generate orbits with widely varying longrun behavior. (For example, a marble balanced on the tip of a cone is unstable in this sense.) What was unexpected, however, was the discovery that this type of instability can occur *throughout* the state space—sometimes actually at every point, but often in strangely patterned, fragmented subsets of the state space—subsets typically of noninteger dimension, called *fractals*. Once investigators knew what to look for, they found this phenomenon, termed *sensitive dependence on initial conditions*, to be widespread among nonlinear dynamical systems, even among the simplest ones. Though technical definitions vary, systems exhibiting this unstable behavior have generally come to be called "chaotic."

For chaotic systems, *any* error in specifying an initial point—even the most minute error (due to, say, computer rounding in the thousandth decimal place)—can give rise to an orbit whose longrun behavior bears no resemblance

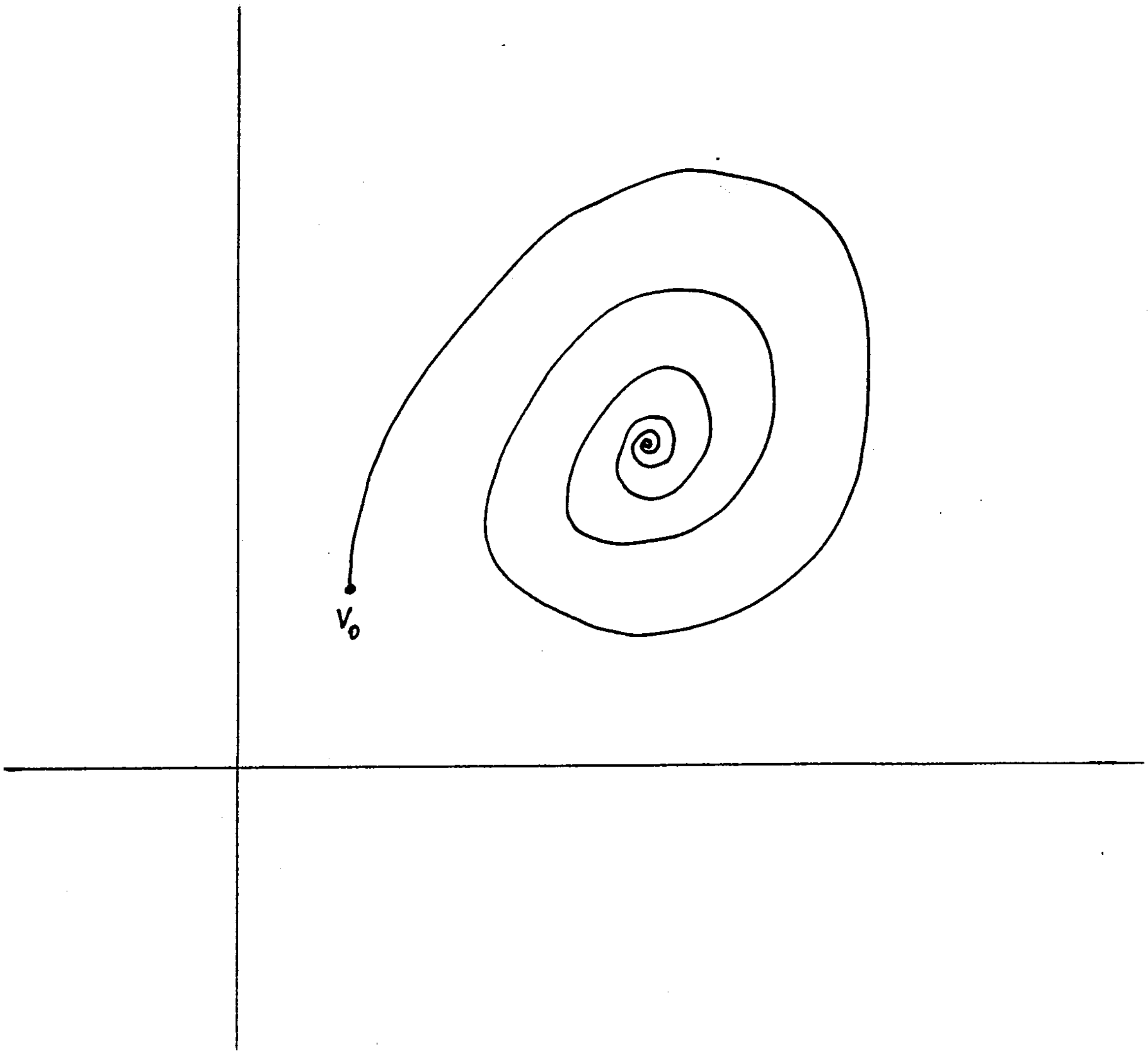

Figure 1: An Orbit in $\mathbb{R}^2$

to that of the orbit of the intended initial point. Since, in the real world, we can never specify a point with mathematically perfect precision, it follows that prediction of the longrun pointwise behavior of a chaotic system is a practical impossibility.

Attractors

For a dynamical system, perhaps the most basic question is "where does the system go, and what does it do when it gets there?" In the earlier view of dynamical systems, the "place where the system went"—the point set in the state space to which orbits converged (called an *attractor*)—was usually assumed to be a geometrically uncomplicated object such as a closed curve or a single point. Economic modelers, for example, have often implicitly assumed that a dynamic economic process will ultimately achieve either an equilibrium, a cyclic pattern, or some other orderly behavior. However, another discovery of chaos theory has been that the attractor of a nonlinear system can be a bizarre, fractal set within which the system's state can flit endlessly in a chaotic, seemingly random manner.

Moreover, just as an economy can have two or more equilibria, a dynamical system can have two or more attractors. In such a case, the set of all initial points whose orbits converge to a particular attractor is called a *basin of attraction*. A recent finding has been that the *boundary* between "competing" basins of attraction can be a fractal even when the attractors themselves are unexceptional sets. A type of sensitivity to the initial condition can operate here too: the slightest movement away from an initial point lying in one basin of attraction may move the system to a new basin of attraction and thus cause it to evolve toward a new attractor.

Chaotic behavior within a working model would be easier to recognize if all orbits initiating near an erratic orbit were also erratic. However, the potentially fractal structure of the region of sensitive dependence can allow initial points whose orbits behave "sensibly" and initial points whose orbits are erratic to coexist inseparably in the state space like two intermingled clouds of dust. Thus, simulation of a model at a few trial points cannot rule out the possibility of chaotic dynamics. Rather, we need a deeper understanding of the mathematical properties of our models. Nor can chaotic dynamics be dismissed as arising only in a few quirky special cases; as we shall see, it arises even when the system's law of motion is a simple quadratic.

THE DISCOVERY OF CHAOS

Recent years have witnessed an explosion of interest and activity in the area of chaotic dynamics. What accounts for this new visibility—a visibility extending even beyond the research community into the public media? To provide an answer, we briefly trace the historical development of the subject.

The first recognition of chaotic dynamics is attributed to Henri Poincaré, a French mathematician whose work on celestial mechanics around the turn of the century helped found the study of dynamical systems, systems in which some structure (perhaps a solar system, perhaps—as now understood—an economy) changes over time according to predetermined rules. Poincaré's writings foresaw the potential for unpredictability in dynamical systems whose equations of motion were nonlinear. However, neither the mathematical theory nor the imaging techniques available at the time permitted Poincaré to explore his intuitions fully.

Following Poincaré's work and that of the American mathematician G.D. Birkhoff in the early part of this century, and despite continuing interest in the Soviet Union, the subject of dynamical systems fell into relative obscurity. During this period, there was some awareness among mathematicians, scientists, and engineers that nonlinear systems were capable of erratic behavior. However, examples of such behavior were ignored, classified as "noise," or dismissed as aberrations. The idea that these phenomena were characteristic of nonlinear dynamical systems and that it was the well-behaved, "textbook" examples that were the special cases had not yet taken root.

Then, in the 1960's and 1970's, there was a flurry of activity in dynamical systems by both mathematicians and scientists working entirely independently. Mathematician Stephen Smale turned his attention to the subject and used the techniques of modern differential topology to create rigorous theoretical models of chaotic dynamics. Meteorologist Edward Lorenz discovered that a simple system of equations he had devised to simulate the earth's weather on a primitive computer displayed a surprising type of sensitivity: the slightest change in the initial conditions eventually would lead to weather patterns bearing no resemblance to those generated in the original run.

Biologist Robert May used the logistic difference equation

$$x_{n+1} = rx_n(1-x_n)$$

to model population level, x, over successive time periods. He observed that, for some choices of the growth rate parameter r, the population level would converge, for other choices it would cycle among a few values, and for still others it would fluctuate seemingly randomly, never achieving either a steady state or any discernible repeating pattern. Moreover, when he attempted to graph the population level against the growth rate parameter, he observed a strangely patterned, fragmented set of points.

Physicist Mitchell Feigenbaum investigated the behavior of dynamical systems whose equations of motion arise from unimodal (hill-shaped) functions. He noticed that certain parameter values that sent the system into repeating cycles always displayed the same numerically precise pattern: no matter which dynamical system was examined, the ratios of successive distances between these parameter values always converged to the same constant, $4.66920\cdots$. Feigenbaum had discovered a universal property of a class of nonlinear dynamical systems. His discovery ultimately clarified how systems can evolve toward chaos.

Thus, as these and other examples demonstrate, while mathematicians were developing the *theory* of nonlinear and chaotic dynamics, scientists in diverse disciplines were witnessing and discovering chaotic phenomena for themselves. Ultimately, researchers learned of one another's findings and recognized their common origin.

The role of the computer in the emergence of the contemporary understanding of dynamical systems is difficult to exaggerate. As we now realize, even the simplest systems can generate bewilderingly complicated behavior. The development of modern computer power and graphics seems to have been necessary before researchers could put the full picture of nonlinear and chaotic dynamics, quite literally, into focus.

THE MATHEMATICS OF CHAOS

We now explain some of the basic mathematical ideas involved in nonlinear dynamics and chaos. We also adopt a slightly different perspective. In our previous discussion, we have implicitly portrayed dynamical systems as being in motion in *continuous* time. However, the equations of motion of such systems typically involve differential equations, and a proper treatment often requires advanced mathematical machinery. It is generally much easier to work with (and to understand) *discrete* time systems—systems in which time takes only integer values representing successive time periods. We now shift our attention to these systems.

We begin by pointing out that, when the law of motion of a discrete dynamical system is unchanging over time, the movement of the system through time can be understood as a process of iterating a function. To establish this point, consider a typical dynamic economic computer model, M, having k endogenous variables. To start the model running, we enter an initial condition vector, v_0, of k numbers. The model computes an output vector, $M(v_0)$, containing the new values of the k endogenous variables at the end of the first time period. The model then acts on $M(v_0)$ and computes a new output vector, $M(M(v_0))$, describing the economy at the end of the second time period, etc. Note that the model itself—the "law of motion"—remains unchanged during this process. In effect, M acts as a *function,* mapping k-vectors to new k-vectors, applying itself iteratively to the last-computed function value. The state space of the economy is the k-dimensional space $\mathbb{R}^k$, and, for each initial condition vector v_0, there is a corresponding orbit

$$v_0,\ M(v_0),\ M(M(v_0)),\ M(M(M(v_0))),\ \cdots$$

describing the future evolution of the economy.

More generally, consider *any* function f. If f maps its domain (the set of all x for which f(x) is defined) into itself, then, for each x_0 in the domain of f, the sequence

$$x_0,\ f(x_0),\ f(f(x_0)),\ f(f(f(x_0))),\ \cdots$$

is well-defined and may be considered an orbit of a dynamical system determined by f through iteration.

Henceforth, for brevity, we denote by

$$f^n$$

the nth iterate of a function f. Thus, $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x)))$, etc. By convention, $f^0(x) = x$. Of course, f^n is itself a function. It should not be confused with the nth *derivative* of f, which is customarily denoted

$$f^{(n)}.$$

We point out that an orbit of a discrete dynamical system of the type being considered can always be expressed as a system of difference equations. In fact, if we relabel an orbit

$$x_0,\ f(x_0),\ f^2(x_0),\ \ldots,\ f^n(x_0),\ \ldots$$

as

$$u_0, u_1, u_2, \ldots, u_n, \ldots,$$

we obtain the system of difference equations

$$\begin{gathered}
u_0 - x_0 = 0, \\
u_1 - f(u_0) = 0, \\
u_2 - f(u_1) = 0, \\
\cdot \\
\cdot \\
\cdot \\
u_n - f(u_{n-1}) = 0, \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{gathered}$$

Conversely, given such a system of equations, we may view the sequence

$$u_0, u_1, u_2, \ldots, u_n, \ldots$$

as the orbit of a discrete dynamical system generated by the function f and initiated at u_0 (i.e., at x_0).

Orbit Diagrams

Fortunately for expository purposes, many of the important features of dynamical systems are present in one–dimensional systems. In fact, one of the important findings of chaos research has been that discrete dynamical systems generated by iteration of even the most elementary nonlinear scalar functions are capable of chaotic behavior. Thus, we shall concentrate on functions that operate on the number line.

For such functions, there is a particularly convenient technique for diagramming orbits. Consider a function f and an initial point x (see fig. 2). Beginning at the point (x, x) on the 45° line, draw a dotted line vertically to the graph of f; the point of intersection will be (x, f(x)). From that point, draw a dotted line horizontally to the 45° line; the point of intersection will be (f(x), f(x)). From there, draw a dotted line vertically to the graph of f; the point of intersection will be $(f(x), f^2(x))$. Continue this pattern of moving vertically to the graph of f and then horizontally to the 45° line. The resulting *orbit diagram* shows the behavior of the orbit originating at x. In particular, the orbit may be visualized from the intersection points marked on the 45° line; the dotted lines indicate the "direction of motion" of the system. Of course, the points (x,x), (f(x), f(x)), $(f^2(x), f^2(x))$, ... only *look like* the orbit. They reside in the plane, whereas the actual orbit, consisting of the numbers x, f(x), $f^2(x)$, ..., resides in the state space, i.e., in the number line.

Dynamics of Linear Systems

Though our basic interest in this paper is in nonlinear dynamics, examination of *linear* systems provides essential intuition about nonlinear ones. Thus, we begin with an exhaustive treatment of the linear case.

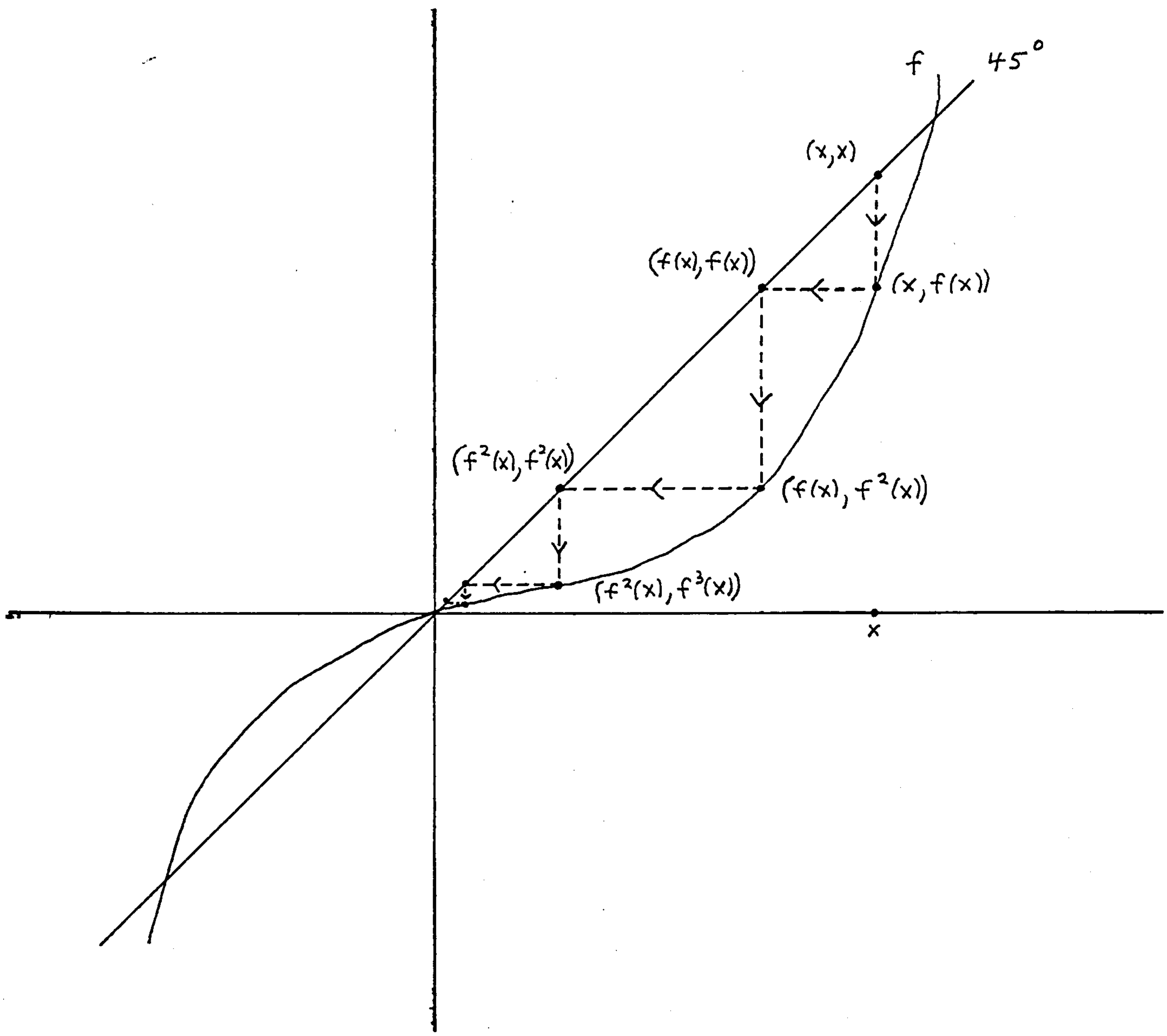

Figure 2: Construction of an Orbit Diagram

Choose any numbers a,b, and consider the function g defined by

$$g(x) = ax + b.$$

To compute a typical orbit of g, observe that

$$g^2(x) = a(ax+b) + b$$

$$= a^2x + b(1+a),$$

$$g^3(x) = a\left[a^2x + b(1+a)\right] + b$$

$$= a^3x + b(1+a+a^2),$$

$$g^4(x) = a^4x + b(1+a+a^2+a^3),$$

and, in general,

$$g^n(x) = a^nx + b(1+a+a^2+a^3+\cdots+a^{n-1}).$$

If a = 1, then

$$g^n(x) = x + bn.$$

However, if $a \neq 1$, the formula for the sum of a geometric series gives

$$g^n(x) = a^n x + b\left[\frac{1-a^n}{1-a}\right]$$

$$= a^n\left[x - \frac{b}{1-a}\right] + \frac{b}{1-a}.$$

Note that, when a is nonnegative, a^n remains nonnegative, while when a is negative, a^n alternates between negative and positive. In particular, when $a = -1$, a^n alternates between -1 and 1. Moreover, when $a \neq \pm 1$, the distance between a^n and 0 either converges monotonically to 0 or diverges monotonically to ∞ as $n \to \infty$ according to whether $|a| < 1$ or $|a| > 1$. Using these facts, we now analyze the behavior of all the orbits generated by g according to the various possibilities for the structural parameters a and b and the initial point x. We shall find it convenient to organize our analysis around the possible value of a. We distinguish seven cases: (1) $a < -1$; (2) $a = -1$; (3) $-1 < a < 0$; (4) $a = 0$; (5) $0 < a < 1$; (6) $a = 1$; and (7) $a > 1$. Within each of these cases, we consider all possible values of the remaining structural parameter b and the initial point x, and we determine the longrun behavior of the orbit originating at x when g has structural parameters a and b.

Let us first dispense with the case $a = 1$. In this case, if $b = 0$, then every x is a fixed point of g (i.e., $g(x) = x$), and (since then also $g^n(x) = x$) the system always remains at any initial point. In contrast, if $b \neq 0$, then *no* x is a fixed point of g; indeed, for any initial point x, $g^n(x)$ diverges monotonically as $n \to \infty$ to either ∞ or $-\infty$ according to whether $b > 0$ or $b < 0$.

In discussing the six remaining cases—that is, the cases in which $a \neq 1$—we take b to be an arbitrary number. In these cases, g has exactly one fixed point, $b/(1-a)$, and any orbit originating there remains there. We next examine the behavior of orbits originating at points other than $b/(1-a)$. For this purpose, we assume that the initial point x is an arbitrary number distinct from $b/(1-a)$.

If $a < -1$, then $g^n(x)$ has no finite or infinite limit; rather, it eventually alternates between positive and negative numbers as its absolute value diverges monotonically to ∞.

If $a = -1$, the fixed point $b/(1-a)$ equals $b/2$, and

$$g^n(x) = (-1)^n(x-b/2) + b/2$$

$$= \begin{cases} b-x & \text{if } n \text{ is odd} \\ x & \text{if } n \text{ is even.} \end{cases}$$

Thus, $g^n(x)$ alternates endlessly between the (distinct) numbers $b-x$ and x.

If $-1 < a < 0$, $g^n(x)$ converges to $b/(1-a)$ while alternating above and below it.

If $a = 0$, then, for all n, $g^n(x) = b$. Thus, the system moves from the initial point directly to b and remains there.

If $0 < a < 1$, $g^n(x)$ converges monotonically to $b/(1-a)$. The convergence is from above if $x > b/(1-a)$ and from below if $x < b/(1-a)$.

Finally, if $a > 1$, then $g^n(x)$ diverges monotonically, to ∞ if $x > b/(1-a)$ and to $-\infty$ if $x < b/(1-a)$.

The possible behaviors of orbits in the one-dimensional linear system are illustrated in figures 3(a)-3(h). From these figures and the preceding discussion, two lessons emerge. First, the fixed point is often at the "center of the action:" it is to or from this point that orbits typically converge or diverge. Second, the slope parameter, a, plays a pivotal role in

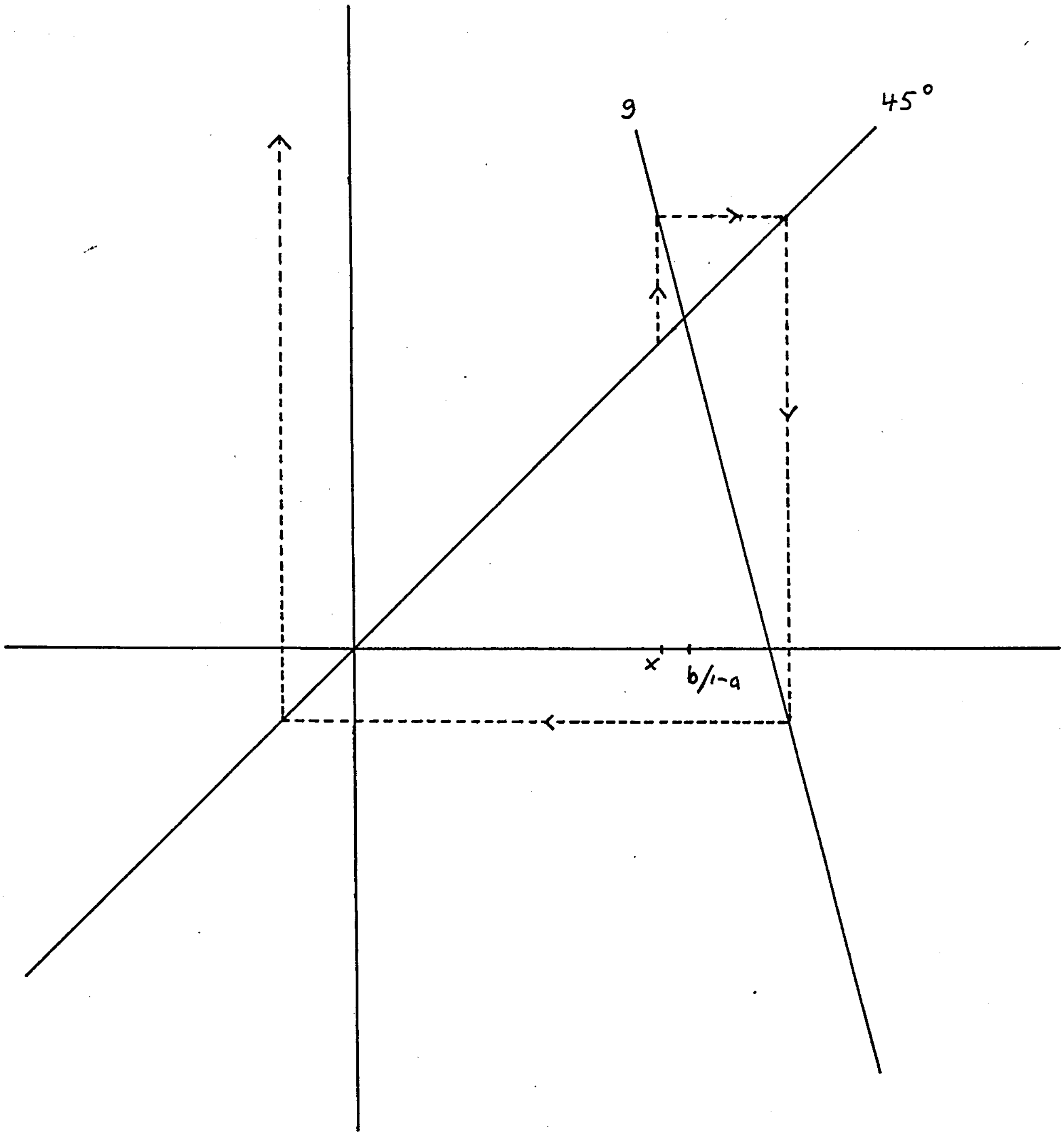

Figure 3(a): Orbit Diagram for Linear Function (a < −1)

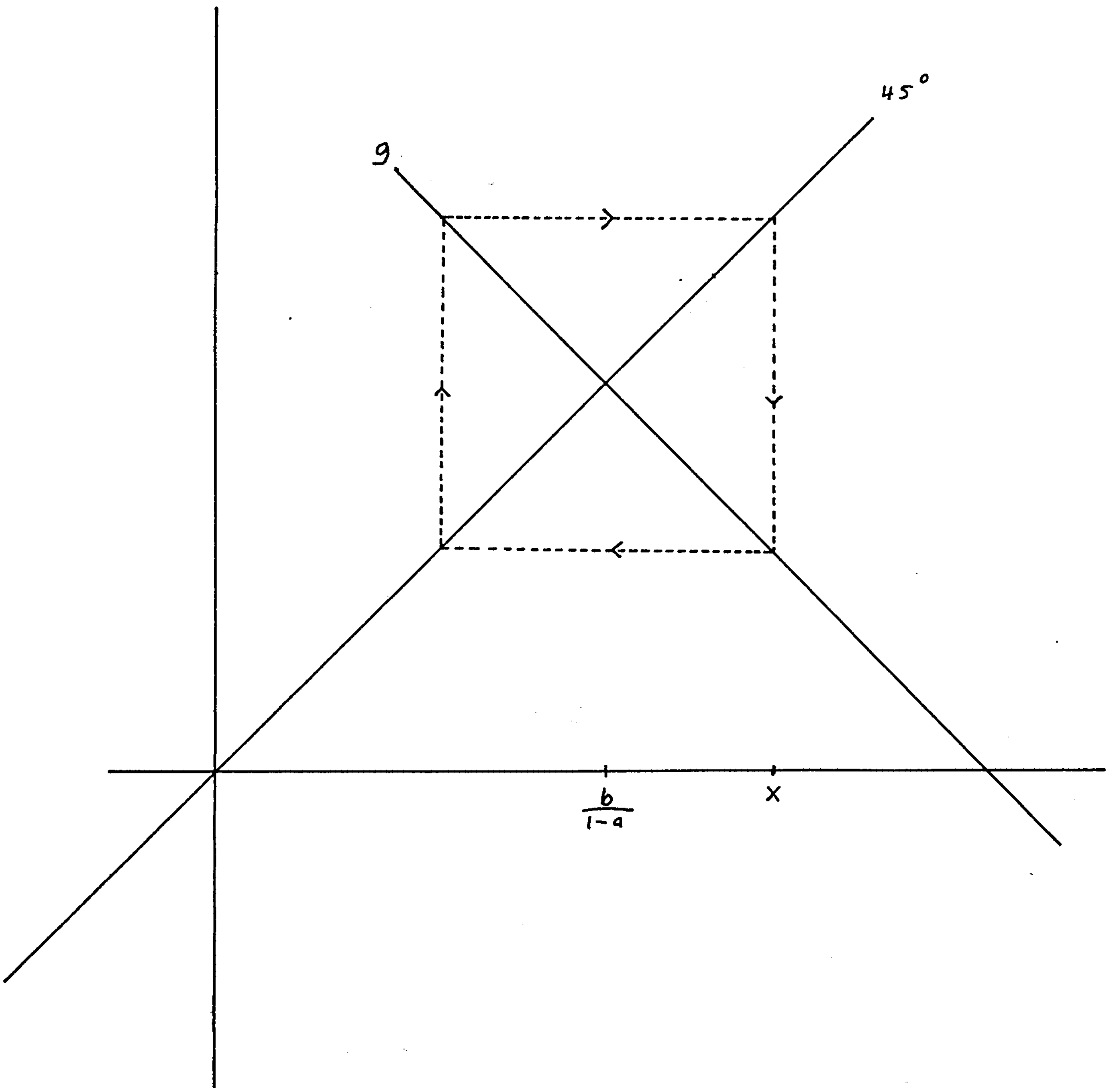

Figure 3(b): Orbit Diagram for Linear Function (a = −1)

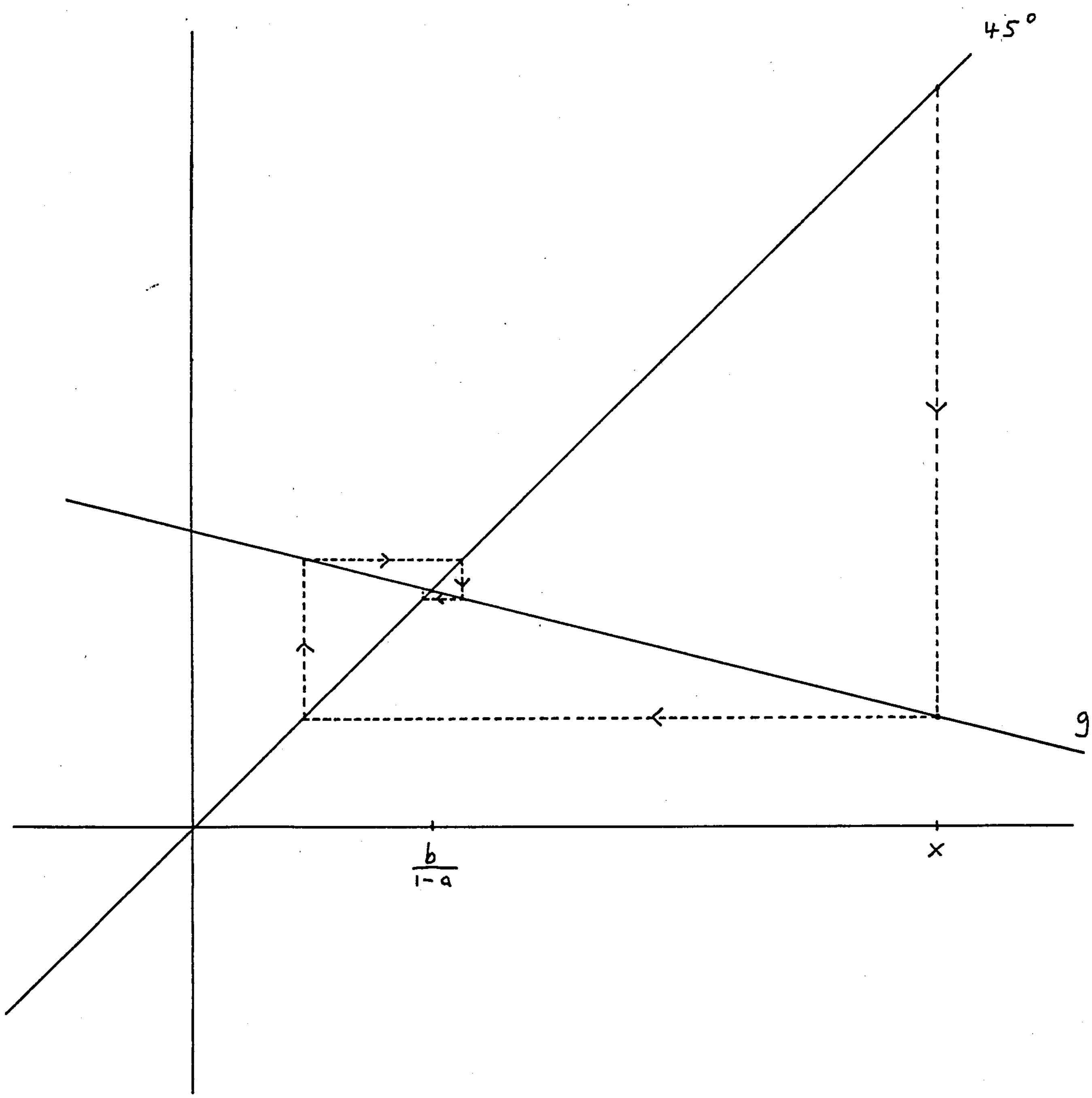

Figure 3(c): Orbit Diagram for Linear Function (–1 < a < 0)

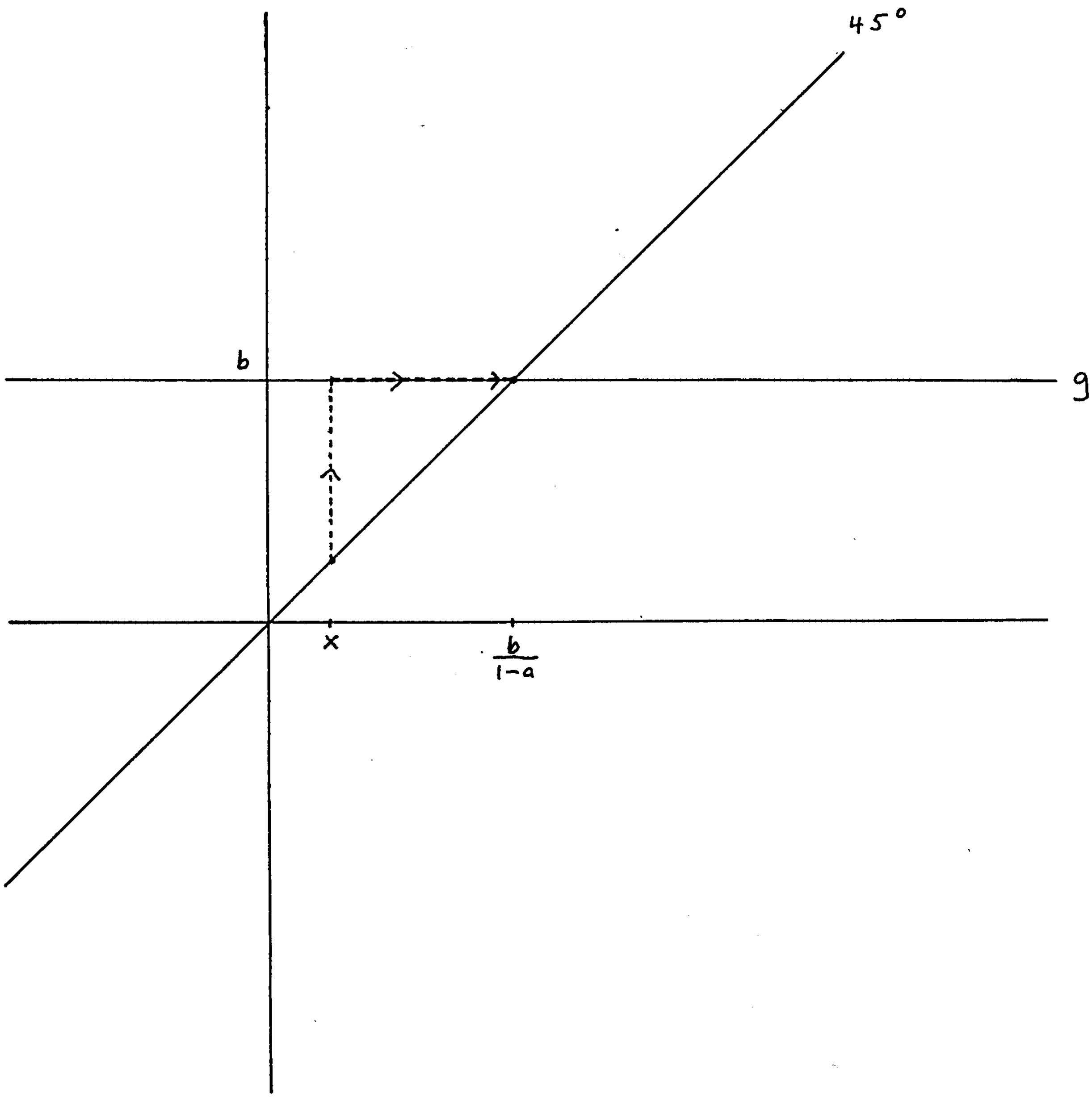

Figure 3(d): Orbit Diagram for Linear Function (a = 0)

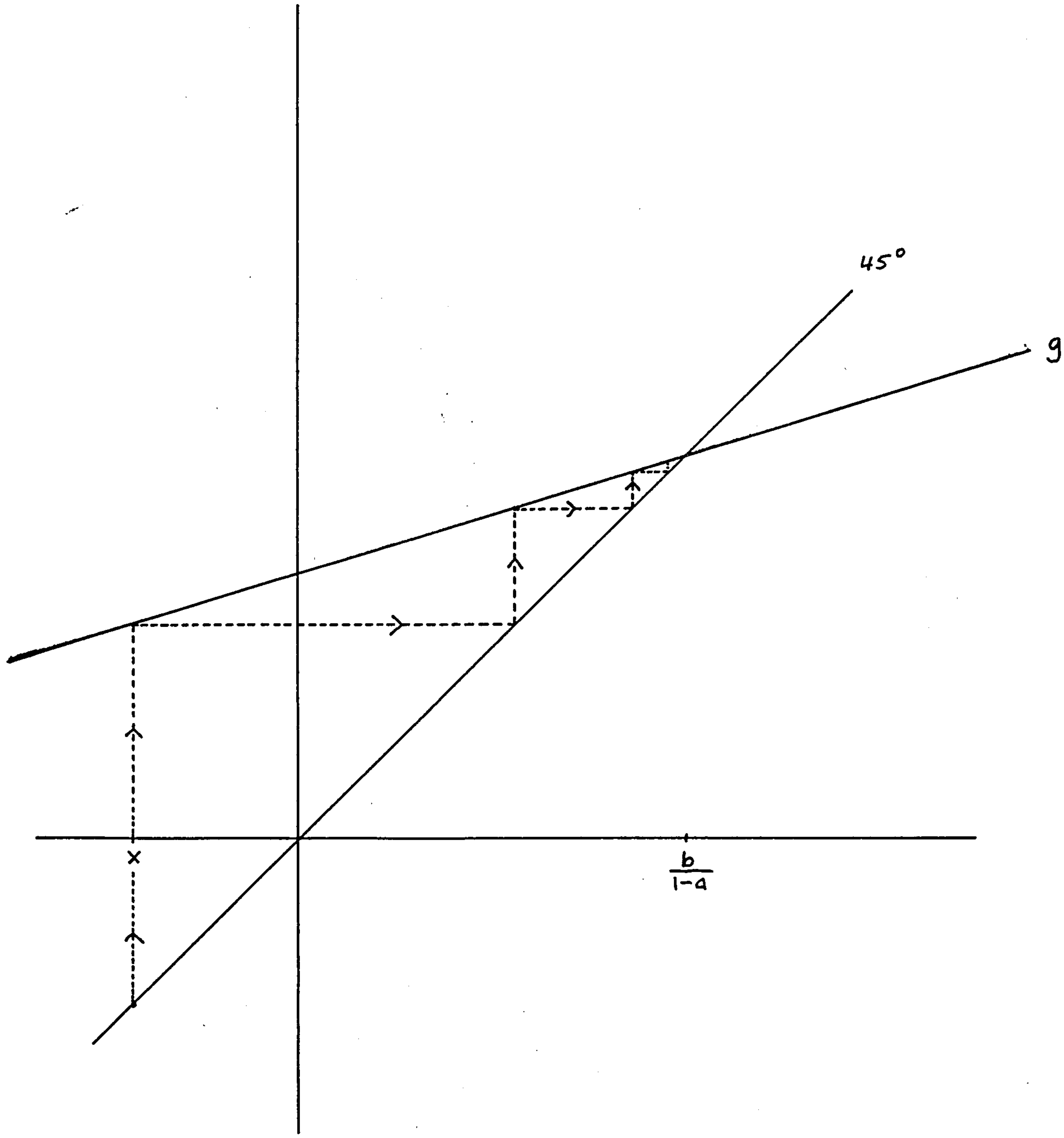

Figure 3(e): Orbit Diagram for Linear Function (0 < a < 1)

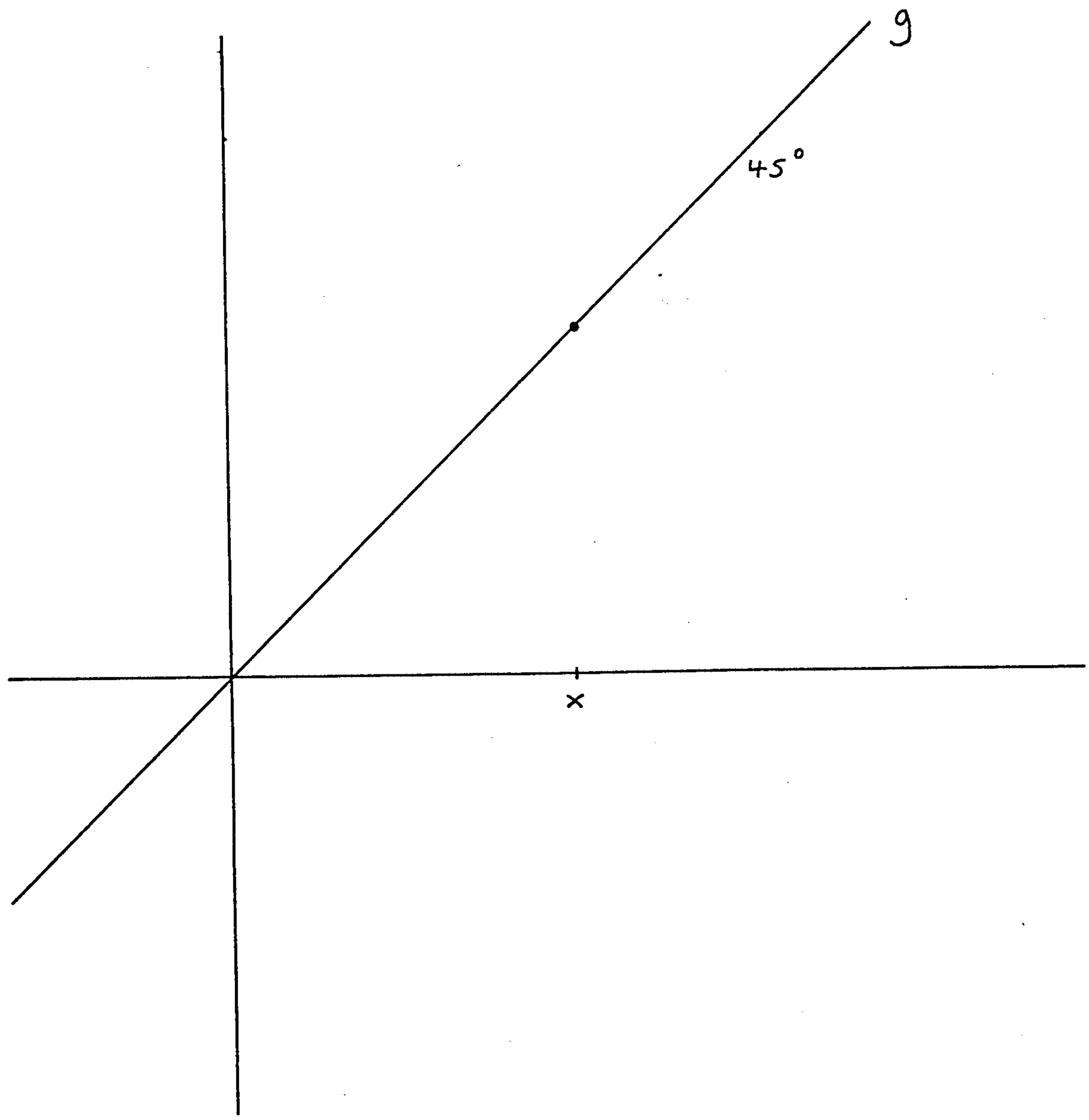

Figure 3(f): Orbit Diagram for Linear Function (a = 1, b = 0)

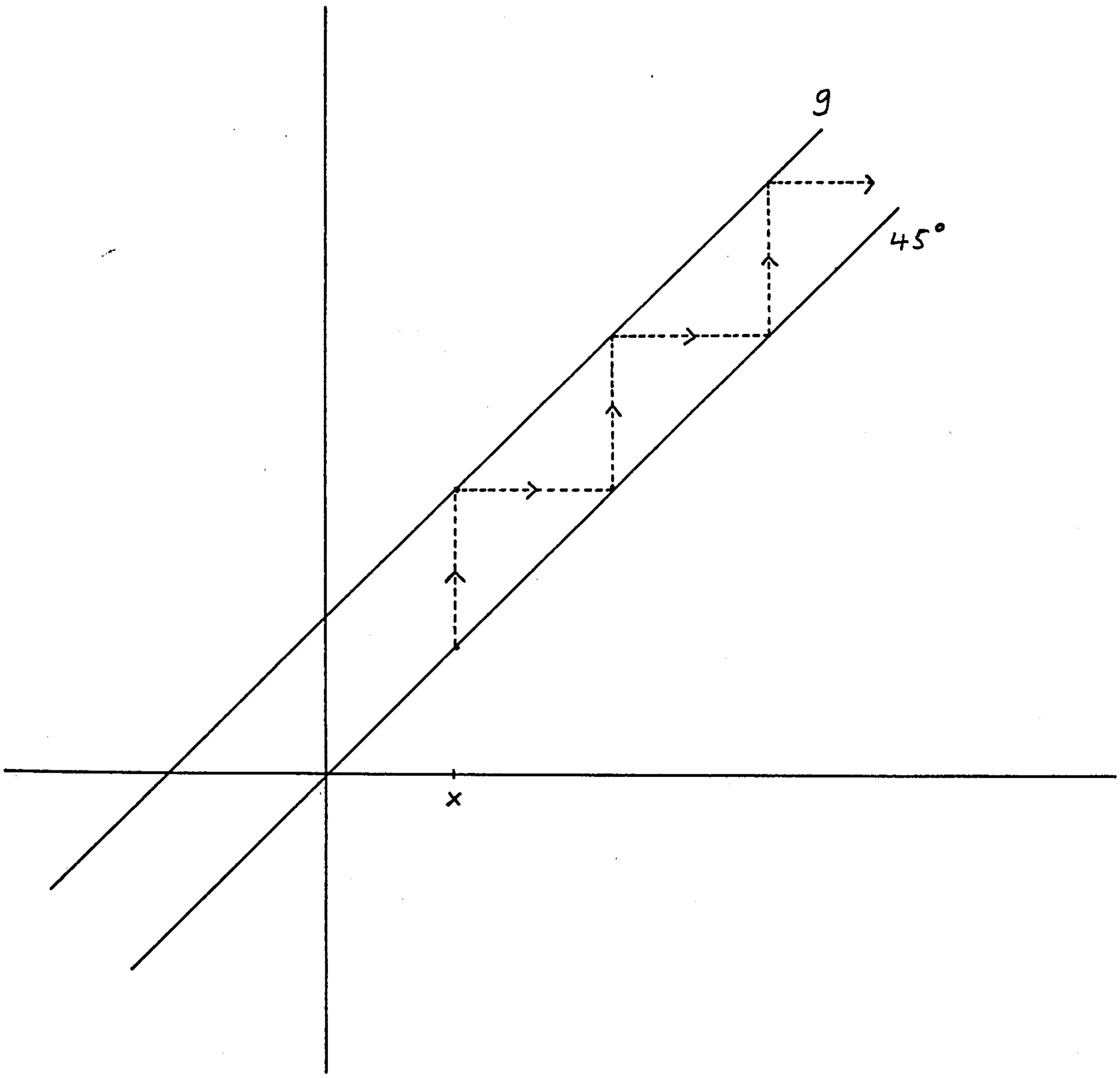

Figure 3(g): Orbit Diagram for Linear Function (a = 1, b ≠ 0)

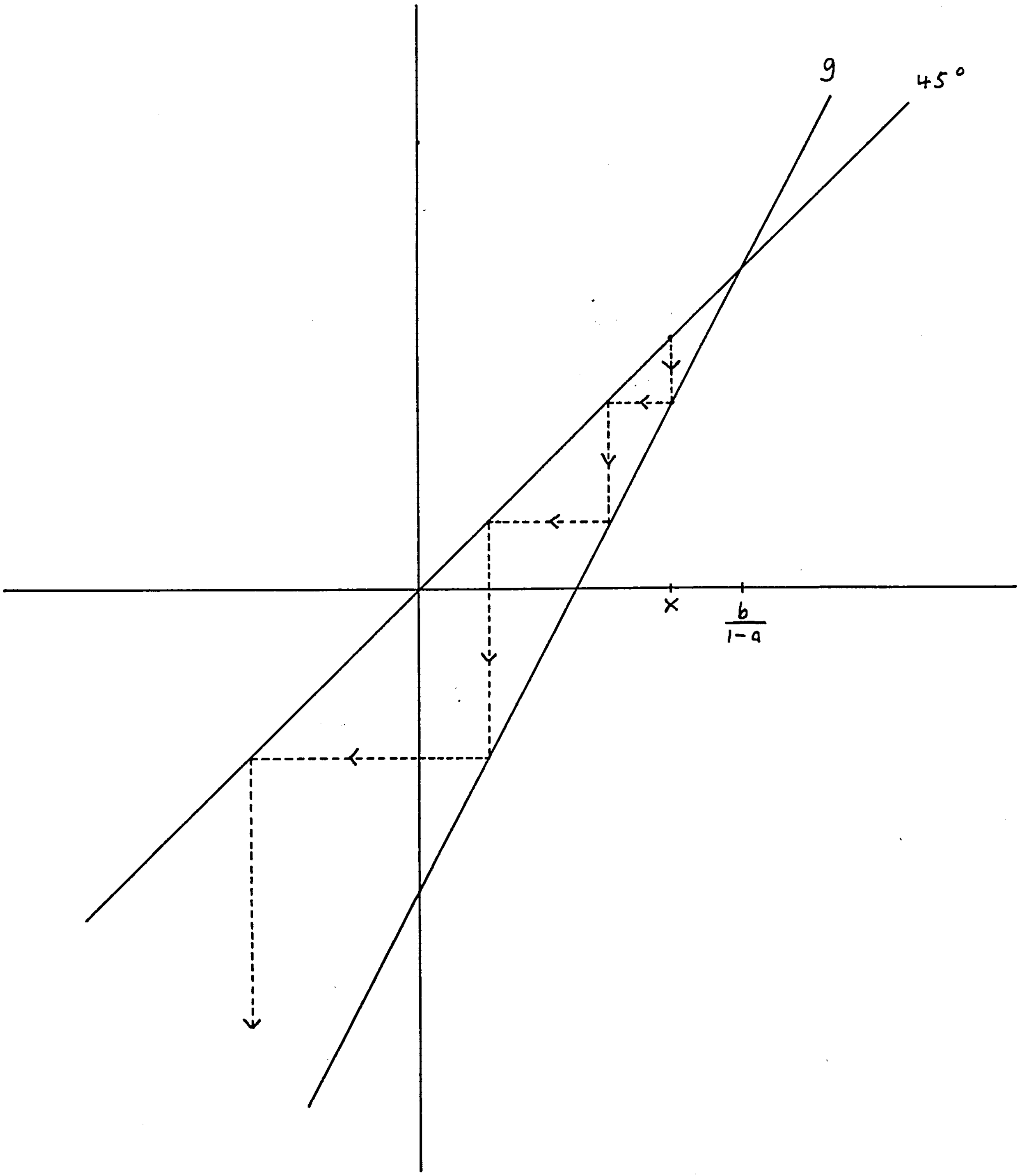

Figure 3(h): Orbit Diagram for Linear Function (a > 1) determining orbit dynamics. These two principles hold as well for nonlinear systems. In the next two subsections, we discuss their role in that context.

Fixed Points and Periodic Points

It is not a coincidence that, in the linear system, convergent orbits always converge to a fixed point of the underlying function. In fact, this property holds in general. For, suppose a continuous function f has a convergent orbit

$$x, f(x), f^2(x), \ldots, f^n(x), \ldots.$$

Let L be the limit. Then

$$f(L) = f\left[\lim_{n\to\infty} f^n(x)\right]$$

$$= \lim_{n\to\infty} f^{n+1}(x)$$

$$= L,$$

so that L is a fixed point of f. Thus, in partial answer to our guiding question, "where does the system go," we can reply: if it converges to any finite limit, that limit must be a fixed point. Correspondingly, if an economy converges to an equilibrium, the equilibrium state must be a fixed point of the system function.

Closely related to fixed points are points whose orbits may leave but later return (see fig. 3(b)). A point x is called a *periodic point* of f *with* *period n* if $f^n(x) = x$. The smallest positive n for which the latter equation holds is called the *prime* period of x. It can be shown that any period of x is a multiple of the prime period.

Every fixed point of a function f is a periodic point of f of prime period 1. It is also true that every periodic point is a fixed point (though not of the same function). For, the periodicity condition

$$f^n(x) = x$$

is nothing but the assertion that x is a fixed point of the function f^n. Thus, properties of fixed points have counterparts for periodic points, and *vice versa*.

If x is a periodic point of f having prime period n_0, then necessarily

$$f^{n_0+1}(x) = f\left[f^{n_0}(x)\right] = f(x),$$

$$f^{n_0+2}(x) = f\left[f^{n_0+1}(x)\right] = f^2(x),$$

etc. It follows that the orbit of x reduces to a finite set consisting of the distinct points x, f(x), $f^2(x)$, ..., $f^{n_0-1}(x)$, through which the system endlessly cycles. (Such an orbit is called a *cycle of length* n_0. An economic example would be a ten-year business cycle.) As a consequence, whenever a system is initialized at a point known to have a small period, the system's entire future evolution can, as a practical matter, be calculated.

Hyperbolic Points

Examination of the linear system reveals that, whenever $|a| < 1$, all orbits converge to the fixed point b/(1–a). However, it is clear from figures 3(c), 3(d), and 3(e) that, if a curve (i.e., a nonlinearity) were introduced into the graph of the function g at some distance from b/(1–a), any orbit originating near enough to b/(1–a) would still converge to b/(1–a). Such local convergence does not depend on the slope of the graph of the function far away from the fixed point; what matters is only the slope—i.e., the derivative—in a neighborhood of the fixed point. In fact, less obviously, but as we shall see momentarily, it is really only the derivative at the fixed point itself that matters.

Similarly, all orbits in the linear system originating elsewhere than b/(1–a) move away from b/(1–a) whenever $|a| > 1$. If a nonlinearity were introduced into the graph of g at a distance from b/(1–a), any orbit originating sufficiently close to (but not precisely at) b/(1–a) would still move away from b/(1–a) (at least initially; the possibility of an eventual return is another issue). Again, for such local "aversion" to b/(1–a), it turns out that only the derivative at b/(1–a) itself matters.

These observations lead to the following definitions. A fixed point p of a function f is called *hyperbolic* if $|f'(p)| \neq 1$. When $|f'(p)| < 1$, p is called *attracting,* while when $|f'(p)| > 1$, p is called *repelling.* These adjectives are justified by the following two propositions, which are readily established: (1) If p is an attracting hyperbolic fixed point, there is an interval containing p such that any orbit originating therein converges to p. (2) If p is a repelling hyperbolic fixed point, there is an interval containing p such that any orbit originating therein (but not at p itself) eventually leaves the interval (at least temporarily). For the function shown in figure 2, 0 is attracting hyperbolic while the other two fixed points are repelling hyperbolic.

In the literature, a periodic point x of f of prime period n is defined as hyperbolic if $|(f^n)'(x)| \neq 1$. The meaning of this definition becomes transparent once it is recalled that x is a fixed point of f^n.

In higher dimensional systems, the notion of the derivative at a point is expressed in terms of a Jacobian matrix, and a periodic point is defined as hyperbolic if none of the eigenvalues of this matrix has complex modulus one (i.e., if none lies on the unit circle in the complex plane).

When a fixed point p is hyperbolic attracting, the system can be considered stable at p with respect to changes in initial conditions. If the system is initialized at p, it will, of course, remain there. More important, though, the system will converge to p even if it is not initialized there, as long as it is initialized sufficiently near p.

In the same vein, a hyperbolic repelling fixed point p can be considered a point of instability of the system with respect to changes in initial conditions. For, while the system will remain at p if initialized precisely there, it will move *away* from p whenever it is initialized sufficiently close to, but not at, p.

One of the consequences of research in nonlinear dynamics has been a deeper understanding of certain sets called *hyperbolic sets*. Within a hyperbolic set, the orbits of any two nearby points initially move away from each other. This behavior is reminiscent of the instability exhibited around individual hyperbolic repelling points. For a hyperbolic set, however, the tendency of nearby orbits to separate occurs *everywhere*. Hyperbolic sets are frequently associated with the appearance of chaotic dynamics. The set Λ of the next section will be a case in point.

An Example of Chaotic Dynamics

In order to gain a qualitative understanding of what is involved in chaotic dynamics, we now examine in detail the class of functions F_μ ($\mu > 1$) defined by

$$F_\mu(x) = \mu x(1-x).$$

Using functions from this class as the law of motion of a discrete dynamical system, we shall investigate the longrun behavior of all orbits. We follow the notation and approach of (*6*).

First, some basic facts (see fig. 4). Let $p_\mu = (\mu-1)/\mu$. Then, $0 < p_\mu < 1$, and p_μ is a fixed point of F_μ. Another fixed point is 0. Since $F_\mu(1) = 0$, the orbit originating at 1 goes immediately to 0 and remains there. Finally, it is easy to show that any orbit of F_μ originating at a point less than 0 (such as the point x_0 of fig. 4) or greater than 1 (such as the point x_1 of fig. 4) diverges to $-\infty$.

Next, suppose $1 < \mu < 3$. Since $F'_\mu(0) = \mu > 1$, 0 is hyperbolic repelling. On the other hand, since $F'_\mu(p_\mu) = 2-\mu$ and $-1 < 2-\mu < 1$, p_μ is hyperbolic attracting. One can show that the basin of attraction of p_μ is precisely the open interval (0,1); any orbit originating in this interval (such as at the point x_2 of fig. 4) converges to p_μ. We have thus determined the longrun behavior of all orbits of F_μ for all values of μ in the range $1 < \mu < 3$, and we have found nothing unusual in the dynamics arising in this parameter range.

However, as μ increases beyond 3, F_μ undergoes various qualitative changes. Among these is a change that occurs as μ passes 4: the maximum value of F_μ (namely $F_\mu(1/2)$, which equals $\mu/4$) increases beyond 1, and some points in [0,1] are thus mapped outside of [0,1] by F_μ. For any such point x

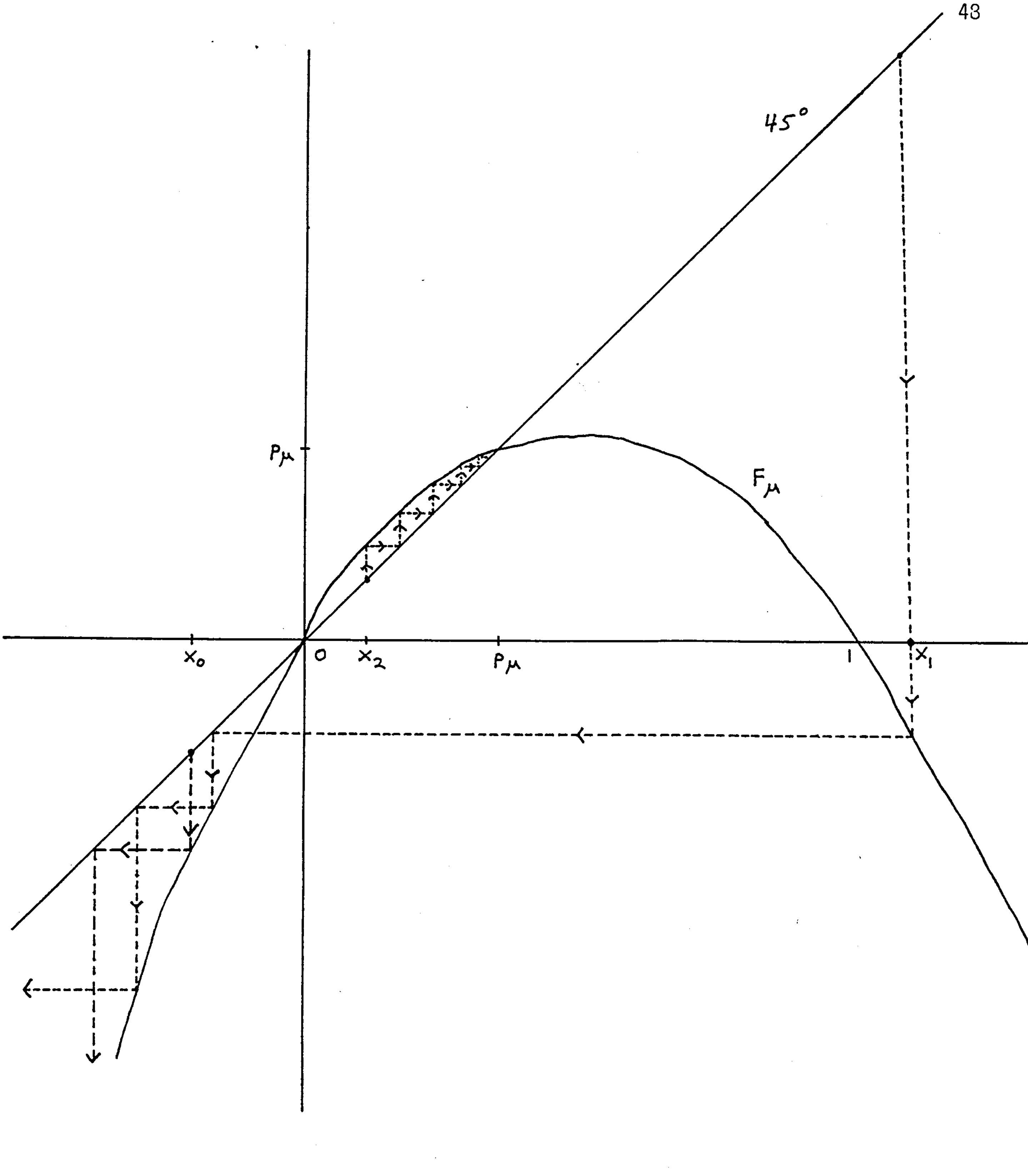

Figure 4: Orbits of F_μ

we have $F_\mu(x) > 1$, and it follows from a previous remark that the orbit of $F_\mu(x)$,

$$F_\mu(x),\ F_\mu(F_\mu(x)),\ F_\mu^2(F_\mu(x)),\ \ldots,\ F_\mu^n(F_\mu(x)),\ \ldots,$$

i.e.,

$$F_\mu(x),\ F_\mu^2(x),\ F_\mu^3(x),\ \ldots,\ F_\mu^{n+1}(x),\ \ldots,$$

must diverge to $-\infty$. Hence, the orbit of x itself must diverge to $-\infty$. More generally, any orbit that originates in [0,1] but does not remain in [0,1] must diverge to $-\infty$.

Of particular interest is the parameter range $\mu > 2 + \sqrt{5}$. Although there are smaller values of μ for which chaotic dynamics appears, the demonstration of chaotic dynamics can be accomplished relatively simply when $\mu > 2 + \sqrt{5}$ (*6*). We turn our attention, therefore, to this case. It will turn out that, on the set of initial points in [0,1] whose orbits never leave [0,1], F_μ behaves chaotically.

Let Λ be this set; that is, let Λ be the set of all x in [0,1] for which each term of the orbit

$$x,\ F_\mu(x),\ F_\mu^2(x),\ \ldots,\ F_\mu^n(x),\ \ldots$$

is in [0,1]. Our first task is to determine the structure of Λ. We shall do so by determining the structure of the *complement* of Λ—the set of those points of [0,1] that are *not* in Λ.

For each n = 0, 1, 2, 3, ..., let A_n be the set of all x in [0,1] whose first n + 1 orbit terms

$$x, \ldots, F_\mu^n(x)$$

are in [0,1] but whose next orbit term, $F_\mu^{n+1}(x)$, is not. Observe that Λ consists precisely of those points of [0,1] that lie in none of the A_n's. Moreover, the A_n's are pairwise disjoint. Thus, one can imagine constructing Λ through the following recursive process: from the interval [0,1], first remove the subset A_0; next, from what remains, remove A_1; etc. In general, when A_0, A_1, ..., A_n have been removed from [0,1], A_{n+1} must still (by disjointness) lie intact in the remaining subset of [0,1]. Remove A_n, and continue this process *ad infinitum.* When all of the A_n's have been removed from [0,1], the subset of [0,1] that remains will be precisely Λ.

To picture what this process actually looks like, we rely on the fact that a point x lies in A_{n+1} if and only if $F_\mu(x)$ lies in A_n.[1] (This property follows from the definition of the A_n's.) Now, A_0 is clearly an open interval of length less than 1 centered at 1/2; thus, removing A_0 from [0,1] leaves two disjoint closed intervals, B_0^1 and B_0^2 (see fig. 5). A_1 (fig. 6) consists of two disjoint open intervals, each lying inside of (and at a positive distance from the endpoints of) one of the closed intervals B_0^1, B_0^2. Thus, removing both A_0 and A_1 from [0,1] leaves behind *four* disjoint closed intervals. This process can be continued. In general, A_n will consist of 2^n disjoint open intervals, each lying inside of, and at a positive distance from the endpoints of, one of the 2^n closed intervals left behind after the removals of A_0, ..., A_{n-1}.

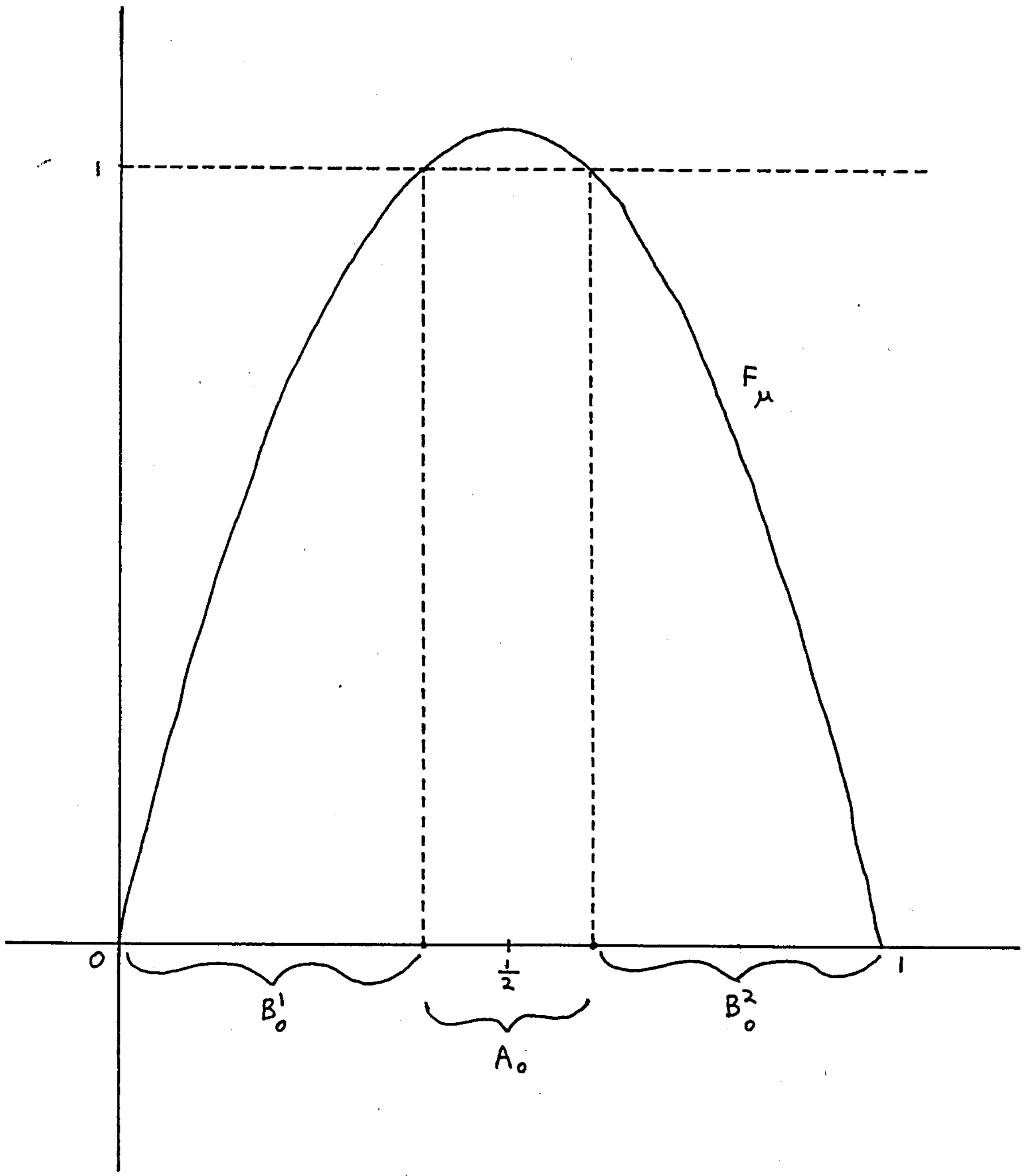

Figure 5: Removing A_0 from [0,1]

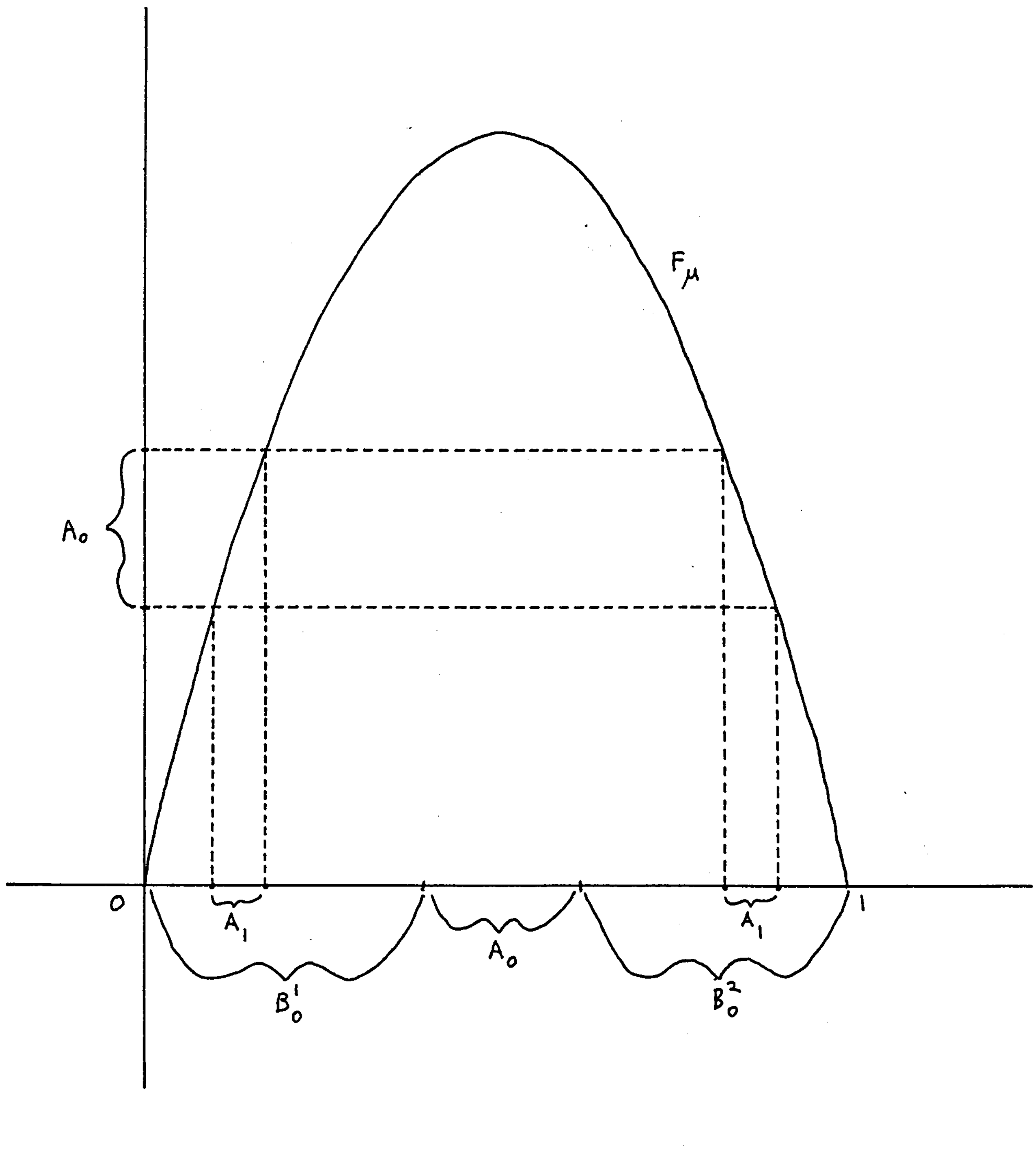

Figure 6: Removing A_1 after A_0

Thus, in brief, Λ is constructed by removing an open interval from the middle of the closed interval [0,1], then removing open intervals from the middles of the remaining closed intervals, etc., *ad infinitum.* This construction bears a striking resemblance to the construction of a classic mathematical object called the *Cantor set*, a set defined by removing from [0,1] the open "middle third " interval (1/3, 2/3), then removing the open middle third intervals (1/9, 2/9), (7/9, 8/9) from the two closed intervals remaining, etc., *ad infinitum,* always removing the open middle third interval of each closed interval remaining after the previous removals. The Cantor set has long been celebrated in mathematics for satisfying the following two conditions: (1) Its "length" is 0 (indeed, by the formula for the sum of a geometric series, the total length of the disjoint intervals removed from [0,1] in constructing the Cantor set is

$$1\cdot(1/3) + 2\cdot(1/9) + 4\cdot(1/27) + \cdots = \sum_{n=1}^{\infty} 2^{n-1}/3^n$$

$$= (1/2)\sum_{n=1}^{\infty} (2/3)^n$$

$$= 1).^2$$

Yet, (2) the Cantor set contains as many points as all of [0,1].[3]

Λ is known to share property (2) with the Cantor set. However, it also shares two further properties having more direct empirical implications: first, it is *perfect*, whose significance here is that, as near as desired to any point of Λ, one can always find another point of Λ; i.e., no point of Λ is isolated. Second, it is *totally disconnected* (it contains no open intervals),[4,5] from which it follows that, as near as desired to any point of Λ, one can always find a point of [0,1] that is *not* in Λ. As a consequence, whenever the dynamical system generated by F_μ is initialized on Λ, its longrun behavior is "infinitely sensitive" to errors in the initial condition; for, within any interval—no matter how small—around an intended starting point in Λ, there exist both points in Λ (whose orbits, by definition, remain in [0,1]) and points not in Λ (whose orbits diverge to $-\infty$). Thus, if one attempted to study this dynamical system on a computer, inevitable rounding errors in determining the points of Λ would make accurate simulation over Λ impossible.

This sensitivity of orbits to the initial condition, while *suggestive* of true "sensitive dependence on initial conditions," must be carefully distinguished from it. The sensitivity just described compares orbits originating in Λ with orbits originating outside Λ. In contrast, and as we shall see shortly, true "sensitive dependence" refers to a kind of separating behavior between orbits originating nearby *within the same set*.

Irregular sets such as Λ and the Cantor set have recently gained the name "fractals."[6] Though the scientific community has not yet arrived at a consistent usage of this term, one often sees the following individual or joint criteria: exhibiting a high degree of jaggedness; self-similar (that is, defined by a recursive process in such a way that any part of the set, when magnified, looks the same as the entire set); and having noninteger dimension.[7]

Until relatively recently, the Cantor-like sets now called fractals were considered exotic structures belonging solely to the world of pure mathematics. The discovery of their intimate connection with nonlinear dynamics has been striking. However, they are now understood to be a typical concomitant of nonlinear dynamical systems. (See, for example, *9*, *11*, *16*, and the references contained therein.) They have been detected in the form of attractors,[8] in the form of the boundary between competing basins of attraction,[9] and—as we shall now see—in the form of the state space region on which chaotic behavior is manifested.

Having characterized the structure of the set Λ of all points whose F_μ-orbits remain in [0,1], we now describe the chaotic behavior of F_μ on this set. First, however, we need a definition. Suppose X' and X are subsets of the real line, $\mathbb{R}$. Then, we say X' is *dense* in X if, for any point x in X, one can find some point from X' as close to x as desired.

Using a technique called *symbolic dynamics*,[10] one can show that the set of periodic points of F_μ is dense in Λ. Thus, cyclic orbits can be found originating arbitrarily near any point of Λ. On the other hand, erratic orbits can also be found originating arbitrarily near any point of Λ. More specifically, one can use symbolic dynamics to show that, arbitrarily near any point of Λ, there is a point of Λ whose orbit is dense in Λ. Such an orbit would appear erratic and essentially random, for it would endlessly "dance" around Λ, visiting and revisiting the vicinity of each point of Λ infinitely many times.

The real hallmark of chaotic dynamics is considered to be sensitive dependence on initial conditions. To define this concept, let X be any set endowed with a distance measure, and let f be a function mapping X to itself. We say f exhibits *sensitive dependence on initial conditions* if there exists a $\delta > 0$ with the following property: for any x in X and any $\epsilon > 0$, there is a point x' in X within a distance of ϵ from x such that, for some n, the distance between $f^n(x)$ and $f^n(x')$ exceeds δ. Heuristically, sensitive dependence means that there is a constant $\delta > 0$ such that, arbitrarily close to any point of X, one can find another point of X whose orbit eventually diverges (even if temporarily) from that of the given point by more than δ.[11] Although this discrepancy in orbits is required only to exceed δ, not to be arbitrarily large in *absolute* terms, it should be noted that the *ratio* of this discrepancy to the distance between x and x′ *will* become arbitrarily large when x′ is chosen arbitrarily close to x. It is in this sense that sensitive dependence implies unpredictable longrun behavior for orbits originating arbitrarily near one another.

F_μ exhibits sensitive dependence on initial conditions. To sketch a proof, let δ be any positive number less than the distance between the intervals B_0^1 and B_0^2, i.e., less than the length of A_0 (fig. 5). Choose any x in Λ and any $\epsilon > 0$. Since, as noted earlier, no point of Λ is isolated, there must exist a point x′ in Λ distinct from x whose distance from x is less than ϵ. However, symbolic dynamics can be used to demonstrate that, for any distinct points of Λ—say, for x and x′—there exists an n such that either $F_\mu^n(x)$ is in B_0^1 and $F_\mu^n(x')$ is in B_0^2 or vice versa. It follows at once that the distance between $F_\mu^n(x)$ and $F_\mu^n(x')$ exceeds δ, and the property of sensitive dependence is established.

While we have discussed chaotic behavior over a fractal set Λ in order to illustrate important aspects of the subject, chaos is not a phenomenon that appears only on unusual sets. One can show, for example, that the function F_4 given by

$$F_4(x) = 4x(1-x)$$

is chaotic on the entire interval [0,1]. Nor is the existence of chaos overly sensitive to functional form. The chaotic dynamics we have described for F_μ

will also be exhibited by essentially any hill-shaped function with sufficiently large slope.

PROBABILISTIC BEHAVIOR OF CHAOTIC ORBITS

We have seen that the orbits of nonlinear models can behave in an erratic, seemingly random manner. This finding suggests that at least some purely deterministic economic models may be capable of generating the sort of jumpiness in economic variables that traditionally has been ascribed solely to external shocks. That is, nonlinear deterministic economic relationships are themselves quite capable of endogenously generating uncertainty. When the initial conditions lie in that portion of the state space over which chaos is manifested, attempts to make point forecasts of longrun economic behavior are, as a matter of mathematical principle, doomed to failure. The uncertainty is intrinsic to the model itself.

There are different types of uncertainty, however, and it is appropriate to inquire whether the erratic quality of chaotic orbits is utterly lacking in discernible structure or whether, rather, it at least satisfies laws of a probabilistic form. Recent work provides encouraging evidence that, in a broad class of economic models, chaotic orbits do indeed behave in a strictly probabilistic manner. Furthermore, the probability distributions of the associated economic variables are potentially computable.

Day and Shafer (4) demonstrate that, in the standard dynamic macroeconomic model developed by Mezler, Modigliani, and Samuelson, chaotic orbits will, under reasonable assumptions, occur for a set of initial conditions of positive measure (length). That is, a randomly selected initial condition will have a positive probability of generating a chaotic orbit. Moreover, when the function f is the system's law of motion, the sequence

$$f, f^2, f^3, \ldots, f^n, \ldots$$

of the iterates of f will behave precisely like a stationary stochastic process. In fact, there will exist a probability distribution on the state space with respect to which the sequence of iterates *is*, in a perfectly rigorous mathematical sense, a stationary stochastic process. Consequently, individual chaotic orbits may be viewed as realizations of such a process.

Day and Shafer not only provide a theoretical treatment of the problem; for a broad range of cases, they determine the actual form of the probabilistic behavior of the chaotic orbits, and they show how the pertinent probability distributions can be estimated numerically. They show that longrun GNP can be characterized by its probabilities of lying in various intervals. In one scenario, they find that GNP cycles among finitely many intervals, varying randomly within each interval. Moreover, normalized sample means of certain model-generated values obey the Central Limit Theorem. This result implies that, even in the absence of exogenous influences, economic variability following a normal probability law arises endogenously as a mathematical consequence of the economics itself.

The authors conclude that chaotic behavior for which orbits act as realizations of stationary stochastic processes is likely to occur for a large class of recursive economies. This viewpoint is supported by recent purely mathematical investigations that have led to the conjecture that "most" chaos is mathematically equivalent to the behavior of stochastic processes of a standard type (see *13*, especially p. 22). It would appear, then, that chaos theory may have led not merely to the negative finding that longrun point forecasting in many nonlinear models is logically untenable, but also to a new understanding that certain longrun economic variables are fundamentally probabilistic, possessing potentially calculable probability distributions that arise from the nonlinear relationships themselves, even in the absence of exogenous sources of uncertainty.

CONCLUSIONS

Recent findings in the field of nonlinear dynamical systems warrant a rethinking of traditional attitudes toward economic dynamics. It is now known that erratic longrun behavior and various forms of sensitivity to initial conditions can arise in even the simplest nonlinear models. Unless there are sound reasons in economic theory to believe that a given dynamic economic process is linear, the process must be viewed as at least potentially liable to the type of chaotic behavior described here.

Chaos theory suggests that the long-range prediction of nonlinear economic processes may be subject to the same basic mathematical limitations as long-range weather prediction. In both cases, future behavior may appear independent of the initial conditions that produced it. A significant and irreducible component of uncertainty may arise from the economic model itself irrespective of any external perturbations, data errors, or limitations of econometric technique. Prediction of point values, even when tempered by reliance on confidence intervals, may have to give way to estimation of longrun probability distributions defined over the appropriate attractor in state space.

It is difficult to study this subject without experiencing a certain humility concerning our ability to control nonlinear economic processes through policy intervention. Nonlinear systems can behave in a counterintuitive manner. The conditions under which we can properly use mathematical models to predict the longrun implications of policy actions need to be clarified.

The discoveries of recent years might seem to have revealed intrinsic mathematical limits to economic prediction. Yet, a deeper understanding of the limitations of longrun point prediction should ultimately enhance, not diminish, the accuracy and credibility of the information we provide. A further enhancement may derive from the replacement, in certain cases, of longrun point forecasts by forecasts based, in part, on endogenous longrun probability distributions.

We have attempted in this paper to sketch some of the major themes of contemporary nonlinear dynamics. Many topics, however, had to be omitted. Suggested further background reading would include (*9*, *6*, *16*, *11*, *7*, *14*). For a sampling of the nascent economics literature on chaos, see (*1*, *2*, *3*, *4*, *5*, *8*, *10*, *15*, *17*).

FOOTNOTES

1. In mathematical parlance, A_{n+1} is the *preimage* of A_n relative to the function F_μ.

2. Modified forms of the Cantor set having positive length can be constructed by removing shorter intervals.

3. Specifically, it can be shown that there exists a one–to–one correspondence between the Cantor set and [0,1]. Since the elements of both sets can thus be paired off, the total number of points in each set must be the same. The fact that this number happens to be infinite should not be held against it. Infinite sets have sizes too.

4. This property should seem at least plausible in view of the method of construction of Λ. Interestingly, the property implies that every point of Λ is on the boundary of Λ.

5. It is in proving this property that the assumption that $\mu > 2 + \sqrt{5}$ is first put to use. See (*6*).

6. From the Latin *fractus*, meaning "broken."

7. There are many ways to extend the usual concept of dimension (0 for a point, 1 for a curve, 2 for a surface, etc.) to more complicated sets. *Hausdorff* dimension (*12*), perhaps the most widely used, assigns to the Cantor set a dimension of ln 2/ln 3, or approximately 0.63. Some other

notions of dimension suggested for application to fractal sets are *information* dimension, *correlation* dimension, and *Lyapunov* dimension (see *16*).

8. A fractal attractor is called a *strange* attractor.

9. Thus, consider an economic model that allows different initial conditions to generate different equilibria. Here the boundary between basins of attraction corresponding to distinct equilibria may be a fractal exhibiting a type of sensitivity to the initial condition (see the "Attractors" section of the Introduction). The equilibrium generated by an initial condition lying on this boundary would be unpredictable.

10. This method is explained in (*18*).

11. *Lyapunov exponents* are sometimes used as a pragmatic measure of this divergence. See (*16*).

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