QUANTIFYING LONG RUN AGRICULTURAL RISKS AND EVALUATING FARMER RESPONSES TO RISK

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Two Applications of Chaos in Economic Theory

by

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Applications of basic mathematical mechanisms that determine the dynamical behavior of systems ranging from planets to weather to biology can also be applied to economics. However, if the theory of chaotic dynamics is to be useful for economic analysis, economists must establish models in which the dynamic behavior of economic agents or aggregates is represented by nonlinear relationships. For such models to be relevant important to show that chaotic trajectories would result from a set of initial conditions of sufficient generality, and that the distribution of sequences describing these trajectories exhibit certain regularity properties so that probabilities concerning the future behavior of an orbit generated by a particular initial condition can be determined. In this paper I will review previous research establishing the existence of chaotic economic models and the conditions under which chaotic trajectories will be generated. I will rely on results presented by Weiss in this volume, particularly with respect to the behavior and stability of the quadratic map. The paper focuses on two examples, a model of pure exchange, and a model of capital accumulation. The models are presented in simple forms that may not be consistent with the real world, but which illustrate fundamental dynamical properties.

Pure Exchange

Sarri (1988, 1989) discusses the properties of the model for a pure exchange economy described in works by Sonnenschein, Mantel, and Debreu. Consider an exchange economy that exists in time. At the beginning of each day the agents in the economy express their desires for the commodities in terms of Walrasian supplies and demands. For an arbitrary price vector **p**, there will be excess supplies and demands in some or all markets. Assume that prices adjust according to the law of supply and demand in the following way. There are c ≥ 2 commodities, and a ≥ 2 agents in the economy. Each agent has an initial endowment of commodities denoted by the vector $\mathbf{w}=(w_1^i, w_2^i, \ldots w_c^i)$ which is a vector in R_+^c, where R_+^c is the positive orthant of the commodity space R^c. Each agents' preferences are given by a smooth concave utility function U that maps the commodity space onto the real line, and where all of the components of the gradient of U, are positive. For any price vector **p**, the budget constraint is given by

$$1.1 \quad (\mathbf{p}\cdot\mathbf{x}) \leq (\mathbf{p}\cdot\mathbf{w}^i) \rightarrow (\mathbf{p}\cdot\mathbf{w}^i - \mathbf{p}\cdot\mathbf{x}) \geq 0, \ \forall\, \mathbf{x} \varepsilon R^c$$

Each agent's demand for goods is given by the point $\mathbf{x}^i$, dependent on **p** where an indifference curve of $U_i(\mathbf{x})$ is tangent to the budget plane. Thus, each agent has $\mathbf{w}^i$ and wants $\mathbf{x}^i$. The difference between what the agent has and what he wants, $\mathbf{x}^i - \mathbf{w}^i$, defines the agent's excess demand vector $\mathbf{g}^i(\mathbf{p}) = (\mathbf{x}^i - \mathbf{w}^i)$, indicating what the agent is willing to trade. If an element of $\mathbf{g}^i(\mathbf{p})$ is positive, demand is greater than endowment and the agent wants to buy. If an element of $\mathbf{g}^i(\mathbf{p})$ is negative the agent has excess supply and wants to sell. Because the vector, $\mathbf{g}^i(\mathbf{p})$, is in the budget plane,

$$1.2 \quad \mathbf{p}\cdot\mathbf{g}^i(\mathbf{p}) = 0$$

each agent's ability to buy and sell commodities depends on the excess demand functions of other agents in the economy. If other agents are willing to buy and sell commodities then a market exists, and we can define an aggregate excess demand function for the market as,

$$1.3 \qquad \boldsymbol{g}(\boldsymbol{p}) = \sum_{i=1}^{c} \boldsymbol{g}^{i}(\boldsymbol{p})$$

The market is in equilibrium, when the total amount of commodities agents want to sell is equal to the total amount of commodities agents want buy. This condition is given by

$$1.4 \qquad \boldsymbol{p}\cdot\boldsymbol{g}(\boldsymbol{p}) = 0$$

If 1.4 does not hold markets will not clear for a given price vector **p**, and a new **p** is needed. But how is that new **p** determined, and does the adjustment of prices lead to an equilibrium?

The price adjustment process is the basic dynamic in the intertemporal exchange economy. Following Sarri (1989), the dynamics of price adjustment are analyzed by considering the tatonnement process. Tatonnement suggests that positive elements of the aggregate excess demand function require a higher price to reach an equilibrium, and negative elements should be priced lower. The price adjustment mechanism is defined as

$$\boldsymbol{P}_{t+1} = \boldsymbol{P}_{t} + h(\boldsymbol{g}(\boldsymbol{P}_{t})) = \boldsymbol{F}_{g}(\boldsymbol{P}_{t})$$

where h is a positive scalar. Using the Sonnenschein-Mantel-Debreu theorem it can be shown that any function satisfying the conditions is an excess demand function for some economy.

1) $\boldsymbol{pg}(\boldsymbol{p}) = 0 \qquad \forall\, \boldsymbol{p} > 0$, 2) if $\boldsymbol{p}_i$, the price of the i^{th} commodity, has a sufficiently small value the aggregate demand

for that commodity will be positive. Thus, there exist excess demand functions that exhibit any particular type of dynamical behavior one would wish to consider (Sarri, 1989).

Investigate the dynamic properties of the tatonnement process requires two standard reductions in the model (Sarri 1989). The first is to normalize the vector **p** and reduce the domain of the problem to the unit interval so that

$$1.6 \qquad \sum_i p_i = 1$$

Applying Walras' Law we know that if we determine the first c-1 prices the last price is determined by 1.6 implying that the first c-1 prices can take any values on (0, 1) so long as the sum of these prices is strictly less than one (we assume there are no free goods). For an economy with only two commodities these reductions make F_g in 1.5 a smooth function defined on [0, 1]. For the case of a two good economy only one price is needed, and **p** becomes a point, denoted p.

The goal of a market dynamic is to find a price that obtains an equilibrium, that is, find a p that makes $g(p_t) = 0$. Such a point is found when $p_{t+1} = p_t$. Graphically, all equilibrium are the set of points where $p = F_g$ on the 45 degree line in figure 1. When p is not an equilibrium point the dynamic process in 1.5 will generate a new price and the process continues until an equilibrium is reached. In some instances 1.5 will converge to a stable equilibrium, while in other cases there may be unstable equilibria. However, there are also points on [0, 1] that do not converge to a stable or unstable equilibria.

To show that there is a set of points on [0, 1] that never obtain an equilibrium, characterize the set of points that are equilibria or converge to an equilibrium in terms of the points that never converge Sarri (1988, 1989). Define the nonconvergent set, S = {of all p ¦ if p is an initial price then the dynamic defined in 1.5 never converges to a zero of g}. Two theorems by Sarri (1989) show that for a g of this kind the set S is large. I will rely on examples from the quadratic difference to illustrate the result, but the result itself is generic.

Theorem 1. For a general function g defined in 1.5 but characterized by the quadratic form, the set S is uncountable.

Remark 1. Demarcate the quadratic difference in figure 1 at 0.5. All points on the p axis to the left of 0.5 are contained in the set denoted a, and all points to the left of 0.5 are in the set denoted b. Choose an initial iterate p_1 and follow the orbit obtaining a sequence of points $\{p_1, p_2, p_3, p_4, \ldots\}$. Each value p_i in the sequence has an address in the corresponding set, a or b where it lives. An alternative way of representing the sequence is to replace each pi with letters that correspond to it's address, that is $\{a, a, b, a, b, \ldots\}$ where $p_1 \varepsilon a, p_2 \varepsilon a, p_3 \varepsilon b, p_4 \varepsilon a, p_5 \varepsilon b$, and so on represents the sequence generated by a particular p_1 . Define $U = \{a, b\}$ as the universal set, or the set of all possible sequences made up of the letters a and b. The assertion is that the dynamics of the quadratic , and by implication, the dynamics of tatonnement for a general g satisifying the Arrow-Debrue-Mantel conditions, are so erratic that they seem to be generated by a random process.

Theorem 2. Let g be a general function as characterized by the quadratic form, and let $Z \varepsilon U\{a,b\}$. There exists an initial iterate p_1 so that the k^{th} iterate of p_1 is in the interval denoted by the k^{th} symbol in Z.

Remark 2. Theorem 2 implies that it is possible to specify the entire future of any initial iterate in advance. That is, it is possible to show how the orbit of any initial seed point will bounce back and forth between the intervals a and b. Furthermore, there exist an unique future for each initial point. If the resulting sequence generated by an orbit does not obtain an equilibrium than the corresponding sequence in U will not become constant in a fixed interval. A nonconstant sequence in U is sufficient to show that the dynamics of the function will not converge. There are an uncountable number of sequences in U that are not eventually constant, so there must be an uncountable number of initial iterates with dynamics that do not converge. Thus theorem 1 is a consequence of theorem 2.

Proof. The proof of theorem 2 relies on the concept of the inverse image of a set. In figure 1 there are points in interval a that will go to interval b after one iteration. The interval a covers the entire interval [0, 1], and a subset of a covers b. The converse of this is also true. There are also points that remain in a after one iteration and then go to b and then back to a and so on. The subset of a that covers b is the inverse image of b denoted $T^{-1}{}_{a}(b)$. Now define a sequence

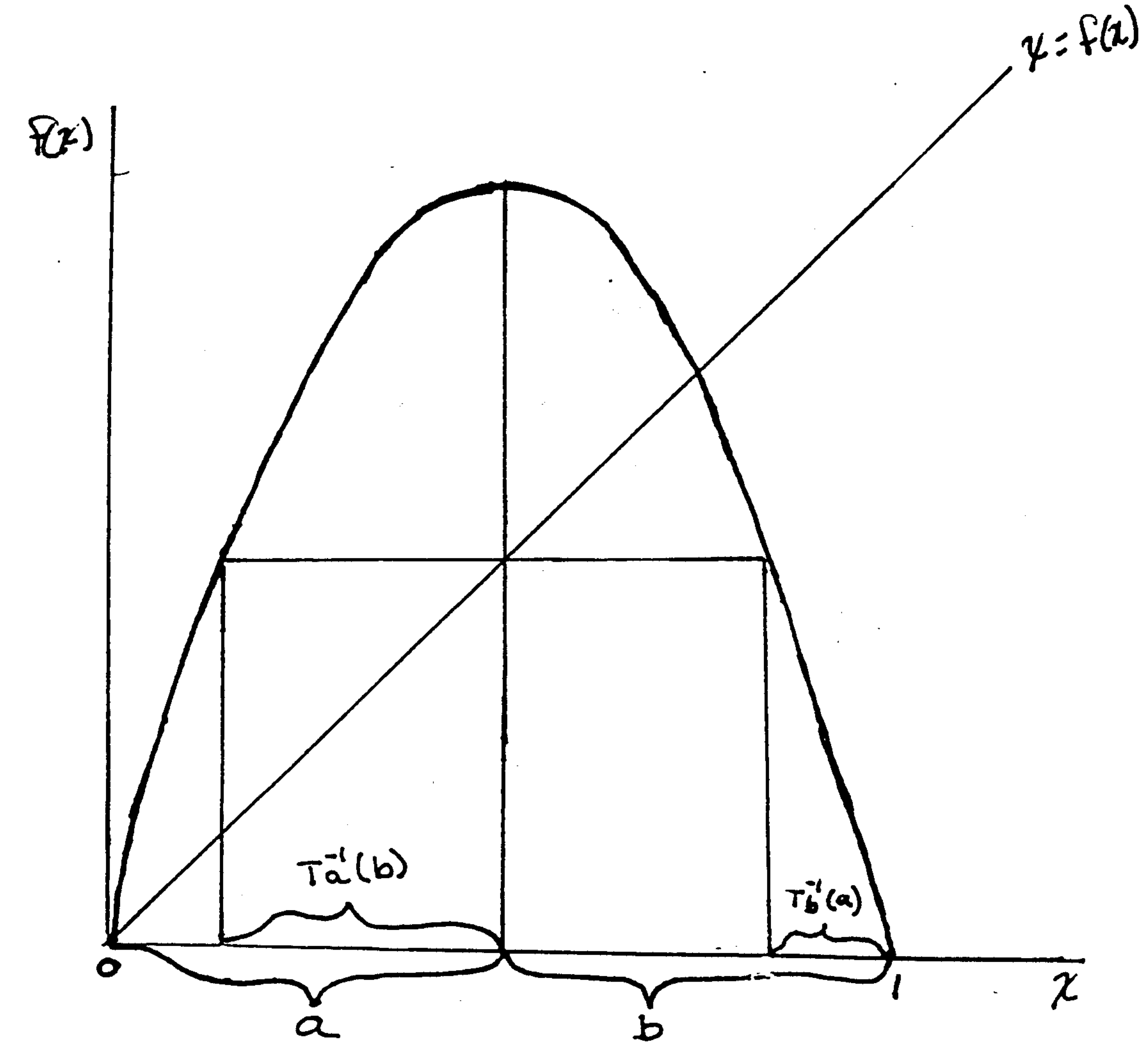

Figure 1. Inverse Images of the Quadratic

$T = \{a, b, a, a, b, \ldots\}$, and let T_n be the listing that specifies the first n terms of T such that $T_2 = \{a, b\}$, $T_c = \{a, b, a\}$. Define the set $C(T_n) = \{ p \mid 1 < i < n\text{-}1,\ p_i$ is in the i^{th} specified interval of T_n and p_n is in the closure of the n^{th} specified interval}. For any sequence T there is a corresponding sequence of inverse images that, by the continuity of F, are nonempty closed subsets. Furthermore, these subsets are nested in smaller and smaller portions of the intervals with the structure, a $C(T_2) \ldots C(T_3) \ldots C(T_n) \ldots$. By definition the set $C(T_n)$ consists of all of the initial points where the dynamics obey the first n steps of the specified sequence. Such points exists and are countable because the intersection of compact nested sets is nonempty. However there are an uncountable number of sequences that will not allow the trajectory of an initial iterate to remain in any one interval, that is not converge. Because there are an uncountable number of points that allow this behavior the set S contains an uncountable number of points. QED.

Sarri's Work on the tatonnement process and his theorems are proofs for general cases with specific applications to economics. These theorems show how tenuous our assumptions of convergence for many dynamical processes may be. In fact, dynamical systems are more likely to be unstable and nonconvergent than to exhibit smooth convergence to an equilibrium.

Capital Accumulation and Business Cycles

Dynamical techniques have been applied to the study of aggregate economic phenomena for quite some time. Two of the most important areas where economic dynamics has been developed have been models of capital accumulation, and business cycles. In both cases there has often been an implicit assumption that dynamical economic systems would ultimately obtain a long-run steady state solution that could be a fixed point, or a limit cycle of some kind. While the discussion of tatonnement and the possibility of chaotic behavior in that process tended toward the abstract, our current discussion will focus on a model with a specific functional form and economic justification.

Bhaduri and Harris investigated the dynamics of the simple Ricardian model for capital accumulation, and found this model to have a rich variety of dynamical behavior for a broad range of economically relevant parameter values. The model is based on a "corn" economy where a single homogeneous output is distributed between wages to labor, profits to capitalists, and factor rents. The model is developed from the notion that land used to produce corn will decrease in quality as the amount of land in production increases. Thus, labor productivity will decrease at the margin. Assume that the marginal productivity of labor can be expressed as a linear function

$$2.1 \qquad \frac{dY_t}{dX_t} = a - bX_t \qquad a>0,\ b>0$$

The corresponding equation for total product is obtained by integrating this equation to yield

$$2.2 \qquad Y_t = \int_0^X MP_x = \int_0^X a\text{-}bX_t = aX_t - \frac{b}{2}X_t^2$$

The constant of integration in 2.2 vanishes if we assume that there is no production of corn if there is no labor. Average product of labor is given by

$$2.3 \qquad AP_X = \frac{Y_t}{X_t} = a - \frac{b}{2}X_t$$

Depending on how much land is in production at any particular time, the rental value of land is given by the difference between the average and marginal product of labor,

$$2.4 \qquad R_t = [\frac{Y_t}{X_t} - \frac{dY_t}{dX_t}]X_t = a - \frac{b}{2}X_t - a - bX_2 = (\frac{b}{2}X_t)X_t = \frac{b}{2}X_t^2$$

Profits to capitalists are the residual after payments have been made to labor's wages and capital rent.

$$2.5 \qquad Y_t - R_t - W_t$$

Changes in capital accumulation in this economy are considered through the dynamics of the wage fund, because the size of the wage fund determines how many workers can be hired to produce corn, and how marginal the land used for cultivation will be. Assume a given wage rate, w, then the total wage fund is

$$2.6 \qquad W_t = wX_t$$

Accumulation in the wage fund are entirely derived from the reinvestments of profits that accrue to capitalists. What will the wage fund be in time t+1 given economic activity in period t? Wages in any future period will equal to profits in the previous period plus wages in the previous period. If profits are positive the wage fund will increase, more labor will be hired, and more land cultivated. If profits are negative, the wage fund will decrease, less labor will be hired, and less land will be cultivated. Thus the dynamics of the wage fund can be expressed simply by

$$2.7 \qquad W_{t+1}-W_t = P_t$$

Using these relationships, we can derive the motion of the economy in terms of employment in the following way.

$$P_t = aX_t-\frac{b}{2}X_t^2-\frac{b}{2}X_t^2-wX_t = aX_t-bX_t^2-wX_t$$

$$P_t = W_{t+1}-W_t = wX_{t+1}-wX_t$$

$$\frac{P_t}{w} = X_{t+1}-X_t$$

$$X_{t+1} = \frac{a}{w}X_t-\frac{b}{w}X_t^2$$

Equation 2.8 has an equilibrium at $X_{t+1} = 0$, and at $X_{t+1} = X_t$ which yields

$$2.9 \qquad X_t = \frac{a-w}{b}X_t^*$$

From equation 2.1 we see that at the equilibrium level X^* the marginal product of labor is equal to the real given wage rate, and that profits go to zero. These are exactly the conditions for a long run competitive equilibrium, all of the rent in 2.5 has gone to capitalists in the form of profits, and 2.7 is zero, the wage bill for each period is unchanged.

Now we have succeeded in deriving the basic dynamics and equilibrium conditions of the simple Ricardian economy, but we need one further step to reduce the system to a well defined dynamic for which there is some economic interpretation. Bhaduri and Harris have derived such a system, one that is equivalent in every way to the quadratic maps Weiss discussed in detail. The analysis relies on the fact that a positive marginal product of labor in 2.1 requires that a > bX for all economically meaningful levels of employment. Using this fact we define a new variable, x, such that, $0 \le x_t = \frac{b}{a}X_t \le 1$, which allows us to rewrite 2.8 as

$$2.10 \qquad x_{t+1} = Ax_t(1-x_t)\,,\ A = \frac{a}{w} \qquad 0 \le x_t \le 1$$

This equation is the result of manipulating a Simple Ricardian system with standard neoclassical assumptions and reducing its dynamics to a function defined on [0, 1] that maps itself into [0, 1]. An equilibrium condition for 2.10 is clearly $\frac{A-1}{A}$ which corresponds to the Ricardian steady state.

To investigate additional economic interpretations of the model it is useful to point out that for the economy to be viable the coefficient, a, in equation 2.1 must be greater that the wage rate. If it were not the level of employment would go to zero where it would stay. Thus, it will always be the case that the parameter, $A = \frac{a}{w} > 1$, if the economy is viable. Because A is the tuning parameter in the quadratic equation, it is useful to rewrite it in different forms to consider how economic relationships between technology and wages affect the dynamics of accumulation. One way to rewrite A is in the form

$$\frac{a}{w} = \frac{a-w+w}{w} = 1+e, \quad e = \frac{a-w}{w}$$

Bhaduri and Harris define e as the exploitation parameter because it is the measure of exploitation measured in corn given the assumption of linear marginal productivity of labor. While there is no further explanation of exploitation, a closer look at e provides some insight. The technical coefficient, a, in equation 2.1, and 2.2 is the dominant coefficient in the technology. The size of a relative to w tells us how much capitalists are exploiting labor. If the value of a is large relative to w then the technology is highly productive relative to the level of wages. As the value of the e approaches and exceeds 3 accumulations would become erratic, and the trajectory would eventually become a cantor set. Thus, if capitalists exploit labor with the thought of rapid and ever growing accumulations the result will be erratic and often negative accumulation that appears to run counter to optimal behavior. The message from this simple economy seems to be that if capitalists do not provide for a reasonable distribution of corn to their labor, their lives will become chaos.

Another interpretation of A comes from expressing it in terms of the AP and MP of labor at the Ricardian equilibrium level. Recall that $AP^* = \frac{(a+w)}{2}$, and $MP^* = w$. Thus, $A = \frac{a}{w}$ can be expressed as

$$2.12 \qquad \left(\frac{a+w}{2}\right)\frac{2}{w}-1 = \frac{2-\eta^*}{\eta^*}$$

where η^* is the output elasticity of labor at the steady state $\eta = \eta^*$. If A must be greater than one as a viability condition, then η must be less than one. When values of η^* , the economy will always converge to a steady state equilibrium, but when values go beyond this range, approach, and recede from then we observe increasingly complicated behavior culminating in chaos. Bhaduri and Harris point out that a value of η^* is a very strong condition that would require the presence of fixed factors in the technology. Such a condition would be "...highly comparable with the classical analysis of income distribution and growth in which limitations of nonproducible natural resources like land are assigned a significant role" (Bhaduri and Harris, p. 899).

The exposition of the classical Ricardian economy by Bhaduri and Harris is important for our study of chaos because it shows that there is a correspondence between simple economic models and mathematical formulations that can exhibit chaotic dynamical behavior. While the model is of the simplist form, it suggests that more complicated models could have ranges of parameters that are economically meaningful and also generate very complicated dynamics. This model also shows, in the strongest terms, that the classical assumption of convergence, or simple oscillation, should be viewed with skepticism.

Conclusions

Two examples of chaotic dynamics in economic systems have been presented in this paper. These examples have been simplified so that they are compatable with the most easily understood mathematical exposition of chaos, the quadratic difference equation. I can not stress enough, however, the generality of these results, and the importance they should be accorded in future analysis of dynamical systems. As Mike Weiss has pointed out in his paper, also presented in this volume, the implications of chaos theory extend beyond these simple models, and include models of consumer behavior, decision analysis and forecasting. I also suspect applications exist in capital replacement theory and asset management problems. Much additional research on applications of chaos theory, and development of empirical methodologies must be done. Until these techniques are developed, applied researchers will find applications difficult. However, knowledge of the basic mechanisms of chaotic dynamics will broaden our range of understanding and deepen our appreciation of possible behaviors in dynamical systems.

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