Investment Decisions in Vintage Models

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This paper is circulated for discussion purposes only and its contents should be considered preliminary.

I. Introduction

The putty-clay vintage model of production as pioneered by Salter {14} and Johanson {8} is increasingly well established as the basis from which to approach both theoretical growth models of the real sector of an economy and empirical problems of investment, productivity and technical change. In both contexts the model is attractive on two counts. First, it avoids arbitrary assumptions about an instantaneous elasticity of substitution between labour and capital by restricting the choice of factor proportions to the current choice of technique and the rate of retirement through either obsolescence or decay of previously selected technologies. Secondly, the putty-clay vintage model obviates the need to measure capital as a stock, which is a blessing for empirical work if not de rigour for theory.

This last point was demonstrated in my paper {11} and subsequently used by Solow {15} as the basis of a simulation study of labour's share of value added and by Solow, Tobin, von Weizsacker and Yaari in their exploration of growth theory assuming a clay-clay technology {16}. It has featured in my own empirical work both in an attempt to measure capital's contribution to output growth {2} and in exercises which called for a surrogate for the capital stock {3} and {4} . All this work, so far as it relates to the point under discussion, depends on demonstrating that under appropriate conditions within a putty-clay vintage model, the rates of growth of aggregate output, y, and employment ℓ are related as

$$y = \alpha \ell + r i \qquad (1)$$

where α is labour's share of value added, i is the proportion of value added currently invested, and r is the immediate rate of profit on new investment. It follows that if r can be determined, then the vintage model gives a neat and concise statement of the contribution of inputs to output growth.

Since the reciprocal of r is the pay-off period in the absence of growth in wages and discounting, it is tempting to assume r is forced by competitive pressures to a lower bound set by investors for behavoural reasons. This approach was adopted in {3} and {4} while a variant of it is used in the well-known Kaldor-Mirrlees model {9}.

More general explanations of the determination of the immediate rate of profit, r, have to involve both the spectrum of alternative techniques from which a choice is made at any moment, and the criterion on which that choice is made. For the most part discussion has been restricted to the latter question of the choice criterion, and the present paper is no expection. However, an earlier paper {12} is concerned with the interrelationship between the two.

It is apparent from the above that the choice of technique, and with it the determination of r, plays an important part in vintage production models. Some early results, subsequently developed in {13} are contained in {4} . These exercises ignore problems of taxation which are taken up by Harcourt {6} and which are central to a recent empirical study by King {10}. The earlier exercises were also limited in the range of investment criteria considered which has been approached somewhat differently and extended by Harcourt in a second paper {7} .

The major weakness, however, of these exercises has been their lack of rigour as evinced by Bliss {1} in his development of the one sector putty clay vintage growth model.

The analysis by Bliss serves to emphasise the importance of choice of technique in vitage models and exposes problems of multiple maxima that had not previously been suspected. One purpose of the present paper is to elaborate these problems. In so far as Bliss's paper considers only present value maximation given perfect foresight and constant returns to scale, a second objective is to extend the analysis. Commenting on the formulations of Bliss and Kaldor/Mirrlees, Hahn {5} writes: 'The "pay-off" notion (of Kaldor/Mirrlees) is certainly attractive since many investigators have reported this kind of behaviour. But there are difficulties. One must remind oneself that one is dealing with steady state. Should an economy be in steady state (so that it has been in approximate steady state for "most of its history"), one would imagine uncertainty considerations to be rather unimportant - after all, everything is essentially being repeated over and over again. .... it is not clear that if profits are to be made by being a little less cautious, no one will be found to make them. On the other hand, it would be hard to maintain that the pay-off criterion is less "realistic" than the thorough-going maximisation assumption of Bliss". While these comments are well taken in the steady-state context, and are perhaps strengthened here by a demonstration that the Kaldor/Mirrlees criterion does not necessarily have all the properties they assume, they can be read, and perhaps should be, as an invitation to explore something other than 'thorough-going (profit) maximisation' in the quest for realism outside the steady state. In any event, the present paper does look at other criteria, specifically

the internal rate of return and the pay-off period, without assuming either a steady state or constant returns to scale. Indeed, the first part of the analysis covers a wide class of choice criteria and is directed towards establishing a vintage model analogue of the Law of Diminishing Returns. This is the subject of the next section following this introduction.

The third section of the present paper is concerned with choice under certainty given specific choice criteria. This is followed by a section exploring the effects of various modifications on the choice of technique, these modifications relating to taxation, the investment lag, uncertainty and risk aversion. The final section, section V. is concerned with sufficient conditions for a positive investment decision at a moment in time and is restricted to choice based on net present value. Discussion of the restraints on growth of both the necessary and sufficient conditions concludes the paper.

II. Vintage Law of Diminishing Returns

II.1 Notation and definitions

Throughout this paper we assume a putty-clay vintage technology and are concerned unless otherwise stated only with the choice of technique at a particular moment in time, which is to be designated time zero. The technology is characterised by

X : an index of physical output

N : an index of labour input

I : an index of cost of the technique chosen

F : the ex ante production function given by

$$X = F(N, I) \tag{2}$$

Once a technology is chosen, X and N remain constant throughout its working life. Thus no allowance is made for physical deterioration of plant reducing output over time in this part of the analysis, although this assumption is dropped when we come to consider uncertainty. Meanwhile, the need for increasing labour input over time for maintenance work, or for a decreasing input through learning by doing can be accommodated to the extent that N is interpreted as an input of labour services rather than of man-hours.

The units in which X, N and I are measured are deflated expenditures. X is the revenue generated at time zero divided by unit product price at time zero; N is the wage bill at time zero divided by the same deflator; and I is the cost of the technology, again deflated by unit product price at time zero. In consequence, we have

$x = X/N$: average product of labour = reciprocal of labour's share of value added by the new technology at time zero.

$r = \frac{X - N}{I}$: the immediate rate of profit = initial profit flow ÷ the cost of the technology.

The properties of the production function (2) need to be carefully defined. We specify that

(i) : $X > 0$ for all $N, I > 0$

(ii) : F is twice differentiable with respect to both N and I and that in particular

$$f_N = \frac{\partial X}{\partial N} > 0 \quad \text{for all } N, I > 0$$

$$f_{NN} = \frac{\partial^2 X}{\partial N^2} < 0 \quad \text{for all } I > 0, \; N > N_o > 0$$

so that the marginal product of labour is always positive and diminishes for values of N such that $N > N_o$ where N_o is non-negative.

(iii): There exists a value of N, denoted $\overline{N}$, such that

(a) $\overline{N} \geq N_o$

(b) For each $I > 0$, $\frac{\partial x}{\partial N} \gtrless 0$ depending on $N \lessgtr \overline{N}$

(c) $\overline{x}$ (= x at $N = \overline{N}$) > 1

Hence for fixed I the average product of labour , x, increases with N over the range $(0, N_o)$, continues increases over the range $(N_o, \overline{N})$, and subsequently declines. At $\overline{N}$ the average product is a maximum and greater than 1. This ensures that for at least some technique the initial profit flow is positive.[1]

Within these restrictions on the production function F it is important to distinguish two main classes:

Classes of F : If there exists a finite value of N, denoted $\tilde{N}$, such that

$f_N \leq 1$ for all $N \geq \tilde{N}$

then F is of Class I. Otherwise F is of Class II

Note that if $\tilde{N}$ exists, then $\tilde{N} > \bar{N}$. For at $\bar{N}$, x is a maximum so that $x = f_N$. But also $x > 1$, so $f_N > 1$. For $N > \bar{N}$ we have $f_{NN} < 0$. Hence, if $f_{\tilde{N}} = 1 < f_{\bar{N}}$, then $\tilde{N} > \bar{N}$

The general shape of F and the distinction between Classes I and II of F are illustrated in Figures 1 and 2. Figure 1 shows three alternative schedules of X versus N for fixed I which satisfy the above restrictions. At point A on all schedules the average product, x, is a maximum and greater than one. Again, on all schedules the marginal product f_N diminishes subsequently, but only on Class I schedules does it diminish sufficiently to become less than one at some point B. It can be noted that B is the point at which r is a maximum. This can be seen graphically from the fact that rI is the vertical intercept of the line with slope one which passes through any given point. More formally, we have by definition

$$r = \frac{X - N}{I}$$

so that

$$dr = \frac{dX - dN}{I}$$

since I is fixed. Now X and N must lie on F, so from (2)

$$dX = f_N \, dN$$

Consequently

$$dr = \frac{(f_N - 1)}{I} \, dN$$

and

$$d^2r = f_{NN} \, (d_N)^2 + (f_N - 1) \, d^2N$$

The conditions for $r = \max r$ are therefore

$$f_N = 1 \; ; \; f_{NN} < 0$$

and these conditions are satisfied only at B. Moreover, it is apparent that if $f_N > 1$, then r increases with N. Consequently for Class II functions, r increases without bound as N increases.

In Figure 2 the points A and B correspond to the same points A and B in Figure 1. In interpreting the diagram two details should be noted. First, the Class II schedule terminates at the point C, while the Class I schedules continue unless they reach the 45° line. Secondly, the curvature of the schedules is not necessarily as regular as shown.

Since
$$\frac{df_N}{dx} = \frac{Nf_{NN}}{f_N - x}$$

the restrictions on F are sufficient to ensure that f_N is an increasing function of x for all $N \geq \overline{N}$. The curvature of the function, however, is not determined and will depend on f_{NNN} which we have not assumed to exist.

To conclude this section we specify some further useful notation and a definition.

$W(t)$: the real wage at time t. Note that by definition of units, $W(o) = 1$.

$\pi(t)$: profits at time t deflated by unit product price at time zero. Hence

$$\pi(t) = X - NW(t) = N(x - W(t)) \tag{3}$$

and
$$r = \frac{\pi(o)}{I}$$

T : the date at which a technique is scrapped. If a technique is scrapped when it ceases to earn profits,

then

$$\pi(T) = 0 \quad \text{and}$$

$$x = W(T)$$

With I fixed, a technique can be characterised by the two variables N and T. However, not all techniques are equally of interest. In particular some techniques are inferior to others under quite general conditions as defined by the following concept of dominance.

Definition of Dominance: A technique (N',T') is dominated by some other, technique (N,T) if both

(1) $$\int_0^T \pi(t)dt \geqslant \int_0^{T'} \pi'(t)dt \qquad (4)$$

and

(2) $$\int_0^{\tau} \pi(t)\,dt \geqslant \int_0^{\tau} \pi'(t)\,dt \qquad \text{for all } \tau\ ; \quad 0 < \tau < \min(T,T') \qquad (5)$$

and one or other of (4) and (5) is a strict inequality for some τ. Rule (1) states that technique (N,T) earns more profits over its life than technique (N',T'). Rule (2) states that technique (N,T) must have earned more profits than technique (N',T') at any moment at which both are in operation. Thus given two techniques of which neither dominates, the one which earns less profit in total must have earned more profit up to some particular moment in time. Given this definition we can define:

$\mathcal{N}$: the set of potentially optimal techniques, defined as the set of techniques which is not dominated.

An obvious restriction on the set $\mathcal{N}$ is obtained by noting that for given N rule (1) for dominance requires that T should be such as to maximise $\int_0^T \pi(t)\,dt$. Thus we require T to be such that

$$\pi(T) = 0 \text{ and } \dot{\pi}(T) < 0$$

From (3) it is apparent that these conditions are met if

$$x = W(T) \text{ and } -N\dot{W}(T) < 0 \qquad (6)$$

i.e. if plants are scrapped when they cease to earn profits (quasi-rents) given that real wages are increasing monotonically with time.

Since the conditions (6) determine T for each N in the set $\mathcal{N}$ it is sufficient from now on to consider N as a complete description of a technique without explicit reference to the associated value of T, which will be assumed to be determined according to (6).

II.2 Weak law of diminishing returns

The weak law of diminishing returns for vintage models constitutes a restriction on the set $\mathcal{N}$ of techniques which are not dominated. It is developed here first in the context of assuming a choice of technique, given I fixed, and requires the following Lemma.

Lemma 1 : for any two techniques

{1} $\pi(t)$ and $\pi'(t)$ can cross at most once as t varies.

{2} If $\pi > \pi'$ then $N' \not\in N$

{3} If π and π' cross once as t varies either

$r' > r$ and $T' < T$ or

$r' < r$ and $T' > T$

To establish this Lemma, note that if $\pi(t)$ and $\pi'(t)$ cross at $t = \tau$

then $\pi(\tau) = \pi'(t) \leftrightarrow \dfrac{X - X'}{N - N'} = W(\tau)$

For $t < \tau$, we have $\dfrac{X - X'}{N - N'} > W(t)$ given $\dot{W} > 0$ for all t.

Therefore $x - NW(t) = \pi(t) \gtrless X' - N'W(t) = \pi'(t)$

depending on $N \gtrless N'$

Similarly, for $t > \tau$

$\pi(t) \lessgtr \Pi'(t)$

depending on $N \gtrless N'$

Hence π and π' do not cross a second time, which establishes {1}.

If $\pi > \pi'$ for all t, then it follows that $T > T'$.

Consequently

$$\int_0^T \pi \, dt > \int_0^{T'} \pi \, dt > \int_0^{T'} \pi' \, dt$$

and $$\int_0^{\tau} \pi \, dt > \int_0^{\tau} \pi \, dt \text{ for } \tau \leq T'$$

Hence by definition N' is dominated, i.e. $N' \notin \mathcal{N}$, as stated in {2} of Lemma 1.

If π and π' cross at least once, then they can only cross once from {1}. Either π has the greater intercept (= rI) and shorter life, or π' has. This is what {3} states.

The weak law of diminishing returns which follows from the Lemma can be stated as follows:

The weak law of diminishing returns : If $N \notin \mathcal{N}$, then for all production functions F, $N \geq \bar{N}$; and for F of Class I, $N \leq \tilde{N}$.

In terms of Figures 1 and 2, this states that the set of techniques which is not dominated lies to the right of point A in Figure 1 and is to the left of B if B exists. Accordingly there is an important restriction on the choice of technique given a criterion of choice consistent with our definition of dominance.

A proof of the law is as follows

If $N \in \mathcal{N}$ then π and $\bar{\pi}$ cross once from {2} of Lemma 1. By definition x is a maximum at $\bar{N}$, so $x \leq \bar{x}$. From (6) this implies $T \leq \bar{T}$, therefore $r \geq \bar{r}$ by {3} of Lemma 1 . But

$r \lesseqgtr \bar{r}$ implies $X - N \geqslant \bar{X} - \bar{N}$ which can be written $N(x - 1) \geqslant \bar{N}\ (\bar{x} - 1)$

Since we have $x \leqslant \bar{x}$, the inequality $r \geqslant \bar{r}$ implies $N \geqslant \bar{N}$ which establishes the first part of the Law.

To establish the second part the argument follows similar lines. If $\tilde{N}$ exists, then at $\tilde{N}$, r is a maximum $= \tilde{r}$. Hence from the Lemma, if both N and $\tilde{N}$ are elements of $\mathcal{N}$, then $T \geqslant \tilde{T}$. This implies $x \geqslant \tilde{x}$ which can be shown to be equivalent to $\tilde{N}r \geqslant N\tilde{r}$. Thus both N and $\tilde{N}$ can be elements of $\mathcal{N}$ only if $\tilde{N} \geqslant N$.

This completes the proof of the Law. It can be noted that at no point does the argument depend on the proposition that real wages are growing at a constant rate: it is sufficient for the theorem that real wages be a monotonic increasing function of time.

The Law as established applies to the case in which I is fixed and diminishing returns are with respect to N for $N > \bar{N}$. This is the case with which we shall be concerned throughout this paper. However it is apparent that there must be an analogue for the Law in the case where N is fixed and X increases with respect to I. This analogue can be sketched with the aid of Figures 3 and 4.

In Figure 3 it is assumed that there exists a range of techniques for which the initial rate of profit, r, is positive as in Figure 1. Obviously without such a range no technique will

be chosen since no technique could be profitable. The initial rate of profit is shown as being a maximum at a point B, while the marginal product $f_I = \partial X/\partial I$ is assumed to vanish at a point A. The points A and B in Figure 3 have a correspondence with the points similarly labelled in Figure 1 as will be explained later. Meanwhile their existence in Figure 3 constitutes an assumption about the shape of the ex ante production function F analogous to the earlier assumptions for the case where I is fixed.

For fixed N the weak law of diminishing returns takes the form that the optimal technique must lie in the interval AB in Figure 3. An outline proof is as follows.

Consider any two techniques (X_1,I_1) and (X_2,I_2) such that $r_1 = r_2$, e.g. points C_1 and C_2 in Figure 3. Then technique C_1 will yield a flow of profits per unit of investment over time which can be represented as in Figure 4 assuming that real wages are a monotonic increasing function of time. Technique C_2 will earn a similar flow of profits per unit investment which cannot be less than the flow from C_1 at any moment and accordingly involves a positive flow over a longer period. Purchase of the technique C_2 can be regarded as purchase of the profit flow from C_1 for a sum I_1 plus purchase of the difference in profit flow between C_2 and C_1 for a sum $I_2 - I_1$. This incremental flow per unit of investment has the form shown in Figure 3. Accordingly, if it is attractive to buy the technique C_1, it is yet more attractive to buy the increment $C_2 - C_1$. Thus C_2 is preferable to C_1, and more generally the optimal technique will lie to the right of point B in Figure 3. Given that it is trivial to show

that the optimal technique will lie to the left of point A in Figure 3, it emerges that the interval (B,A) defines a range within which I must lie.

If the production function F is homogeneous of degree one it is easily shown that points A and B in Figures 1, 2 and 3 correspond exactly. Thus we have as restrictions on the optimal choice of technique in this special case:

$$\text{w.r.t. } B: \quad x \geq f_N \quad \leftrightarrow \quad f_I \geq 0$$

and

$$\text{w.r.t. } A: \quad f_N \leq 1 \quad \leftrightarrow \quad f_I \leq r \tag{7}$$

In both his analyses of choice of technique, {6} and {7} Harcourt chooses to regard the level of output , X, rather than of either of the inputs, N and I as fixed, and thus considers choice of a point along an isoquant. Our weak law of diminishing returns has an analogue in this case also, and again the demonstration of this is only sketched out below, this time with the aid of Figure 5.

In Figure 5 the techniques C_1 and C_2 have the same value of r and their properties generally correspond to the properties of C_1 and C_2 in Figure 4. Accordingly the same arguments suffice to show that C_2 is preferable to C_1 and more generally that the optimal technique must lie to the right of B along the isoquant. Since it must also lie to the left of A our weak law is established for fixed X and, as before, it can be easily shown that under constant returns to scale points A and B correspond in each of the Figures 1, 2, 3 and 5.

Any of the Figures 1, 3 and 5 suffice to show that the vintage model formulation has a more restricted range of potentially optimal techniques than is the case for the conventional neo-classical model. Thus as Figure 5 makes clear, the putty-clay formulation restricts the aggregate factor ratio not only to the extent that this ratio is determined for the most part by investment decisions at earlier dates but also to the extent that it is restricted for new investments by our weak law.

Figure 6 illustrates the restrictions of the weak law on choice of technique for the case where I is taken as fixed in the particular instance where the ex ante production function is of the constant elasticity form

$$X = A\,N^{b}\,I^{c} \qquad \text{for } 0 < b < 1, \quad 0 < c \tag{8}$$

Given the restrictions on b and c specified in (8), the restrictions of the law, as given by (7) amount to the requirement $bx \geqslant 1$ as shown in the Figure.

II.3 Strong Law of diminishing returns

The weak law of diminishing returns can be strengthened by noting that there may exist a technique N, denoted $\hat{N}$, such that the total profit it earns over its lifetime exceeds that of all other techniques. Thus

$$\int_0^T \pi(t)\, dt \qquad \text{subject to} \quad x = W(T)$$

is a maximum with respect to N for $N = \hat{N}$.

In general if $\hat{N}$ exists its location will depend on the time-path of real wages W(t). However, as was noted by Bliss, (who attributes the point to Mirrlees), in the special case where real wages grow at a constant rate, say w, then $\hat{N}$ is independent of w. Accordingly, the significance of $\hat{N}$ is developed here only in relation to this special case which will be assumed throughout the remainder of this paper. Given this we have

$$W(t) = e^{wt}$$

so that

$$\pi(t) = X - Ne^{wt}$$

and

$$x = e^{wT}$$

whence

$$T = \frac{1}{w} \log x$$

it being assumed that $w \geqslant 0$.

Total profits over the lifetime of a plant can now be expressed as

$$\int_0^T \pi(t)\, dt = \int_0^{\frac{1}{w}\log x} X - Ne^{wt}\, dt \tag{9}$$

so that

$$d\int_0^T \pi(t)\, dt = \frac{dX}{w}\log x - \frac{dN}{w}(x-1) \tag{10}$$

Since X, N and I are restrained by the function F, and since we are considering alternative techniques with I fixed, then dX and dN are simply related as

$$dX = f_N\, dN \tag{11}$$

Substituting (11) into (10) gives as a necessary condition for (9) to be a maximum the condition

$$f_N = \frac{x-1}{\log x} \tag{12}$$

This condition must clearly be satisfied by $\hat{N}$ if it exists. However (12) may be satisfied by other values of N as we shall see.

The properties of the function $(x - 1)/\log x$ are most easily approached by noting that if y is a random variable with density function $g(y)$ given by

$$g(y) \propto y^{-1} \quad \text{for} \quad \min(1,x) < y < \max(1,x)$$

then

$$E(y) = \frac{x-1}{\log x} \tag{13}$$

where E(y) denotes the expected value of y. From this several properties of the function follow. First the function maps x into itself at x = 0 and x =1. Secondly, it is a monotonic increasing function of x. Thirdly we must have

$$\min(1,x) \leq \frac{x-1}{\log x} \leq \max(1,x)$$

Indeed, this property can be strengthened by noting that since the density function of y is monotonic decreasing it must have a mean less than that of a rectangularly distributed variate over the same interval, i.e. less than $\frac{1}{2}(x+1)$. Hence

$$\frac{x-1}{\log x} < \frac{1}{2}(x+1) \tag{14}$$

Finally, it can be shown that the function is concave from below. Differentiating twice gives

$$\frac{d^2}{dx^2}\frac{x-1}{\log x} = \frac{\frac{x-1}{\log x} - \frac{1}{2}(x+1)}{2x^2(\log x)^2} \tag{15}$$

From (14) the numerator of (15) is positive. Since the denominator is also positive, the function (13) must be concave. It is graphed in Figure 7.

From equation (12) we require $f_N = \frac{x-1}{\log x}$

for $N = \hat{N}$, if $\hat{N}$ is to be the technique which maximises undiscounted profits. A graph of f_N, taken from Figure 2, is also shown on Figure 7. Comparing the graphs of the two sides of (12) it is clear that this

equation may have multiple solutions. More specifically, if F is of Class I, then (12) must have at least one solution: if F is of Class II then equation (12) may or may not have at least one solution, and therefore $\hat{N}$ may not exist.

To distinguish local maxima from local minima of (9) we need the second order condition for a maximum, which is

$$(f_N - x)^2 + f_{NN} X \log x < 0 \tag{16}$$

and is equivalent to the requirement that f_N must intersect $(x - 1)/\log x$ from below. It should be noted that it is not enough for (16) to be satisfied that f_{NN} should be negative; it must be strongly negative.

In the constant returns to scale case the restriction (16) can be expressed as a restriction on the substitution elasticity, σ, of the ex ante production function, viz:

$$\sigma < \log x \frac{Nf_N}{If_I} \tag{17}$$

which is not a condition that all conventional production functions can satisfy.

To explore further the question of whether $\hat{N}$ exists for a production function, we can first note from Figure 7 that if (12) has at least one solution, then it has at least one local maximum solution, and hence it has a global maximum solution, i.e. $\hat{N}$ exists. It is also apparent that $\hat{N}$ will exist if F is of Class I. If F is of Class II and $\hat{N}$ exists, then F will be said to belong to Class IIa. Otherwise F will belong to Class

IIb.

A possible schedule for f_N vs. x for a Class IIb function is shown in Figure 7. From this it is apparent that for a function to be of Class IIb the marginal product f_N must fail to diminish sufficiently rapidly for f_N to intersect with $(x - 1)/\log x$. In this sense then, the marginal product of labour does not diminish fast enough. In such cases the total profit (9) will continue to increase as N increases without bound to infinity. Accordingly, in such cases N may be said to exist in the special sense that $\hat{N} = \infty$

When $\hat{N}$ exists it cannot be dominated by rule (1) for dominance. More generally, for any technique N not to be dominated by some local maximum at which (12) holds, the technique N must have earned more profits by some date than the technique corresponding to the local maximum. Since we have seen that for two techniques within $\mathcal{N}$ their respective functions $\pi(t)$ must cross once, this means that a technique not dominated by a local maximum must have a higher value of r, a shorter life, lower x and consequently larger N. It follows that all points not dominated by the global maximum $\hat{N}$ must involve $N \geq \hat{N}$. Further, if there is a local maximum for some $N \geq \hat{N}$, then not all $N > \hat{N}$ belong to $\mathcal{N}$. This can be seen with the aid of Figure 8.

From the above arguments it is apparent that no point to the left of $\hat{N}$ can lie within $\mathcal{N}$. The Figure 8 shows two intervals of N corresponding to $\mathcal{N}$, the interval to the right of the local maximum N_2, (which must be subject to $N < \tilde{N}$ where appropriate), and an interval $(\hat{N}, N_1)$ where N_1 is constructed to the left of N_2 as shown. Points in

(N_1, N_2) do not belong in $\mathcal{N}$ since they are dominated by N_2: they yield a lower total profit over their lives and, since N_1 is smaller than N_2 but greater than $\bar{N}$, they have a lower r and so cannot have earned greater profits at any moment in time.

The above arguments can be gathered together in a theorem we may call the strong law of diminishing returns:

The strong law of diminishing returns: the set $\mathcal{N}$ of techniques which is not dominated is restricted as follows:

If F is of Class I; $\mathcal{N} \in \{ N;\ \hat{N} \leq N \leq \tilde{N} \}$

If F is of Class IIa; $\mathcal{N} \in \{ N;\ \hat{N} \leq N \}$

If F is of Class IIb; $\mathcal{N} \equiv \infty$

This theorem is stronger than the weak law by virtue of the replacement of $\bar{N}$ by $\hat{N}$ where the latter exists. Conditions for this existence have been derived only for the case where real wages grow at a constant rate w, in which event $\hat{N}$ is independent of w. Analogous results for $\hat{N}$ can be obtained by assuming that N rather than I is the fixed factor input.

The restrictions imposed by the strong law of diminishing returns are illustrated for the constant elasticity production function (8) in Figure 6. A lower bound on the elasticity b is provided by (12). Since this equation can be satisfied by one and only one value of x for each b, the constant elasticity function is not of Class IIb; it is in fact of Class I. Moreover, the unique solution for x must correspond to a local maximum and be a global maximum. Accordingly for points (b,x) above the

lower bound in Figure 6 the second order condition (16) is satisfied. As can be seen from the diagram, the range to which x is restricted by the strong law can be quite narrow for the larger feasible values of b.

III Various Objectives

III.1 Introduction

In this section we explore the effects of choosing a technique according to specific criteria namely net present value, the internal rate of return, and the pay-off period. Throughout the choice is assumed to be made with respect to a fixed amount of investment, I, and real wages are expected to grow at a constant rate w. With respect to net present value the analysis is closely akin to that of Bliss {1} although the ex ante production function is not assumed to be homogeneous here. The results for pay-off period criteria have a bearing on the formulation of Kaldor and Mirlees {9}.

To develop the analysis some further notation is called for. Thus we define:

λ : the real rate of discount

z : the present value of the profit stream $\pi(t)$ over the period $(0,T)$

Hence

$$z = \int_0^T (X-Ne^{wt})e^{-\lambda t}dt$$

$$= x\left(\frac{1-e^{-\lambda T}}{\lambda}\right) - N\left(\frac{1-e^{-(\lambda-w)T}}{\lambda-w}\right)$$

$D_T(\alpha)$: the present value of a constant unit flow of income over the period $(0,T)$ discounted at rate α.

Thus

$$D_T(\alpha) = \frac{1-e^{-\alpha T}}{\alpha}$$

and $$z = xD_T(\lambda) - ND_T(\lambda-w)$$

If $D_T(\alpha)$ is evaluated subject to $x = e^{wT}$ then it is denoted $D^*(\alpha)$ and we have

$$D^*(\alpha) = \frac{1}{\alpha}(1-x^{-\alpha/w}) \tag{18}$$

Similarly, if z is subject to the restraint $x = e^{wT}$ we have z^* given by

$$z^* = \int_0^{\frac{1}{w}\log x} (X-Ne^{wt})e^{-\lambda t}dt$$

$$= XD^*(\lambda) - ND^*(\lambda-w) \tag{19}$$

A useful result to note at this point which can be obtained by manipulation of (19) is that

$$wXD^*(\lambda) = (w-\lambda)\ Z^* + rI \tag{20}$$

where r is the immediate rate of profit as before.

The net present value of a technique is given by

$$Z-I = XD(\lambda) - ND(\lambda-w) - I$$

from which it follows, given

$$dD(\alpha) = e^{-\alpha T}dT - d\alpha \int_0^T te^{-\alpha t}\ dt$$

that

$$\begin{aligned} d(Z-I) &= D(\lambda)\ dX - D(\lambda-w)\ dN - dI \\ &- d\lambda \int_0^T (X-Ne^{wt})te^{-\lambda t}dt \\ &- Ndw \int_0^T te^{(w-\lambda)t}dt \\ &+ (X-Ne^{wT})\ e^{-\lambda T}dT \end{aligned} \tag{21}$$

If this relationship is now constrained to satisfy $x = e^{wT}$ we get

$$\begin{aligned} d(Z^*-I) &= D^*(\lambda)\ dX - D^*(\lambda-w)\ dN - dI \\ &- d\lambda \int_0^{\frac{1}{w}\log x} (X-Ne^{wt})te^{-\lambda t}dt \\ &- Ndw \int_0^{\frac{1}{w}\log x} te^{(w-\lambda)t}dt \end{aligned} \tag{22}$$

From this result an important property of the function Z^* can be demonstrated namely that with respect to λ and w, Z^* is a homogeneous function of degree minus one. This can be shown as follows. Since

$$\int_0^T te^{-\alpha t}dt = \frac{1}{\alpha}\,(D(\alpha) - Te^{-\alpha T})$$

the result (22) reduces for given X,N and I to

$$dZ^* = -\frac{d\lambda}{\lambda}\,X\,(D^*(\lambda)-Te^{-\lambda T})$$

$$+\ \frac{dw}{w}\,N\,(D^*(\lambda-w) - Te^{(w-\lambda)T})$$

where

$$x = e^{wT} = X/N$$

Thus $$dZ^* = -\frac{d\lambda}{\lambda}\ \{X\,D^*(\lambda) - ND^*(\lambda-w)\}$$

$$+\ (\frac{dw}{w} - \frac{d\lambda}{\lambda})\ N(D^*(\lambda-w) - Te^{(w-\lambda)T})$$

Under the restriction that $dw/w = d\lambda/\lambda = \beta$, say, this reduces in view of (19) to simply

$$dZ^* = -\beta\ Z^*$$

i.e. the effect of scaling up both λ and w by a factor β is to scale down Z^* by the same factor. This homogeneity property of Z^* is important for subsequent analysis.

III.2 Net present value

Properties of the technique, to be denoted N^*, which has the highest net present value, Z-I, which are to be demonstrated here are gathered together in the following theorem:

Theorem 1: If N^* is the technique for which the net present value $Z-I$ is a maximum for fixed I, λ, $w > 0$, then

(i) $N^* \in \mathcal{N}$ and if N^* is finite

(ii) $$Z^* = \frac{r^*I}{\lambda + w\frac{f_N}{x-f_N}}$$

and

(iii) $$\frac{\Delta N^*}{\Delta \lambda} > 0 > \frac{\Delta N^*}{\Delta w}$$

The first part of the Theorem states that N^* is not dominated. This can be demonstrated with the aid of Figure 9 which shows the flow of profits over time for two techniques N^* and N'. In the diagram the initial rate of profit r' is shown as being greater than r^*. If this was not the case, then N' could not dominate N^* since the profit flow from the latter would exceed that from N' over some initial period. Similarly the diagram shows technique N^* earning positive profits over a longer period than technique N'. If this was not so, and given $r' > r^*$, then we would have $\pi'(t) > \pi^*(t)$ for all t. In this event the profit flow $\pi'(t)$ would have a higher present value than that for technique N^* for any discount rate λ. Accordingly N^* could not be the technique which maximises net present value.

The remaining possibility for N' to dominate N^* given that the latter maximises net present value is the one illustrated in Figure 9 with $r' > r^*$ and $T^* > T'$. In this case there will be a moment in time, τ, at which $\pi' = \pi^*$ as shown in the figure. If N' is to dominate N^* then the area α in Figure 9 must exceed the area β according to Rule 2 for dominance. Following the introduction of a discount factor, λ, the area α will reduce to $A > \alpha e^{-\lambda t}$ while the area β will reduce to $B < \beta e^{-\lambda t}$. Thus if $\alpha > \beta$, then $A > B$ i.e. if N' dominates N^* it must have a higher net present value. This contradicts the definition of N^*. Hence (i) of the Theorem is established.

To establish (ii) of the Theorem we can assume that N^* is finite and return later to examine the conditions under which this is valid.

Since maximisation of $Z-I$ has to involve $x = e^{wT}$ as can be seen from (21), we have from (21) that for fixed I, λ and w

$$d(Z^*-I) = D^*(\lambda)dX - D^*(\lambda-w)dN$$

Under the same condition of fixed I the fact that X,N and I must satisfy the ex ante production constraint implies

$$dX = f_N \, dN$$

Accordingly the first order condition for Z-I to be a maximum with respect to N is

$$f_N = \frac{D^*(\lambda-w)}{D^*(\lambda)} \tag{23}$$

The three results (19), (20) and (23) can be combined to eliminate $D^*(\lambda)$ and $D^*(\lambda-w)$ and yield

$$Z^* = \frac{r^*I}{\lambda + w \frac{f_N}{x-f_N}} \tag{24}$$

Thus part (ii) of our theorem is established.

The result (24) implies that if w is zero then $Z^* = r^*I/\lambda$. Since rI is the initial profit flow from a technique and this flow remains constant for $w = 0$ it is trivial that for $w = 0$, the present value Z is given by

$$Z = rI/\lambda$$

With λ and I fixed this means that maximising Z is equivalent to choosing the technique which maximises r i.e. the technique which minimises the crude (undiscounted) pay-off period. This is the technique $\tilde{N}$ so we have

$$N^* = \tilde{N}$$

for $w = 0$ if $\tilde{N}$ exists.

Of course the result (24) is more general. If profits over time are eroded by increasing real wages, then assuming that Z^* is maximised, the result shows how this erosion is equivalent to an increase in the discount rate by an amount $wf_N/(x-f_N)$. This amount is simply proportional to w, the constant of proportionality being $b/(1-b)$ in the simple case (8) of a constant

elasticity ex ante production function.

The result (24) also yields a simple expression for the condition that net present value be positive:

$$Z^* > I \text{ if and only if } r^* > \lambda + w \frac{f_N}{x-f_N} \tag{25}$$

which can be written for the constant returns to scale case as

$$r > \lambda + w \frac{Nf_N}{If_I}$$

To establish (iii) of our Theorem we assume again that N^* is finite. The first part of the (iii) states that the effect of an increase in the discount rate is to move the optimal technique to a higher value of N. This can be proved with the aid of Figure 10.

If N_1^* is optimal at discount rate λ_1, and N_2^* is optimal at $\lambda_2 > \lambda_1$ then either $r_1^* > r_2^*$ and $T_1^* < T_2^*$, or $r_1^* < r_2^*$ and $T_1^* > T_2^*$. In any other event one of N_1^* and N_2^* would dominate and so by (i) of Theorem 1 could not be optimal. If $r_1^* < r_2^*$, then $N_1 < N_2$. Consequently $\Delta N > 0$ and this is consistent with (iii) of Theorem 1. Accordingly, to establish (iii) we need to show that the remaining possibility $r_1^* > r_2^*$ with $T_1^* < T_2^*$ cannot maintain.

Figure 10 shows the flow of discounted profits over time for the two techniques N_1^* and N_2^* when the rate of discount is λ_1 and $r_1^* > r_2^*$ with $T_1^* < T_2^*$. The argument follows the same lines as that associated with Figure 9. The schedules for the two discounted profit streams must cross at a single moment in time, τ : the area α must be greater than β if N_1 is to be optimal at discount rate λ_1. When the discount rate is raised by an amount $\Delta\lambda$ to λ_2, then α is reduced to $A > \alpha e^{-\tau\Delta\lambda}$ while β is reduced to $B < \beta e^{-\tau\Delta\lambda}$. Thus $\alpha = \beta$ implies $A > B$, i.e. the profit flow from N_1^* has a higher present value than that of N_2^* at the higher discount rate λ_2. This contradicts the definition of N_2^* so the case is impossible. Accordingly we establish that

$$\frac{\Delta N^*}{\Delta\lambda} > 0 \tag{26}$$

i.e. an increase in the discount rate results in the choice of a more labour intensive production technique, that is in a technique which has a shorter (crude) pay-off period. It may also be the case that an increase in λ results in the impossibility of satisfying the condition (25) that net present value be positive.

The remaining part of (iii) of Theorem 1 states that an increase in the rate of growth of real wages, w, results in a less labour intensive technique being chosen. This can be shown given the result (26) for an increase in the discount rate and recalling that Z^* is homogeneous of degree minus one in λ and w.

It follows from the homogeneity property of Z^* that if both λ and w are increased so as to keep their ratio constant, then Z^* is scaled down by a factor that is independent of the technique N. Thus equal proportionate increases in λ and w change the absolute level of Z^* but do not change the designation N^* of the technique which is optimal.

This result indicates that the effect of an increase in w can be considered as being that of first an equal proportionate increase in λ and w followed by a proportionate decline in λ back to its original level. The first of these two steps results in no change in N^*: the second step lowers N^* as indicated by (26). Hence the proof of (iii) of Theorem 1 is complete.

The homogeneity property of Z^* implies that the location of N^* depends only on the ratio of λ and w and not their absolute value. This ratio is denoted by

$$\mu = \lambda/w \tag{27}$$

and plays an important part in subsequent analysis. Thus the effect on N^* of $w = 0$ is equivalent to that of $\lambda = \infty$ and implies $N^* = \tilde{N}$ when the latter exists as has been shown. Similarly the effect of $w = \infty$ is the same as having $\lambda = 0$ i.e. no discounting of profits. Present value maximisation is simply equivalent to maximising total profits over the lifetime of a technique in this extreme and leads to $N^* = \hat{N}$. More generally, as μ increases from 0 to ∞ the optimal

technique N* moves monotonically across the range $\hat{N}$ to $\tilde{N}$ of techniques which are potentially optimal as specified by the strong law of diminishing returns. However, as we shall see, this movement is not necessarily continuous.

It has been shown that the first order condition for Z* to be a maximum is given by (23);

$$f_N = \frac{D^*(\lambda-w)}{D^*(\lambda)}$$

From (18) and (27) the right-hand side of this expression can be expressed as a function, ϕ say, of x and μ

$$\phi = \frac{D^*(\lambda-w)}{D^*(\lambda)}$$

$$= \frac{\mu}{\mu-1} \quad \frac{x^{\mu}-x}{x^{\mu}-1} \tag{28}$$

We now need the properties of ϕ stated in the following Lemma

Lemma 2: The function ϕ defined by equation (28) has the properties

(i) $\min(1,x) < \phi < \dfrac{x-1}{\log x}$

and

(ii) $\dfrac{\partial\phi}{\partial x} > 0 > \dfrac{\partial\phi}{\partial\mu}$

To establish the lemma consider a random variable, y, with density function g(y) such that

$$g(y) \propto y^{\mu-2} \text{ for } \min(1,x) < y < \max(1,x) \tag{29}$$

It can be shown that ϕ is related to E(y) as

$$\phi = \frac{x}{E(y)}$$

Since E(y) is bounded by the range of y we have

$$\min(1,x) < \phi < \max(1,x)$$

Moreover, an increase in μ decreases the skewness of the distribution of y and hence can be shown to increase E(y). Thus E(y) is a minimum, and ϕ therefore a maximum with respect to non-negative μ for $\mu = 0$. This can be shown to yield from (29)

$$\phi < \frac{x}{E(y/\mu>0)} = \frac{x-1}{\log x}$$

To complete the Lemma we need to show that

$$\frac{\partial\phi}{\partial x} = \frac{\mu}{E(y)} \frac{E(y)-1}{x^{\mu}-1}$$

is positive. This follows at once from the fact that for $\mu > 0$, $E(y)-1$ and $x^{\mu}-1$ have the same sign.

From the Lemma we have that ϕ is bounded from above by $(x-1)/\log x$; that it is an increasing function of x; and that ϕ approaches its upper bound as μ declines toward zero. Looked at in terms of Figure 11, this means that even if

$$f_N = \frac{x-1}{\log x}$$

has a solution, the first-order condition

$$f_N = \phi \tag{30}$$

may not have a solution. And if the first order condition has a solution, then it may have more than one solution. Specifically, if F is of Class I then (30) has at least one solution : if
F is of Class II then (30) may have no solution or one or more solutions:
if F is of Class IIb then (30) has no solution.

Situations where the first-order condition has no solution are ones where f_N fails to diminish sufficiently rapidly as N increases. As N increases the net present value Z^*-I increases without limit. Thus $N^* = \infty$ in these cases, and the optimal technique exists only in this limiting sense.

Where one or more solutions to the first-order conditions exist they must include at least one local maximum, and hence there is a global maximum. To see this we need the second-order condition to distinguish local minima and maxima. This can be obtained by differentiating (21) and imposing $x = e^{wT}$ and (23) and takes the form

$$e^{-\lambda T}(f_N-x)^2 + f_{NN}\, wXD^*(\lambda) < 0 \tag{31}$$

where T is subject to $x = e^{wT}$. The condition (31) can be shown to be equivalent to the requirement that f_N must cut ϕ from below, i.e. f_N must be

more steeply sloped than ϕ at their point of intersection. From Figure 11 it is apparent that if (30) has one or more solutions it must have some local maximum solution(s) and hence a global maximum.

Under constant returns to scale the second order condition (31) can be expressed in terms of the substitution elasticity of the ex ante production function, i.e. σ, as

$$\sigma < \frac{x^{\mu}-1}{\mu} \quad \frac{Nf_N}{If_I} \tag{32}$$

It is in essentially this form that the condition appears in Bliss {1}. He comments that the condition depends on factor prices, i.e. on μ, and the units in which x is measured.

This dependence of the conditions (31) and (32) on factor prices clearly introduces complications. However, it can be shown that for a wide class of cases these complications will not arise in practice. Specifically we can establish the following theorem:

Theorem 2: If N is a solution of the first order condition (23) and $N \in \mathcal{N}$, then a sufficient condition for N to be the unique local maximum solution of (23) for $\mu > 0$ is

$$\frac{\partial}{\partial N} \left(\frac{f_N}{x}\right) \leqslant 0$$

which reduces to

$$\sigma \leqslant 1$$

if the ex ante production function is homogeneous of degree one.

Obvious illustrations of the theorem are that if the ex ante production function is of the constant elasticity or CES (weak substitutes) form, then any solution of (23) which satisfies the Law of diminishing returns will be a unique local maximum solution and therefore a global maximum.

To prove the theorem we write the condition (31) in the form

$$f_{NN} < \frac{f_N}{X} (f_N - x) \psi \tag{33}$$

where

$$\psi = \frac{\mu}{x^{\mu}-1} \cdot \frac{x-f_N}{f_N}$$

and procede to show that $\psi < 1$. To see this note that if N satisfies the first order condition (23) then

$$f_N = \frac{x}{E(y)}$$

where y is the random variable defined by (29). Thus we have

$$\psi = \frac{\mu}{x^{\mu}-1} \; E(y-1) \tag{34}$$

and if $N \in \mathcal{N}$

$$E(y) = c \int_1^x y^{\mu-1} \, dy = c \, \frac{x^{\mu}-1}{\mu} \tag{35}$$

and

$$1 = c \int_1^x y^{\mu-2} \, dy = c \, \frac{x^{\mu-1}-1}{\mu-1} \tag{36}$$

The results (34) and (35) yield

$$\psi = c\left(1-\frac{1}{E(y)}\right)$$

so that

$$\psi < 1 \text{ if and only if } \frac{1}{c} + \frac{1}{E(y)} > 1$$

Now from (36) it is apparent that $^1/c$ increases with μ. Therefore it is a minimum for non-negative μ for $\mu = 0$. Thus

$$\frac{1}{c} + \frac{1}{E(y)} \geqslant \left.\frac{1}{c}\right|_{\mu=0} + \frac{1}{E(y)}$$

$$= 1 + \frac{1}{E(y)} - \frac{1}{x}$$

$$> 1$$

since

$$E(y) < x$$

Accordingly we have $\psi < 1$ so that from (33) the second order condition for a maximum must be satisfied if

$$f_{NN} \leqslant \frac{f_N}{X} (f_N - x) \tag{37}$$

given

$$x > f_N > 1$$

It is now straightforward to show that the condition (37) can be expressed in the form stated in the theorem and reduces to the requirement $\sigma \leqslant 1$ given constant returns. Expressed verbaly it states that for $N \in \mathcal{N}$, the elasticity of output with respect to the variable input (labour) must not increase with respect to the variable input.

Bliss {1} illustrates the possibility of multiple solutions for the first order conditions by considering a production function which for a particular value of μ, satisfies the first order condition for all values of x within some range. In such case we have

$$f_N = \phi \tag{38}$$

for all x in the relevant range. However we can easily show that such cases are of little interest. For it follows from the formulation

$$\frac{f_N}{x} = \frac{1}{E(y)}$$

that the condition of Theorem 2 will be satisfied only if

$$\frac{\partial E(y)}{\partial N} > 0$$

But since

$$\frac{\partial E(y)}{\partial N} = \frac{\partial E(y)}{\partial x} \cdot \frac{f_N - x}{N}$$

and

$$\frac{\partial E(y)}{\partial x} > 0$$

it follows that the Theorem 2 condition can be satisfied only if $f_N > x$, which violates our Law of Diminishing Returns. Accordingly intervals of x over which (38) holds exactly have to be minima of net present value.

Notwithstanding that Bliss's example will not suffice to illustrate the point, multiple maxima of net present value may exist. Accordingly unless we are prepared to restrict the ex ante production function as required by Theorem 2, the possibility exists of more than one competitive

equilibrium in a growth mode. However, Theorem 1 will always hold good.

III.3 Internal rate of return

As an alternative to maximisation of net present value we now turn to consider the choice of technique based on the internal rate of return as defined by λ in the equation

$$I = \int_0^T (X - Ne^{wt})e^{-\lambda t}dt \qquad (39)$$

where

$$T = \frac{1}{w} \log x$$

As before, the choice will be assumed to be made in terms of a choice of N given fixed I.

The analysis of this choice problem has much in common with maximisation of net present value. In particular imposing the restraint $Z = I$ on the general formulation of section III.1 above easily yields first and second order conditions for the maximisation of λ which are of the same form as those for maximisation of $Z-I$. Indeed if λ^o is the maximum attainable internal rate of return and is realised by some technique N^o, then this same technique is chosen if the object is to maximise net present value given a fixed discount rate $\lambda = \lambda^o$.

From this equivalence and Theorem 1, the following Theorem for maximisation of the internal rate of return is easily established:

Theorem 3: If N^o is the technique for which the internal rate of return λ, defined by equation (39) has its maximum value, λ^o, for fixed I, $w > 0$, then

(i) $N^o \in \mathcal{N}$ and if N^o is finite

(ii) $\lambda^o = r^o - w \dfrac{f_N}{x - f_N}$

(iii) $N^o \gtrless N^* \Leftrightarrow \lambda^o \gtrless \lambda^* \Leftrightarrow Z^* \gtrless I$

where N^* is the technique which maximises $Z-I$ for fixed discount rate λ^* and for the same fixed w and I

and

(iv) $\frac{\Delta N^o}{\Delta w} < 0$

The proof that $N^o \in \mathcal{N}$ follows the same lines as the proof of (i) of Theorem 1 or from the equivalence of net present value and internal rate of return maximisation already discussed. Accordingly there is no need to develop the proof explicitly.

Part (ii) of Theorem 3 also follows by analogy with its counterpart of Theorem 1. Its interpretation is that the difference between the initial and average rates of profit is proportional to the rate of growth of real wages when the average rate, λ, is a maximum. The coefficient of proportionality depends as previously on the elasticity of output with respect to the variable input (labour) as evaluated at the optimal point on the ex ante production function.

This result provides an alternative theory about the determination of r to assuming simply that r is maximised. As such it can be substituted in equation (1) to provide an explanation of the contribution of capital to growth in terms of λ and w. It has been used in this way in {4}.

Part (iii) of Theorem 3 also follows easily from the analogy with present value maximisation given the result (iii) of Theorem 1; $\frac{\Delta N^*}{\Delta \lambda} > 0$. Accordingly in the interest of economy the proof is only outlined here.

Starting from maximisation of net present value Z^*-I for $\lambda = \lambda^*$ we have technique N^*. If $Z^* > I$, then λ can be raised; N will increase; and Z^* will decline. If λ is raised until $Z^* = I$, then $\lambda = \lambda^o > \lambda^*$ and $N - N^o > N^*$ Parallel arguments cover the case where initially $Z^* < I$. Further, if $\lambda^o > \lambda^*$, then maximum net present value at discount rate λ^o is less than at discount rate λ^*. Since the former is zero the latter must be positive, i.e. $\lambda^o > \lambda^* \Rightarrow Z^* > I$. Moreover, N^* at discount rate λ^o exceeds N^* at discount rate λ^*. But the former is N^o, so $\lambda^o > \lambda^* \Rightarrow N^o > N^*$. Similar arguments cover the case $\lambda^o < \lambda^*$. Finally, if $N^* > N^o$ then from (iii) of Theorem 1,

$\lambda^* > \lambda^o$, and conversely for $N^* < N^o$.

Part (iv) of Theorem 3 can be proved employing both of the results (iii) of Theorem 1. Starting with maximisation of net present value at discount rate λ^o we have technique N^o and $Z = I$. If w is now decreased, N will increase and so will Z. The latter can be brought back to its original value by raising λ and thus further increasing N. At the value of λ which achieves this $Z = I$ so λ is the maximum internal rate of return at the lower value of w.

For future reference the most important part of Theorem 3 is (iii). From this we have that if present value maximisation results in a technique for which $Z^* > I$ then a switch to internal rate of return maximisation will yeild a rate above the exogenous discount rate and result in a move towards a more labour intensive technique.

To conclude this stage of the discussion of internal rate of return maximisation it must be recognised that there may be more than one local maximum. There cannot be a continuum or plateau of local maxima for the same reason as was discussed with respect to present value maximisation: a range of N over which the first order conditions are identically satisfied has to be a continuum of minima. However there may be two local maxima of equal magnitude and hence no unique global maximum technique. Even in this case, however, there is of course a global maximum for λ^o and the choice of either local maximum for N^o will not result in a violation of Theorem 3.

III.4 Pay-off period

While consideration of present value maximisation and maximisation of the internal rate of return is prompted by theory, it is largely empirical observation of behaviour which has drawn attention to the choice of technique based on a pay-off period criterion. However, the implications of such criteria are being fed back into theory, for example by King {10} and notably by Kaldor and Mirrless {9}.

For analytic purposes there is some ambiguity about the appropriate definition of the pay-off period. It is defined here by Θ in the expression

$$I = \int_0^{\Theta} (X-Ne^{wt})\, e^{-\lambda t} dt \qquad (40)$$

so that Θ is the length of time it takes for a technique to earn enough profits for their present value to equal the cost of the technique. Most authors, however, and notably Kaldor/Mirrlees and Harcourt, define the pay-off period as above but subject to the restriction $\lambda = 0$. Clearly this is a special case which our analysis will endeavour to embrace, not least because it is the case that corresponds most closely to empirical observation of behaviour.

There is also ambiguity about the choice criterion to be formulated relative to the pay-off period. It will be assumed here that the object of the exercise is to find thatN, to be denoted N^{Θ}, for which Θ is a minimum. However others regard the pay-off period as an argument of a lexicographic criterion of choice. Thus Harcourt assumes that there is an exogeneously given horizon, H and that firms choose a technique first by restricting their choice to the set of techniques for which $\Theta < H$ (given $\lambda = 0$) and then by choosing the technique from within that set for which the accumulated profits over the period (0,H) is maximum. Kaldor/Mirrlees formulate the problem similarly: choice is first restricted to a set of techniques in the same way as specified by Harcourt. However they now argue that competitive pressures will reduce this set to a single element so that no further criterion is needed. In effect, therefore, they have an equation of form (40) above. Both these formulations are commented on later.

The first point to note about the pay-off period as we have defined it (including the special case $\lambda = 0$) is that it may not exist. It is quite possible for there to be no technique which earns profits which will cover the capital cost. In this event the technique with highest net present value

has a net present value less than the cost I. Thus we have as a necessary and sufficient condition for N^{Θ} to exist that $Z^* > I$. This is the first part of our theorem on the properties of N^{Θ} as set out below

Theorem 4: If N^{Θ} is the technique for which the pay-off period Θ defined by equation (40) is minimum for fixed I, λ and w, then

(i) N^{Θ} exists if and only if $Z^* > I$

(ii) if N^{Θ} exists $\Theta \not\sim$ and if N^{Θ} is finite

(iii) $N^{\Theta} > N^{o}$

and

(iv) $\frac{\Delta N}{\Delta \lambda} < 0 > \frac{\Delta N}{\Delta w}$

It can be noted that the existence condition of Theorem 4 implies for the case $\lambda = 0$ that the technique which earns maximum undiscounted profits over its lifetime, i.e. technique $\hat{N}$, must earn profits in excess of I. Further, when a technique has a well-defined pay-off period, Θ, then Θ must be less than or equal to the economic life of the plant, i.e.

$$\Theta \leq T = \frac{1}{w} \log x \qquad (41)$$

if Θ exists.

To prove that if N^{Θ} exists then N^{Θ} is not dominated and so satisfies our law of diminishing returns is very simple for the case $\lambda = 0$. The more generally case can be demonstrated with the aid of Figure 12. It is easily shown that if technique N_2 is to dominate N_1 while the latter has the smaller pay-off period, then N_2 must earn higher profits initially and have a shorter economic lifetime: all other cases are easily eliminated and Figure 12 is drawn accordingly.

In Figure 12 area $\alpha <$ area β since because $\Theta_2 > \Theta_1$ it follows that by time Θ_1 technique N_1 has earned more cummulative discounted profits. If now the discount rate is lowered to zero, $\alpha \rightarrow A < \alpha e^{\lambda\tau}$ and $\beta > B > be^{\lambda\tau}$. Hence

$B > A$, i.e. technique N_1 has earned more cummulative undiscounted profits by time Θ_1 than has technique N_2. Accordingly the latter cannot dominate.

The second part of (iv) of Theorem 4 can be proved similarly. Let technique N_1 be the technique with smallest pay-off period given w_1 and let N_2 be smaller than N_1. If now w is decreased the reduction in pay-off period for N_1 can be shown to be greater than that for N_2. Accordingly N_1 is still to be preferred at the lower level of w. Thus lowering w cannot lower N^{Θ}.

In the limit where $w = 0$, equation (40) can be reduced to

$$1 = \int_0^{\Theta} re^{-\lambda t}\, dt$$

Thus r and Θ are inversely related and minimum Θ corresponds to maximum r, i.e. to the choice of technique $N^{\Theta} = \tilde{N}$. Accordingly minimising the pay-off period is equivalent to maximising the immediate rate of profit if no allowance is made for growth in real wages.

The remaining parts of Theorem 4 yet to be discussed are illustrated in Figure 13. The figure incorporates the restraint (41) and embodies the assumption that Θ is a convex function of N. However such an assumption is hard to justify and convexity is not assumed in the arguments of the text.

The Figure 13 shows Θ as a function of N for various values of λ. The fact that the contours corresponding to particular values of λ do not cross can be shown from the total differential of (40) which yields for fixed I and w

$$(X-Ne^{w\Theta})e^{-\lambda\Theta}d\Theta = d\lambda \int_0^{\Theta} t(X-Ne^{wt})e^{-\lambda t}dt + dN \int_0^{\Theta} (e^{wt}-f_N)e^{-\lambda t}dt \qquad (42)$$

Since $\Theta < T$ we have $X > Ne^{w\Theta}$ and therefore for fixed N as λ increases so does Θ.

Moreover, as $\Theta \to T$, $d\Theta/d\lambda \to \infty$ as shown in the diagram.

Two further points illustrated in the diagram are easily established. The first is the obvious one that as λ increases so the range of N for which Θ exists is diminished. The second point is that as λ increases without bound, the last technique to survive in the sense that for it Θ exists, is the technique N^o. This is the only technique (assuming N^o is unique) which has a defined pay-off period for $\lambda = \lambda_o$ since for all other techniques $Z < I$. At λ_o the pay-off period for technique N^o is of course $T = \frac{1}{w} \log x^o$, i.e. the economic life of the technique.

The diagram shows that N^* increases with λ as required by Theorem 1 and that for each λ, $N^* < N^\Theta$. To see that this must be so, consider a technique $N' < N^*$. For N' initial profits must be lower than for N^* and so must ultimate cummulative discounted profits. Indeed the whole schedule of cummulative discounted profits for N' must lie below that for N^*. Accordingly N' cannot have a smaller pay-off period.

It now only remains to show that N^Θ decreases with λ to complete the proof of the theorem. For if this can be established, then it must follow from the fact that $N^\Theta = N^o$ for $\lambda = \lambda^o$ that $N^\Theta > N^o$ when the former exists i.e. when $\lambda < \lambda^o$. Accordingly (iii) of Theorem 4 follows directly from the first part of (iv).

To establish the first part of (iv) of Theorem 4 note that from (42) it follows that Θ is stationary with respect to N for fixed λ if

$$f_N = \int_0^\Theta e^{(w-\lambda)t} dt \Big/ \int_0^\Theta e^{-\lambda t} dt$$

$$= \psi, \text{ say} \qquad (43)$$

From (40) and (43) it follows that ψ can be written as

$$\psi = \frac{1}{N} \left(X - I \Big/ \int_0^\Theta e^{-\lambda t} dt\right) \qquad (44)$$

so that for a given technique

$$d\psi = \{e^{-\lambda\Theta}d\Theta - d\lambda\int_0^\Theta te^{-\lambda t}dt\} \frac{(X-N\psi)^2}{NI} \tag{45}$$

Accordingly since Θ increases with w, ψ has its minimum value with respect to non-negative w when w = 0. In this event it is obvious from (43) that ψ is one in this limit and stationary Θ corresponds to $f_N = 1$, i.e. to the technique $\tilde{N}$ and hence to maximisation of the immediate rate of profit as previously discussed.

An upper limit to ψ can be found by considering its behaviour relative to changes in λ for a given technique, N. Specifically, substituting for $d\Theta$ in (45) according to (42) with fixed N yields

$$d\psi = \frac{(X-N\psi)^2 d}{I(X-Ne^{w\Theta})}\int_0^\Theta te^{-\lambda t}(e^{w\Theta}-e^{wt})dt \tag{46}$$

Since the integral is necessarily positive, ψ must increase for a given technique with respect to λ. Thus it is a minimum with respect to λ for w > 0 when $\lambda = 0$, in which event (44) yields the first part of

$$\psi = \frac{e^{w\Theta}-1}{w\Theta} < \frac{x-1}{\log x}$$

while the second follows from the fact that (41) must be satisfied. It follows then that for $\lambda = 0$, ψ is analogous to the function ϕ defined by (28) which plays such a large role in the analysis of present value maximisation. Indeed the analogy goes further and extends to the case where $\lambda > 0$. For it is straightforward to show that as λ increases so Θ increases towards $T = \frac{1}{w}\log x$. Accordingly

$$\psi \to \phi$$

as λ increases, and the approach is monotonic and from below in view of (46). Thus we obtain that for all w and $\lambda \geq 0$

$$1 \leq \psi \leq \phi \leq \frac{x-1}{\log x}$$

Consider now the following. Let N_1 be the technique which has minimum Θ for discount rate λ_1 and let λ_2 be some lower discount rate. Then from (43) and (46)

$$f_{N_1} = \psi(\lambda_1, N_1) > \psi(\lambda_2, N_1)$$

Suppose further that N_2 is a technique which has the same pay-off period as N_1 at discount rate λ_2. Then from (44)

$$\frac{X_1 - X_2}{N_1 - N_2} = \psi(\lambda_2, N_2) < f_{N_1}$$

Thus since we assume $f_{NN} < 0$ it must be the case that $N_2 > N_1$. This result would be sufficient to ensure that N^{Θ} decreases as λ increases if stationary values of Θ with respect to N are minima ie if Θ has only one local minimum. To see that this is in fact the case, note that from (42)

$$\frac{d\Theta}{dN} = \frac{\int_0^{\Theta} (e^{wt} - f_N) e^{-\lambda t} dt}{(X - Ne^{w\Theta}) e^{-\lambda\Theta}}$$

so that

$$\frac{d}{dN}\left(\frac{d\Theta}{dN}\right) = \frac{-f_{NN}\int_0^{\Theta} e^{-\lambda t} dt}{(X - Ne^{w\Theta}) e^{-\lambda\Theta}} - 2\frac{d\Theta}{dN}\,\frac{f_N - e^{w\Theta}}{X - Ne^{w\Theta}}$$

$$+ \left(\frac{d\Theta}{dN}\right)^2 \left(\lambda + \frac{wNe^{w\Theta}}{X - Ne^{w\Theta}}\right)$$

Accordingly any point at which $d\Theta/dN = 0$ must be a minimum point given $f_{NN} < 0 < x - e^{w\Theta}$. Together with our earlier result this implies that N^{Θ} increases as λ decreases.

The above completes the proof of Theorem 4, two points from which are worth emphasising. The first is that decreases in both w and λ effect N^{Θ} in the same direction. For w a decrease not only results in more techniques for which

Θ is defined but also in less discouragement for high values of N. Accordingly N^{Θ} increases. A decline in λ might be thought to encourage techniques with longer lives. However what happens for $t > \Theta$ is immaterial and declining λ results in Θ being defined for techniques with higher immediate profit rates. This last turns out to be the dominating factor so N^{Θ} increases.

The second point to note from Theorem 4 is the simple ordering of the techniques N^*, N^o and N^{Θ} which is now established and illustrated in Figure 13. As we move from undiscounted profit maximisation ($\hat{N}$), through discounted profit maximisation (N^*), to internal rate of return maximisation (N^o), and on to pay-off period minimisation (N^{Θ}) first with a discount rate and then without , finally finishing up with maximisation of the initial rate of profit ($\tilde{N}$), so throughout we are concerned with techniques which are not dominated and which are progressively more labour intensive.

In their formulation of growth with a vintage technology Kaldor and Mirrlees assume that the choice of technique depends first on whether the undiscounted profits within some horizon, H, exceed the cost of the technology. In our terms this means $\Theta < H$ for $\lambda = 0$. They then assume that if this condition is satisfied, the present value of a technique will exceed its cost. Again translating to our terms., the assumption is that $Z^* > I$ and therefore Θ exists for $\lambda > 0$. Figure 13 shows clearly that this assumption is not merited. For example, if $\lambda = \lambda^o$ and $H < H^o$, then $Z^* < I$ for all techniques which satisfy $\Theta < H$ for $\lambda = 0$. More generally, the assumption of Kaldor and Mirrlees will be valid for given H only if λ is restricted to some range of non-negative values for given w.

Harcourt's formulation {7} is similar to that of Kaldor/Mirrlees as we have seen. Harcourt assumes that the objective is to choose that technique for which the undiscounted profits within the horizon H are a maximum, subject to their being greater than the cost of the technique. This can be shown to involve a choice of technique which moves from A to B in Figure 13 as H increases, and hence clarifies the uncertainty expressed by Harcourt about the relative labour intensity of a technique chosen by this criterion.

IV. Various Modifications

IV.1 Introduction

The analysis of the previous section assumes a number of simplifications both in the investment decision itself and in the context in which it is made. Such is the way in which theory is most readily developed. However, at some point it is desirable to relax the simplifications to obtain insights into the direction in which important magnitudes will be modified in a move towards greater realism. In this spirit various modifications are considered in this section of the paper.

The first general area to be explored is taxation. In so far as investment decisions are motivated by a quest for private gain, only those parts of the benefits which accrue to the investor are relevant to him. This may or may not introduce distortions relative to the 'no tax' case. In point of fact we find below that while profits tax and investment grants may not distort, the same cannot be said for value added taxation.

A second general area for consideration is uncertainty. This has been considered here only for a simple case where a plant might 'die' for reasons other than technological change. The formulation has 'radioactive decay' as a special case. Uncertainty attributable to future factor prices is not considered although it must clearly come into account at some point as growth theory develops. Indeed since capitalism is hard to justify in the absence of uninsurable risks it is hard to see the relevance of growth models in general, and investment decisions in particular, to the real world situation without some

introduction of a stochastic element. Accordingly we have attempted some generalisations in this direction, first in terms of uncertainty and then of risk aversion not least with the familiar Keynesian formulation of the importance of a risk premium in mind. The results lend support to Keynes' formulation of the problem in terms of the internal rate of return rather than net present value and suggest that pay-off period criteria provide a surrogate for the modifications which uncertainty otherwise introduces.

IV.2 Taxation

With respect to each of our three criteria of investment choice the effects of profits tax, investment grants and a gestation period on new capacity can be considered simultaneously. Introducing these three considerations modifies the expression for net present value to

$$Z' - \alpha I e^{\lambda \ell} = \beta Z - \alpha I e^{\lambda \ell} \qquad (47)$$

where $100\,(1 - \alpha) = \%$ investment grant

$100\,(1 - \beta) = \%$ rate of profits tax

$\ell =$ gestation period

it being assumed that the costs of investment, I, have to be met at a time ℓ units before quasi-rents begin to be earned.

A number of points follow immediately from (47). First it is apparent that for fixed λ, the effects of a gestation lag and of investment grants are comparable and work in opposite directions: increasing ℓ is equivalent to increasing α. Secondly, these factors have no

influence on the choice of technique with respect to fixed I. Indeed since (47) is simply a linear transformation of Z - I, maximisation of (47) involves the same choice of technique as is obtained in the absence of profits tax, investment grants, and a gestation period. All that is affected is the size of net present value at the optimum, and hence whether or not it is positive. These simple results for the optimum technique are expressed in the following theorem:

Theorem 5: (1) The effect of introducing profits tax on the optimum technique is

(a) to lower N when the internal rate of return is maximised;

(b) no change in N when net present value is maximised;

(c) to raise N when the pay-off period is minimised;

(2) The effect on a gestation period on the optimum technique is qualitatively the same and opposite in sign to that of introducing investment grants, and is equivalent to that of introducing a profits tax for all three investment criteria.

(3) The effect of introducing value added tax on the optimum technique is to lower N according to all three investment criteria.

The results in parts (1) and (2) of Theorem 5 relating to internal rate of return maximisation are almost as easily derived as those for net present value maximisation. From (47) it is now required to choose N so as to maximise λ in the expression

$$\beta Z - \alpha I e^{\lambda \ell} \tag{48}$$

where Z is, of course, a function of λ

From (48) it follows at once that increasing α is equivalent to decreasing β. The effects of introducing a profits tax are isolated by having $1 - \alpha = 0 = \ell$. Because of the tax, for any λ we have $\beta Z < Z$. Accordingly since $Z = I$ for $\lambda = \lambda^o$, equation (48) can be satisfied only at some $\lambda < \lambda^o$. The highest value of λ at which it can be satisfied therefore leads through net present value maximisation to a value of N below N^o; and at this value λ is a maximum subject to (48).

Similarly, the effect of introducing a gestation period is to make the right hand side of (48) negative at $\lambda = \lambda^o$. Accordingly λ must be lower, leading to a lower optimal value of N.

If the criterion of choice is minimisation of the pay-off period, then before and after introduction of profits tax this period is given respectively by Θ and Θ' in

$$I = \int_0^{\Theta} (X - Ne^{wt})\, e^{-\lambda t}\, dt = \beta \int_0^{\Theta'} (X - Ne^{wt})\, e^{-\lambda t}\, dt$$

for any given technique. To ascertain the effects on optimal N of the tax it is useful to consider first the effects of changing w and λ simultaneously with the introducing of the tax. Thus supposing w and λ are increased to $w' = w/\beta$ and $\lambda' = \lambda/\beta$ at the same time as the tax is introduced. Then the new pay-off period is given by Θ'' where

$$I = \beta \int_0^{\Theta''} (X - Ne^{w't})\, e^{-\lambda t}\, dt$$

By changing the variable of integration now to τ ; $\tau\beta = t$, it is easily shown that $\beta\theta'' = \theta$ i.e. θ'' is simply proportional to θ. Accordingly the technique which gives minimum $\vee\theta$ also gives minimum θ'' and so that optimum does not change if w and λ change as specified. Since the result we require is with respect to fixed λ and w, the optimum will in fact shift corresponding the effects of lowering w' and λ' to their original levels. From Theorem 4 this implies that optimum N increases.

To complete the proof of parts (1) and (2) of Theorem 5, it can be seen at once from

$$\alpha I e^{\lambda \ell} = \beta \int_0^{\theta} (X - Ne^{wt})\, e^{-\lambda t}\, dt$$

that the effects of investment grants and gestation lags correspond as stated in the theorem to the introduction of a profits tax.

The effects of a value added tax are more complex and are introduced in so far as they relate to present values in Figure 14. The complications arise from the fact that unlike profits tax which influences only the sharing of quasi-rents between the investor and the tax authority, an effect of VAT is to reduce the economic life of a plant at fixed prices and hence the total sum of quasi-rents which it earns. Thus if the tax is levied at rate $1 - \alpha$ we have

$$\int_0^T (X - Ne^{wt})\, e^{-\lambda t}\, dt = \int_0^{T'} (\alpha X - Ne^{wt})\, e^{-\lambda t}\, dt + (1 - \alpha)\, X \int_0^T e^{-\lambda t}\, dt$$

$$+ \int_{T'}^{T} (X - Ne^{wt})\, e^{-\lambda t}\, dt \qquad (49)$$

where the left-hand side is the present value of profits before introduction of the tax and the right-hand side shows first profits remaining with the investor after imposition of the tax, then the tax revenue, and finally the lost value through reduction in economic life from T to T', i.e. from

$$w\,T = \log x \quad \text{to} \quad w\,T' = \log(\alpha x)$$

From this it is straightforward to show that if the tax authority wished to obtain the same present value of revenue from the given investment, the investor would prefer this revenue to be raised by a profits tax rather than by value added tax. For a given technique, under profits tax, a part of the total tax revenue is paid from profits earned between times T and T' and the remainder of these profits are retained by the investor. Under VAT not only are there no profits retained from this period but also all revenue for the tax authority has to come out of earnings prior to T'. Since this preference for profits tax maintains for any given technique, it will hold in general. In some cases it will imply that investments worthwhile under profits tax will no longer be worthwhile under VAT. This makes no difference if the tax authority is concerned to raise revenue in proportion to realised investment. However, if their concern is to raise an absolute amount of revenue then it implies that tax rates under VAT would need to raise more revenue from each realised investment than would a profits tax. Accordingly the preference among investors for profits tax would be strengthened.

If Z' is the present value of profits after imposition of the tax, then Z' is the first term on the right-hand side of (49).

There is a reasonably simple relationship between present value after tax, denoted Z', and its value before tax, i.e. the left-hand side of (49) which is Z. The relationship is

$$Z = \alpha^{\mu-1}\left[Z' + \int_{\frac{1}{w}\log\alpha}^{0} (\alpha X - Ne^{w\tau})\, e^{-\lambda\tau} dt\right] \tag{50}$$

and can most easily be derived by transforming the variable of integration in the expression for Z from t to $\tau = t + \frac{1}{w}\log\alpha$.

The integral appearing in (50) must be positive for $\alpha < 1 < \alpha x$, this second condition being necessary for $T' > 0$. The derivative of the integral with respect to N will be positive if the condition

$$f_N > \frac{\mu(1/\alpha)^{\mu} - (1/\alpha)}{\mu-1(1/\alpha)^{\mu} - 1} \tag{51}$$

is satisfied. But we know from (28) that the condition for Z to be stationary with respect to N is

$$f_N = \emptyset = \frac{\mu}{\mu-1}\,\frac{x^{\mu} - x}{x^{\mu} - 1}$$

and that from Lemma 2, $\partial\emptyset/\partial x > 0$

Thus if Z is stationary with respect to N, the condition (51) will be satisfied given $\alpha x > 1$. Accordingly the difference between Z and Z' is an increasing function of N throughout a range of values of N which includes all the stationary values of Z. Hence the maximum value of Z' in this range is at a lower value of N than the maximum value of Z, i.e.

imposition of a value added tax lowers the value of N corresponding to the optimal technique, as stated in Theorem 5.

The effects of VAT when the criterion of choice is maximisation of the internal rate of return can be derived from the following argument. We start with the technique corresponding to maximisation of λ given VAT and with the maximum discount rate, say λ^*. If we now switch to present value maximisation given VAT and $\lambda = \lambda^*$, we have $Z = I$ and no change in technique. Keeping this technique a switch to profits tax which yields the same revenue will yield $Z > I$ since profits tax is preferred. Keeing to the same rate of profits tax and to $\lambda = \lambda^*$ we now maximise present value Z. There are two effects. First, $Z - I > 0$ is increased. Secondly, the choice of technique moves to a higher N since while profits tax at any rate makes no difference to optimal N, VAT at any rate shifts N down relatively. Given that we had previously maximised Z with respect to VAT, the move to higher N on switching to profits tax is inevitable. This move will also have had the effect of increasing tax revenue since under profits tax the revenue in proportional to Z. Thus in relation to our original position, N and Z and tax revenue are all larger while $\lambda = \lambda^*$. Suppose now we increase the profits tax rate so as to make maximum $Z = I$. This does not change N but it does of course raise tax revenue. Our next move is to simultaneously raise λ and lower the profit tax rate, maintaining throughout $Z = I$. Since we are maximising Z the change in tax rate does not change N: but raising λ does effect N - it increases N. This simultaneous increasing of λ and lowering the tax rate so as to maintain $Z = I$ is continued until the tax revenue is back to the original level at which we started. The point reached is equivalent to maximising λ subject to profits tax at whatever rate is necessary to yield the same revenue as we started with given λ

maximisation under a VAT regime.

Since all changes in N in the above sequence of events are increases in N it follows that N is larger under profits tax than under VAT. Moreover, since N is smaller with profits tax than without from Theorem 5, it follows that N is reduced by imposition of VAT as stated in Theorem 5.

To complete Theorem 5 we need to explore the effects of VAT when the criterion of choice is minimisation of the pay-off period. To do this we note that Θ under VAT is given by

$$I = \int_0^{\Theta} (\alpha x - Ne^{wt}) e^{-\lambda t} dt$$

From this it is easily shown that for a given technique Θ decreases with α. Moreover, for that technique which minimises Θ with respect to N for given α, $d\Theta / d\alpha$ is decreasing with respect to N. Accordingly, since Θ has a unique minimum with respect to N, the effect of raising α is to raise N corresponding to this minimum. Thus raising VAT lowers the optimal N.

IV.3 Uncertainty

Uncertainty can be introduced into the investment decision model in a variety of ways and there can be little doubt that its implications will vary according to the means chosen. In the present case uncertainty is introduced in a very simple form: it is assumed that the physical life of plant, τ , (as opposed to its economic life, $T = \frac{1}{w} \log x$), is exogeneously determined according to some stochastic distribution which is independent of the technique chosen. Accordingly, the actual life of a plant will be the minimum of its economic life and physical life, and it follows at once that our introduction of uncertainty can only detract from the attractiveness of an investment.

If $F(\tau)$ is the distribution function of the random variable τ and $S(\tau)$ is its complement, then $S(\tau)$ is the probability that a plant is still earning quasi-rents at time τ if its economic life is not yet over. Accordingly $S(\tau)$ can be referred to as the survival curve for the plant and enters simply into the expression for expected present value of all quasi-rents as

$$E(Z) = \int_0^T \pi(t)\, S(t)\, e^{-\lambda t}\, dt$$

where $T = \frac{1}{w} \log x$

If $S(\tau) = 1$ for all τ in the interval $(0,T)$, the plant is sure to survive its economic lifetime and $E(Z) = Z$, i.e. there is no uncertainty which matters. This is, of course, a limiting case. More generally

$S(\tau) < 1$ so that $E(Z) < Z$, i.e. uncertainty reduces the expected value of Z.

Since $0 \leq S(\tau) \leq 1$, $S(\tau)$ can be transformed into a function $e^{-\delta\tau}$ such that

$$\int_0^T \pi(t)\, S(t)\, e^{-\lambda t}\, dt = \int_0^T \pi(t)\, e^{-(\lambda + \delta)t}\, dt \qquad (52)$$

where $\delta > 0$. Hence δ is a surrogate discount factor which has the same effect on present value as does the survival function, $S(\tau)$. In general δ will depend on N for fixed λ and w since from (52)

$$d\delta \int_0^T \pi\, t\, e^{-(\lambda+\delta)t}\, dt = dN \int_0^T \frac{\partial\pi}{\partial N} (e^{-\delta t} - S)\, e^{-\lambda t}\, dt$$

It follows that δ will be independent of N in general only in the special case

$$S(\tau) = e^{-\delta\tau}$$

which corresponds to what is usually known as radio-active decay and implies that the physical life expectation of a plant is independent of its age.

If the survival curve is exponential and we have therefore radio-active decay it is obvious that the introduction of uncertainty is equivalent to raising the discount rate, λ. From Theorem 1 this implies that E(Z) is a maximum for a more labour intensive technique

than N*. (We shall see later that this result holds without restriction on $S(\tau)$.) Moreover, if $E(Z) > I$ then maximising the internal rate of return, λ subject to $E(Z) = I$ not only results in a further shift to a more labour intensive technique but also leads to the same choice of technique as in the certainty case, i.e. technique N^o. Thus uncertainty in the form of radio-active decay makes no difference to the choice of technique if the criterion of choice is maximisation of the internal rate of return.

This result is one of two invariances with respect to uncertainty which are stated in Theorem 6.

Theorem 6: (i) If the survival curve is exponential, then the technique which maximises the internal rate of return is technique N^o.

(ii) If N^Θ exists, then N^Θ is the technique which maximises $Pr\ (Z > I)$.

The second invariance result in Theorem 6 applies for any survival curve, S. It states that the technique with minimum pay-off period under certainty is the technique for which, under uncertainty, the probability that net present value is positive is a maximum. This result follows directly from the fact that a technique will have a positive net present if and only if its physical life exceeds its pay-off period. It implies that a pay-off period criterion is a particular way of abstracting from uncertainty which is equivalent to weighing the outcomes of an investment on a binary scale of 'success' ($Z \geqslant I$) and 'failure' and maximising the chances of success. More generally, the restriction of choice of

technique to those which have a pay-off period less than some horizon, H, is equivalent to restricting choice to those techniques which have a probability of at least S(H) of being 'a success'.

Pay-off period criteria are such a complete means of abstracting from uncertainty that there is no way of introducing uncertainty into them beyond the analogy pointed out above. In particular, the expected pay-off period cannot be well defined since for some outcomes of τ which may have non-zero probability of occuring the pay-off period is not defined. The nearest we can come is to consider the probability that it is defined, which is the probability that $Z \geq I$ as already discussed. Accordingly there is nothing more to be said about pay-off period criteria under uncertainty.

Theorem 7 gathers together various results for present value maximisation and internal rate of return maximisation under uncertainty which apply in the general case where the survival function, $S(\tau)$, is restricted only to being monotonic decreasing for non-negative τ and such that $S(0) = 1$; $S(\infty) = 0$.

Theorem 7: (i) The technique which maximises $E(Z)$ is more labour intensive than technique N^*.

(ii) If max $E(Z) \geq I$, then the technique which maximises λ subject to $E(Z) \geq I$ is more labour intensive than the technique which maximises $E(Z)$.

(iii) The technique which maximises $E(\lambda)$ is more labour intensive than technique N^o

The first part of Theorem 7 states that maximisation of E(Z) results in a more labour intensive technique than does maximising Z given certainty. The proof is simple: starting with two techniques with the same Z in the absence of uncertainty it is easily shown that E(Z) must be greater for the more labour intensive of the pair. Analysis on similar lines establishes the next part of the theorem which can be explained in relation to Figure 15. The figure shows two techniques with the same E(Z) at some discount rate, λ , say. Accordingly areas α and β are equal in the Figure. If now the discount rate is raised to $\lambda_2 > \lambda_1$ then area α falls to some are $A > \alpha\, e^{(\lambda_1 - \lambda_2)\,\tau}$. Similarly β falls to some area $B < \beta^{(\lambda_1 - \lambda_2)\,\tau}$. Accordingly $A > B$ and the more labour intensive technique, technique 1, is to be preferred at the higher discount rate. From this point it is straightforward to establish that provided max $E(Z) \geqslant I$, the technique which maximises λ is more labour intensive than that which maximises E(Z). From Theorem 6, the former is of course technique N^o in the special case of radio-active decay.

The final part of Theorem 7 states that maximising the expected internal rate of return, $E(\lambda)$, calls for a more labour intensive technique than maximising λ in the absence of uncertainty. To establish this we can note that the internal rate of return is given by

$$I = \int_0^{\min(\tau,T)} \pi\, e^{-\lambda t}\, dt \tag{53}$$

from which it is straightforward to show that τ is a concave function of λ and hence that $E(\lambda) \leqslant \lambda \big|_{\tau = E(\tau)}$. However this plays no part in our proof which requires from (53) that

$$\frac{d\lambda}{dN} = \int_0^{\min(\tau,T)} \frac{\partial\pi}{\partial} e^{-\lambda t}\, dt \Big/ \int_0^{\min(\tau,T)} \pi\, t\, e^{-\lambda t}\, dt \qquad (54)$$

Since the denominator of (54) is necessarily positive, the sign of $d\lambda / dN$ is that of the numerator of (54). Now if $T = \min(\tau,T)$, then the numerator of (54) will be zero for the technique N^o since N^o maximises λ in the absence of uncertainty. Moreover, again at N^o, the numerator will be positive for $\tau < T^o$ since $\partial\pi / \partial N$ is a monotonic decreasing function of t. Accordingly for $\tau < T^o$, any technique which yields a higher λ than technique N^o yields, must involve $N > N^o$: all $N < N^o$ are inferior for $\tau < T^o = \frac{1}{w} \log x^o$. Further, for $\tau > T^o$, all techniques other than N^o by definition yield a lower λ than λ^o. Accordingly in no circumstances is λ greater than that yielded by N^o if $N < N^o$. Thus the technique which maximises $E(\lambda)$ must involve $N \geqslant N^o$.

From these results we can note that since the effect of uncertainty is to make $E(Z)$ a maximum for some $N > N^*$, a surrogate for uncertainty which would result in choosing the same technique would be to add an uncertainty premium (<u>not</u> a risk premium) to the discount rate and proceed to treat the choice problem under uncertainty as having a certainty equivalent. By making this premium large enough, any technique in the range N^* to $\tilde{N}$ could be reached. However, there is a disadvantage in this since raising the uncertainty premium will lower the certainty equivalent Z. Accordingly, if the restraint $Z \geqslant I$ is not to be violated, the uncertainty premium cannot exceed $\lambda^o - \lambda$ and the chosen technique cannot be more labour intensive than N^o.

To get beyond N^o one possibility is to switch our criterion of choice. Thus maximising $Pr(Z - I > 0)$, which we have seen is equivalent to minimising the pay-off period, results in $N = N^\theta > N^o$. An alternative is to maximise $E(\lambda)$. Both are effectively risk-averse types of behaviour. Yet another procedure is to introduce risk aversion directly as in the next section. Meanwhile it is apparent that risk aversion cannot be treated simply as certainty equivalence by adding a risk premium if its effect is to call for something more labour intensive than N^o while maintaining $Z \geq I$.

IV.4 Risk aversion

The introduction of uncertainty into the analysis of choice of technique allows not only that there may be lack of certainty equivalence but also that there may be specific attitudes among investors towards risk. Here we shall only be concerned with attitudes which take the form of risk aversion defined as:

Risk aversion: A decision maker has risk aversion if his criterion of choice is to maximise the expected value of some function of a positively desirable characteristic, a, given by U(a) with properties

(i) $U(0) = 0$

(ii) $U'(a) \geq 0$

(iii) $\frac{d}{da} \frac{U(a)}{a} < 0$

Property (i) above states that zero satisfaction is obtained when $a = 0$ and essentially defines a scale of measurement for a. Property (ii)

makes 'a' a positively desirable characteristic, or more strictly, not undesirable. Property (iii) states that the average amount of satisfaction obtained from a must fall as a increases and hence must be greater, at positive levels of a and less for negative a than the marginal satisfaction to be obtained from additional a. This then is a weaker requirement than that marginal satisfaction should diminish but is by no means incompatible with it.

The properties of the function U are not retained under an arbitrary monotone transformation of U. However they are retained for a transform W(U) which has the same properties with respect to U as U has with respect to a.

Our interest in risk aversion derives from the following theorem:

Theorem 8: Maximising EU(Z - I) results in the choice of a more labour intensive technique than maximising E(Z) if U is consistent with risk aversion.

To establish Theorem 8 we consider two techniques, N_1 and N_2, for which E(Z) is the same and show that of the two the one which is more labour intensive, denoted N_2, is preferable under risk aversion. From this the theorem follows directly.

The analysis is developed with the aid of Figures 16 and 17. Figure 16 shows how the present value of cumulated profits builds up as a function of the physical life of each plant, τ. For $\tau = 0$ both graphs start at - I and that for technique N_2 has the steeper slope since it

it corresponds to the technique with higher immediate profit rate. Technique N_2 also has the shorter economic life and the lower present value over its economic life time. These properties must hold, as must the property that the two graphs cross once and once only if the two techniques are to have the same E(Z). The point at which the schedules cross is denoted by Z = C so that the probability Pr (Z $\leqslant$ C) = p is the same for both techniques.

This last point is reproduced in Figure 17, which shows the distribution functions, F(Z - I) of cumulative net profits Z - I for each of the techniques. Again the schedules cross at Z = C and we have F(C - I) = p. Below C, $F_1(Z) > F_2(Z)$ implying that low values of Z are more likely for technique 1, and this follows directly from Figure 16.

To go further we note that

$$E_1 (Z - I) = p\, E_1(Z - I/Z < C) + (1 - p)\, E_1 (Z - I/Z > C)$$

and

$$E_1 U(Z - I) = p\, E_1(U(Z - I)/ Z < C) + (1-p)\, E_1(U(Z - I)/Z > C)$$

and similarly for technique N_2. Moreover, by definition $E_1(Z - I) = E_2(Z - I)$. From these results it is easy to show that

$$\Delta E U = E_2 U - E_1 U = p\, \Delta E(Z/Z < C) \left\{ \frac{\Delta E(U/Z < C)}{\Delta E(Z/Z < C)} - \frac{\Delta E(U/Z > C)}{\Delta E(Z/Z > C)} \right\} \qquad (55)$$

Now

$$E_1(U/Z < C) = \int_0^{F(C - I)} U(Z - I)\, dF_1 (Z - I)$$

$$= \int_0^{F(C-I)} \frac{U(Z - I)}{Z - I} \quad (Z - I) \; dF_1 \; (Z - I)$$

So that

$$\Delta \; E(U/Z < C) = -\int_0^{F(C-I)} \frac{U(Z-I)}{Z-I} \; (Z \text{ i } I) \; d(F_1 \; (Z - I) - F_2 \; (Z - I)) \qquad (56)$$

Similarly

$$\Delta \; E(Z/Z < C) = -\int_0^{F(C-I)} (Z - I) \; d \; (F_1(Z - I) - F_2 \; (Z - I)) \qquad (57)$$

$$> 0$$

since

$$E_2(Z/Z < C) \quad > \quad E_1(Z/Z < C)$$

as can readily be seen from Figure 16. From (56) and (57) we have

$$\frac{\Delta \; E(U/Z < C)}{\Delta \; E(Z/Z < C)} = \int_0^{F(C-I)} \frac{U(Z - I)}{Z - I} \quad g(Z) \; d \; (F_1 \; (Z - I) - F_2 \; (Z - I) \qquad (59)$$

where

$$g(Z) \quad = \quad \frac{Z - I}{\int_0^{F(C-I)} (Z-I) \; d \; (F_1(Z-I) - F_2(Z-I))}$$

so that

$$1 \quad = \quad \int_0^{F(C-I)} g(Z) \quad d \; \{F_1 \; (Z-I) - F_2(Z-I)\}$$

It follows at once that the right-hand side of (59) is an average of values of U(Z-I) / (Z-I) taken over the range (-I, C-I). As such it must exceed its smallest value within that range which occurs at Z = C if U has property (iii) for risk aversion. Therefore

$$\frac{\Delta E (U / Z < C)}{\Delta E (Z / Z < C} \geq \frac{U (C - I)}{C - I} \tag{60}$$

An exactly parallel argument establishes

$$\frac{\Delta E (U / Z > C)}{\Delta E (Z / Z > C)} \leq \frac{U (C - I)}{C - I} \tag{61}$$

Hence from (55), (58), (60) and (61)

$$\Delta EU = E_2U - E_1U \geq 0$$

i.e. technique N_2 is preferred to N_1 under risk aversion. From this Theorem 8 follows directly.

V Sufficient Conditions

V.1 Optimal timing

So far throughout our analysis we have been concerned only with the choice of technique so as to maximise some particular choice criterion. Thus we have been concerned with the best choice in some sense without reference to whether the best is good enough. To complete the discussion, therefore, it is necessary to explore the conditions under which a positive investment decision will be made.

The necessary and sufficient conditions for a positive investment decision are discussed here only in relation to the choice of technique based on net present value. Parallel discussion deriving from alternative criteria are clearly possible and not without interest. However, they are not pursued partly because this paper is already long enough, and partly because the lines of argument emerge readily from the case considered.

The obvious necessary condition for an investment selected according to net present value to satisfy is that its net present value, Z-I, should be positive: otherwise the investment is not worth making. This is not a sufficient condition, however. To obtain a sufficient condition we need to demonstrate that it is better to invest now rather than at some later moment. In particular we need to check that the rate of technical progress is not so great that deferment of the investment decision will produce greater returns.

If it is assumed that firms are operating in perfectly competitive markets then the deferrment of an investment will not influence the firms future environment and the net present value of making the decision to invest at time t is given by V(t) where

$$V(t) = (Z(t) - I(t))e^{-\lambda t} \qquad (62)$$

so that

$$\dot{V}(t) = -\lambda V(t) + e^{-\lambda t}\{\dot{Z}(t) - \dot{I}(t)\}$$

The condition that there should be no advantage to delaying the investment is $\dot{V}(t) < 0$ implying

$$\lambda V > (\dot{Z} - \dot{I})e^{-\lambda t}$$

and hence from (62) that

$$\lambda(Z-I) > \dot{Z} - \dot{I}$$

This condition can be further simplyfied by noting that for comparability we require the present value of capital expenditure to be constant in base prices, i.e.

$$I(t) = I(0)e^{\lambda t} \tag{63}$$

Thus our sufficient condition reduces to

$$\lambda Z > \dot{Z} \tag{64}$$

i.e. the rate of growth of Z to be achieved by delaying should not exceed the discount rate λ.

To develop this condition further note that Z(t) is given by

$$Z = \int_t^{t+T} (X(t)-N(t)e^{w\tau})e^{-\lambda(\tau-t)}d\tau$$

so

$$\dot{Z} = \lambda Z - rI + \dot{N}\int_t^{t+T} (f_N - e^{w\tau})e^{-\lambda(t-\tau)}d\tau$$

$$+ (f_I\dot{I} + f_t)\int_t^{t+T} e^{-\lambda(\tau-t)}d\tau \tag{65}$$

This result simplifies considerably if N is now assumed to be such as to maximise Z at a moment in time. Under this condition the first integral on the right-hand side of (65) is zero and the second is equal to $\frac{Z}{X-Nf_N}$. Thus the condition (65) becomes

$$rI > Z\frac{f_I\dot{I}+f_t}{X-N_{f_N}}$$

But if N maximises Z, then from Theorem 1,

$$Z(\lambda + w\frac{f_N}{x-f_N}) = rI$$

while from (63) we have $\dot{I} = \lambda I$. Accordingly the condition (64) further simplifies to yield

$$\lambda(X-Nf_N-If_I) > f_t - wNf_N \qquad (66)$$

as our sufficient condition.

From (66) it is apparent that if X is a homogeneous function of degree one in N and I at each moment in time then the condition (66) reduces simply to requiring $f_t > wNf_N$.

The function f_t that enters into the condition (66) is the partial derivative of X with respect to t. It will be positive, therefore, if there is technical progress. More specifically, if technical progress is Harrod neutral at rate g(t) at time t, then

$$f_t = g(t)Nf_N$$

so that the condition (66) becomes

$$\lambda\ (X-Nf_N-If_I) > Nf_N(g-w) \qquad (67)$$

Accordingly we require the Harrod-neutral rate of progress to be not greater than the rate of growth of real wages if our sufficient condition is to be satisfied under constant returns to scale.

V.2 Feasible growth paths

It is apparent from the above discussion that if we are to have constant growth of real wages, w, then the sufficient condition (67) puts a lower bound on w which must not be violated if investment is not to be deferred. By contrast the necessary condition that net present value must be positive can be interpreted as an upper bound on w. The combination of conditions accordingly constrains the possibilities for any model which is to satisfy both conditions.

To illustrate these constraints we can consider the putty-clay vintage model as set out by Bliss {1} on which we impose constant λ and w and hence constant $\mu = \lambda/w$.

In the model we have techniques chosen so as to maximise net present value which will imply from (28) that

$$b = \frac{\mu}{\mu-1} \cdot \frac{y^{\mu-1}-1}{y^{\mu}-1}$$

where $b = \frac{\partial \log X}{\partial \log N}$ and $y = xe^{-wt}$. The condition that net present value be positive can be expressed from (25) as

$$\Omega = \frac{y-1}{\lambda+w\frac{b}{1-b}} > \frac{I}{Ne^{wt}} = R \qquad (68)$$

while the sufficient condition that investments should not be delayed is $g > w$ given constant returns from (67). It need not be assumed that g is constant to obtain from this last condition by some lengthy manipulations the equivalent expression

$$\frac{\dot{y}}{y} < (1-b)\frac{\dot{R}}{R}$$

An equally tedious but otherwise straightforward series of algebraic manipulations yields

$$(1-b)\frac{\dot{\Omega}}{\Omega} = \frac{\dot{y}}{y}$$

so that combining results we get

$$\frac{\dot{\Omega}}{\Omega} \leq \frac{\dot{R}}{R} \qquad (69)$$

while from (68) $\Omega \geq R$ (70)

The two conditions (69) and (70) are perhaps the simplest expression of the necessary and sufficient constraints imposed by investment theory under the specified assumptions. It can be noted that in no way do these assumptions restrict the path of N. They imply that the ratio of Ω to R must always be at least one and cannot grow. The ratio will be one in the limiting case where net present value is identically zero. In this event Ω and R must have the same growth rate. Accordingly the rate of Harrod-netural progress must be constant and equal to the rate of growth of real wages.

Footnotes

(1) For some production functions which we shall want to regard as admissible there is no initial range over which the average product labour, x, increase with respect to N for fixed I. However since this range is of no significance in subsequent analysis it can be assumed to exist by modification of the production function for low values of N in such cases.

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Figure 5

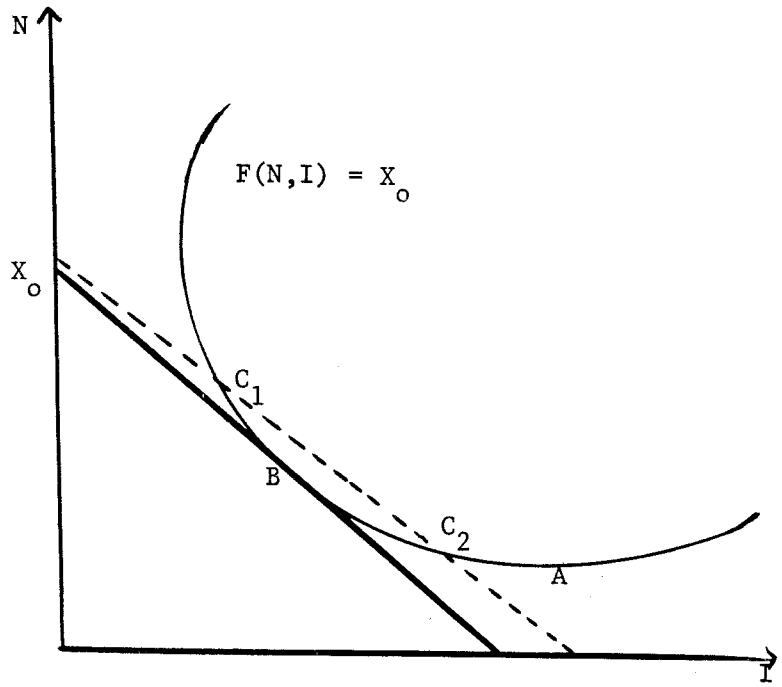

Figure 6

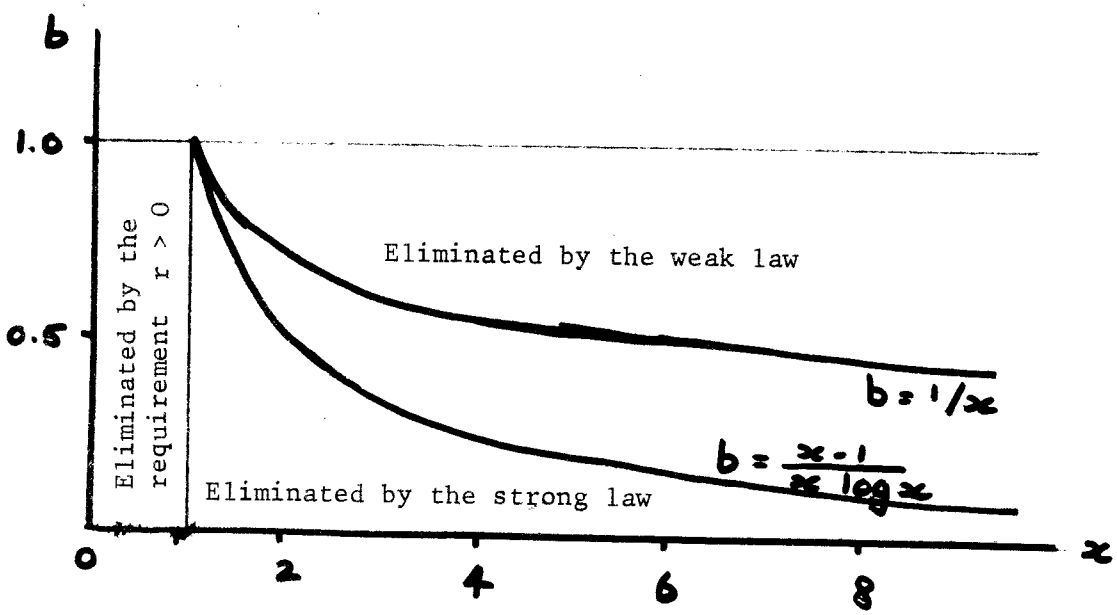

Figure 7

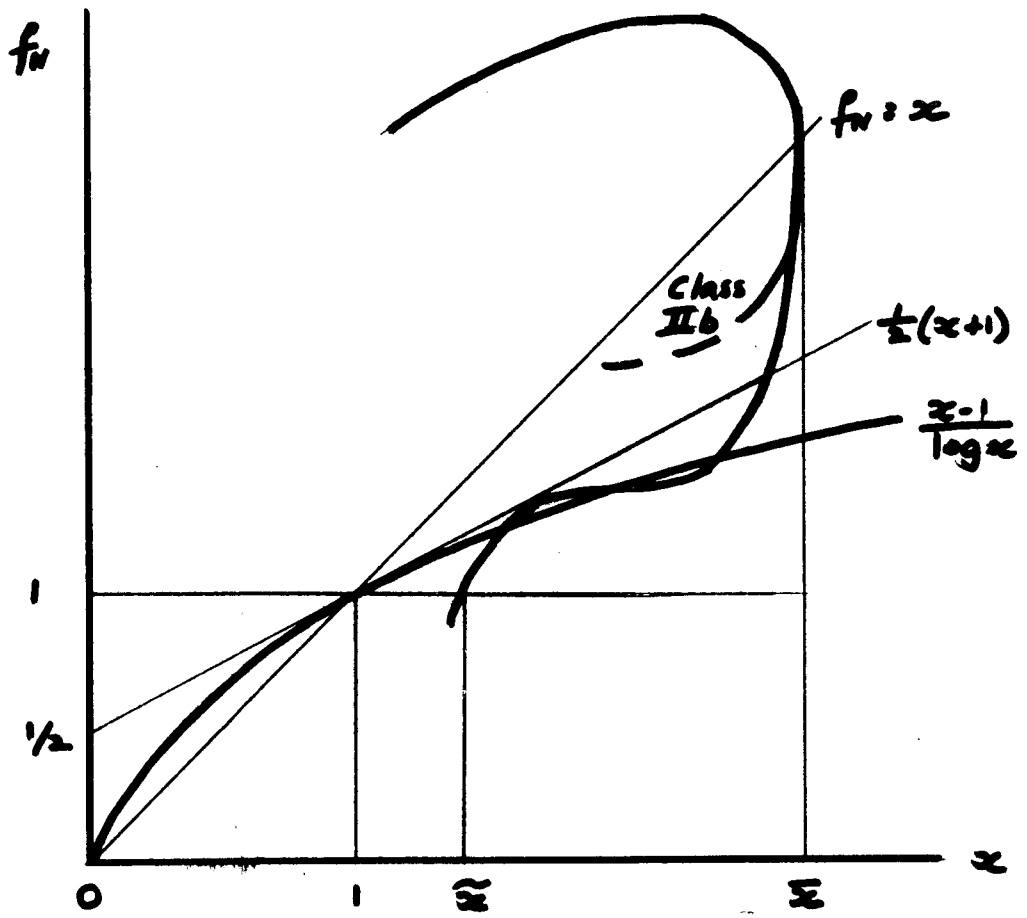

Figure 8

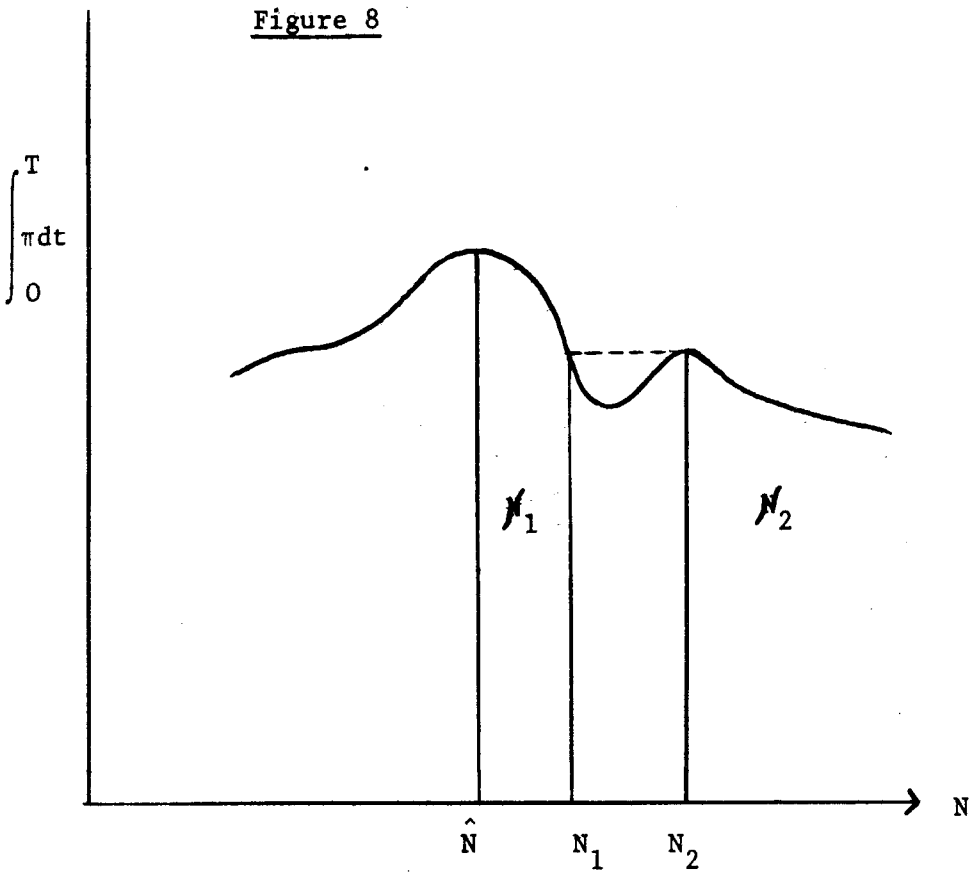

Figure 9

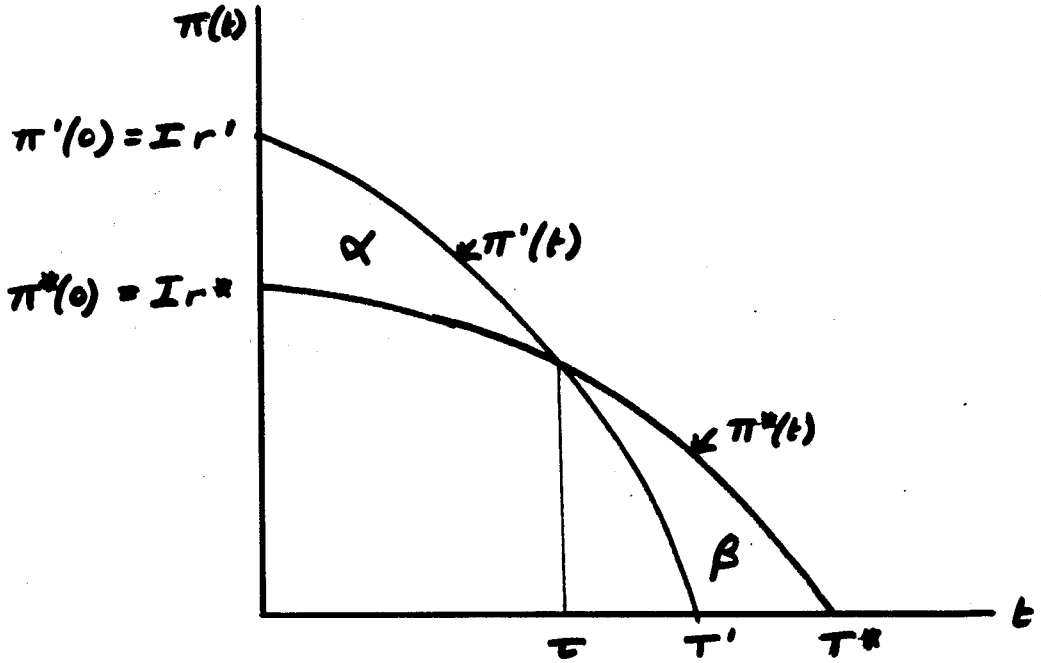

Figure 10

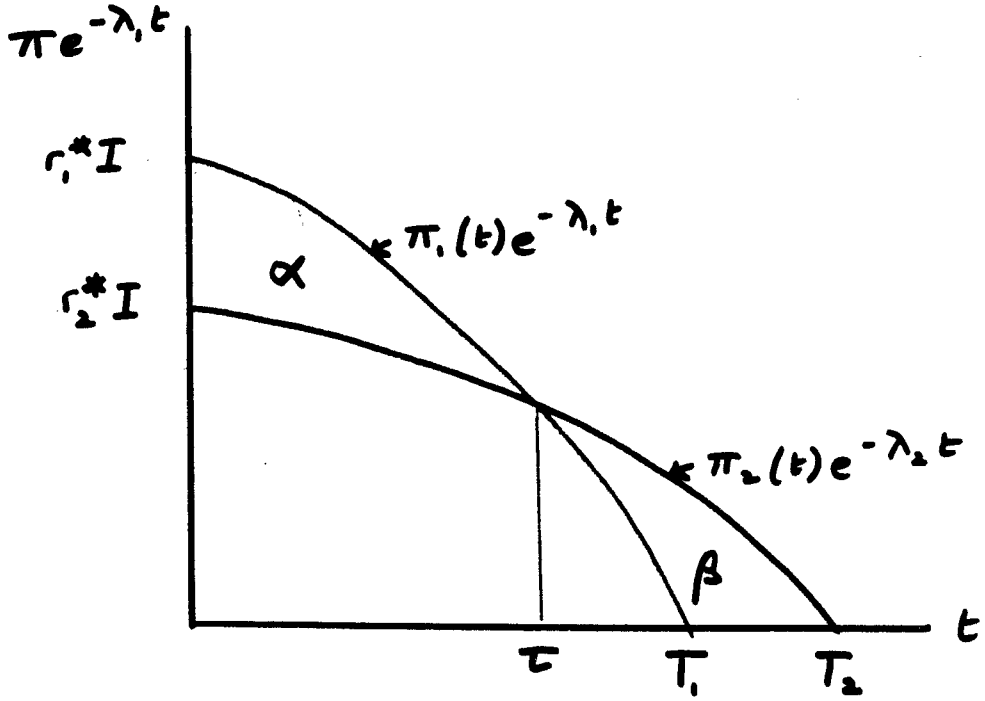

Figure 11

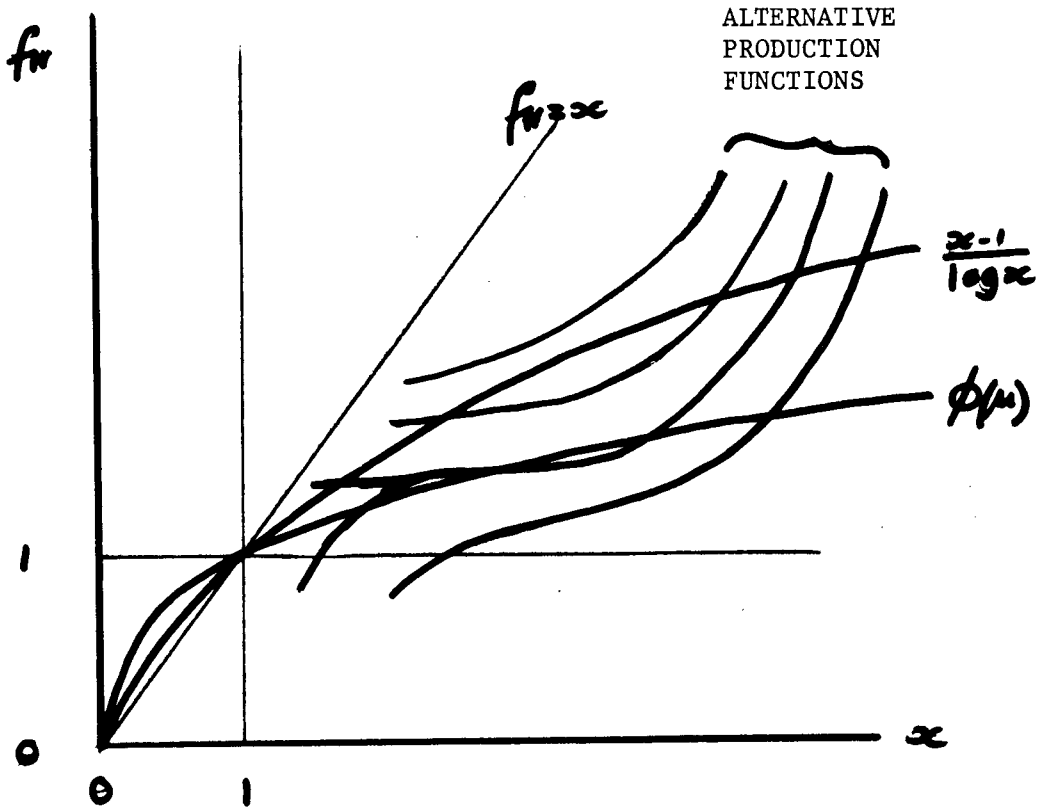

Figure 12

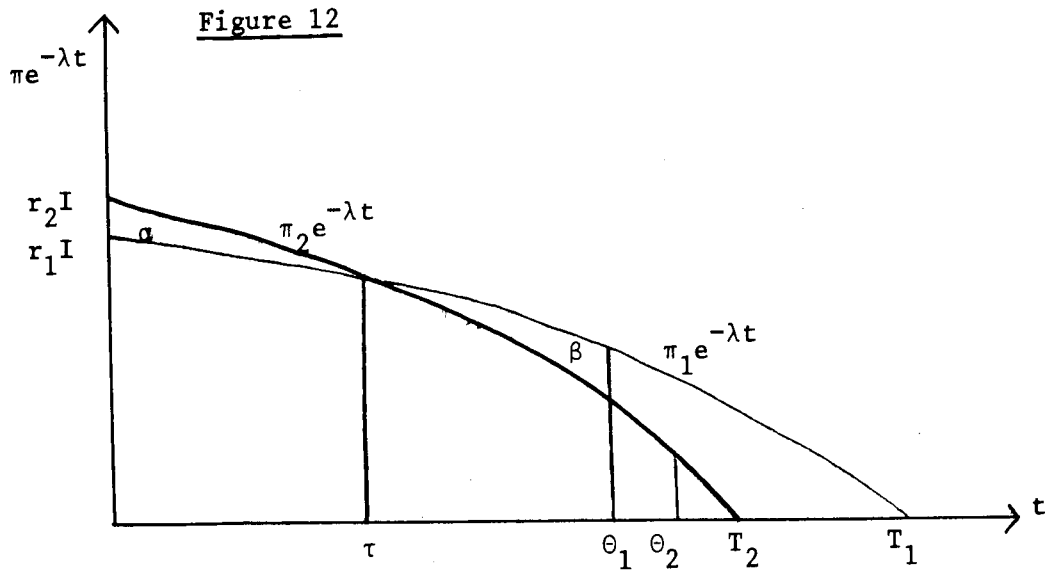

Figure 13

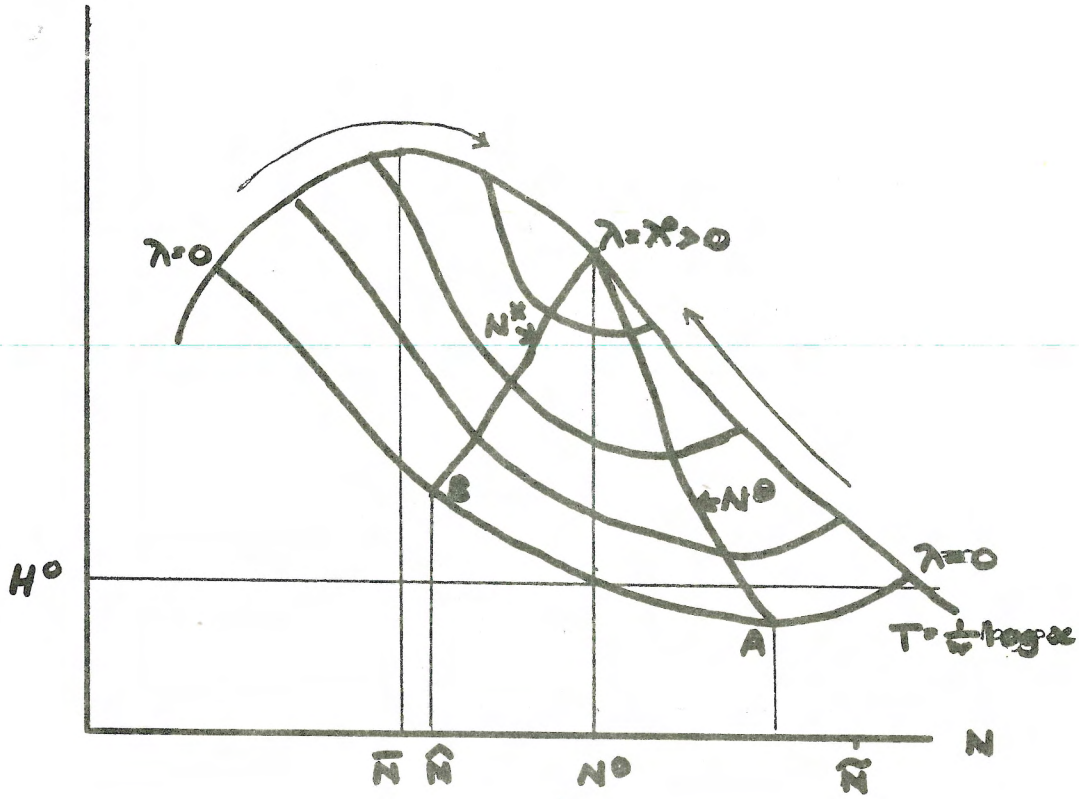

Figure 14

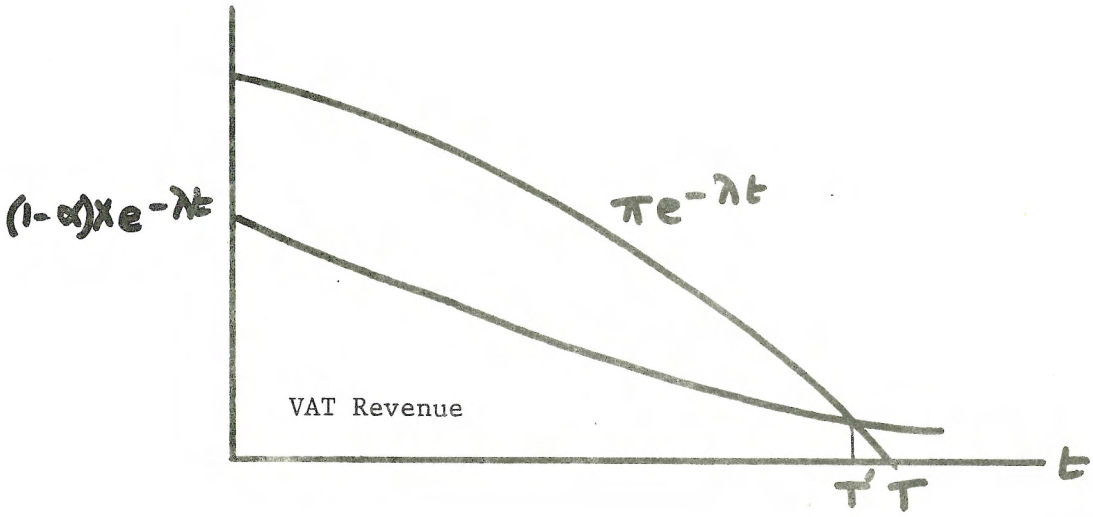

Figure 15

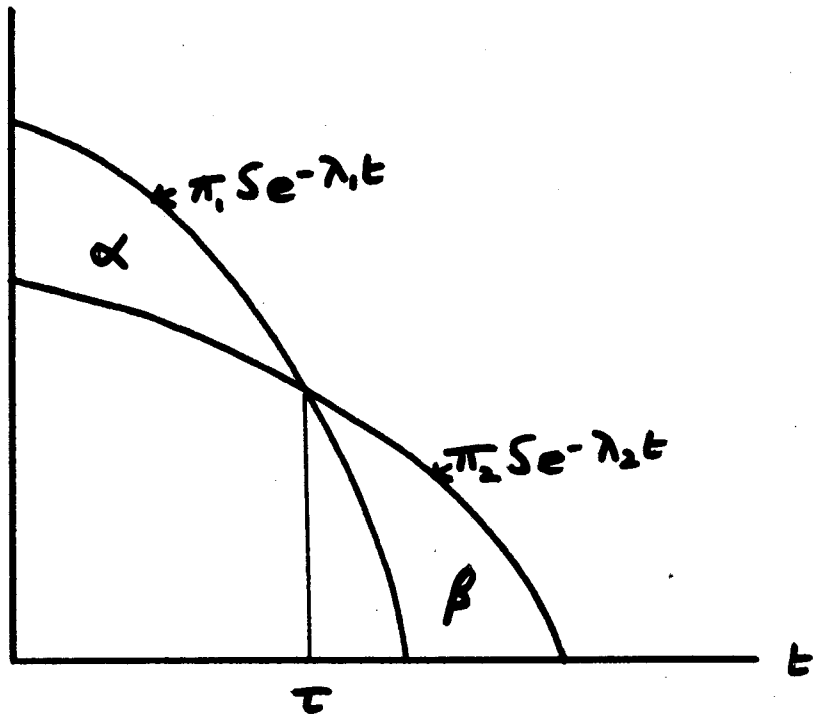

Figure 16

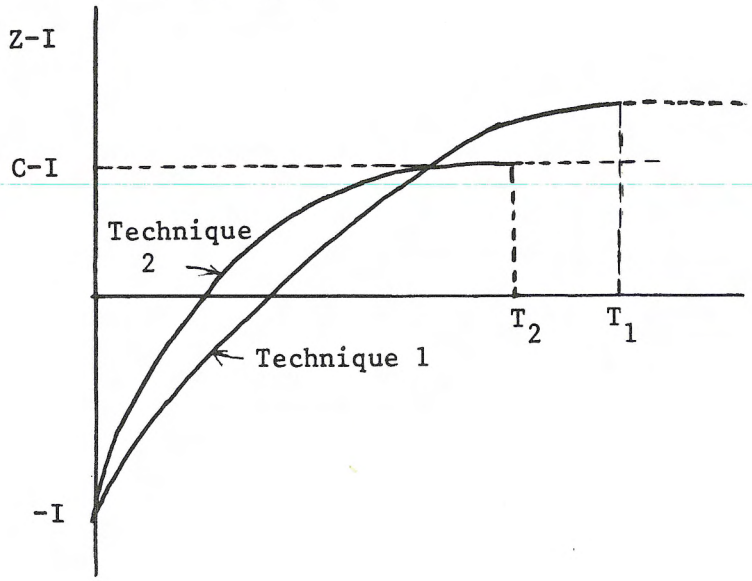

Figure 17