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MEASUREMENT UNIT INVARIANT COEFFICIENTS IN MULTIPLICATIVE-LOGARITHMIC FUNCTIONS

by

David I. Stern

Heslington, York, YO1 5DD
Telephone: 44 (0)904 432 999
Telefax: 44 (0)904 432 998

Abstract

Hunt and Lynk (1993) demonstrate that the coefficients of the non-multiplicative terms of a multiplicative logarithmic function are dependent on the units used. I demonstrate that a unique very simple transformation circumvents this problem.

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1. INTRODUCTION

In a recent paper in Applied Economics, Hunt and Lynk (1993) demonstrate that the coefficients of the non-multiplicative terms of a multiplicative logarithmic function are dependent on the units used. They explain this algebraically and also provide econometric results, using a translog production function to prove the point. All this is correct, but a simple transformation of the data makes the coefficients invariant to the units of measurement. This is simply to subtract the mean of the logarithm of each variable from its logarithm in each period and to use the transformed data in the regression. This transformation is unique. Also I show that in the case of an equation used to estimate the underlying trend in technical change the transformation results in coefficients that have a meaningful interpretation.

2. ALGEBRA

The function investigated algebraically by Hunt and Lynk is :

$$\ln Y_t = \alpha_0 + \alpha_1 \ln(X_{1t}) + \alpha_2 \ln(X_{2t}) + \alpha_3 \ln(X_{1t}) \ln(X_{2t}) \qquad t = 1, \dots, T \qquad (1)$$

Now let us apply an operator M() to $\ln(X_{1t})$ and $\ln(X_{2t})$ so that the coefficients are measurement unit invariant :

$$\ln Y_t = \beta_0 + \beta_1 M(\ln(X_{1t})) + \beta_2 M(\ln(X_{2t})) + \beta_3 M(\ln(X_{1t})) M(\ln(X_{2t})) \qquad (2)$$

An additional condition that we place on the operator is that :

$$M(\ln(Z_i)) - M(\ln(Z_j)) = \ln(Z_i). - \ln(Z_j). \qquad \forall_{i,j} \qquad (3)$$

which follows from the definition of the regression slope coefficient as the partial derivative of the dependent variable with respect to the relevant independent variable. From this it follows that the operator must work through subtracting the same constant from each observation of the variable and, therefore, M() must be a linear operator. This constant can be represented as a function f(Z) mapping from R^T into R^1 and as M() is a linear operator it must also be a linear function ie :

$$M(\ln(Z_t)) \quad = \quad \ln(Z_t). - f(\ln(Z_t)). \tag{4}$$

Now change the units of measurement as in Hunt and Lynk (1993):

$$M(\ln(Z_t / a)) \quad = \quad \ln(Z_t) - \ln(a) - f(\ln(Z_t)) + f(\ln(a)) \tag{5}$$

The multiplicative logarithmic function now is :

$$\ln Y_t \quad = \quad \gamma_0 + \gamma_1 M(\ln(X_{1t} / a)) + \gamma_2 M(\ln(X_{2t} / a)) + \gamma_3 M(\ln(X_{1t} / a)) M(\ln(X_{2t} / a)) \tag{6}$$

If $\gamma_1 = \beta_1$ and $\gamma_2 = \beta_2$ then the operator must map $\ln(Z_t / a)$ into $\ln(Z_t)$. Therefore in equation (5) ln(a) = f(ln(a)). This can only be true if f() is the mean function of a. Other options which could be true for a constant such as the median or mode (not linear functions for a variable) or simply a (not a constant for a variable) do not meet the necessary criteria in the case of a variable. No transformation of the dependent variable is necessary, as as Hunt and Lynk (1993) state this only affects the constant term. The constant term is the level of the dependent variable when all the independent variables are zero so it has an obvious and meaningful interpretation regardless of the units of the dependent variable. So this approach is not simply a regression using the deviations from the means of the dependent variables. Such an equation would have no constant and the multiplicative term would be $(X_{1t}\ X_{2t})'$ rather than $(X'_{1t}\ X'_{2t})$. Also, if the initial multiplicative logarithmic function is transformed in some way to produce the estimating equation, the same procedure should still be applied to obtain unbiased estimates of the coefficients in the function itself (see below).

3. DISCUSSION

The question is, does this transformation produce a meaningful set of coefficients or simply another set of, albeit unit-invariant, essentially meaningless coefficients ? This is a particularly important question in the context of measuring technical change. For example we might be interested in an equation explaining total factor productivity that is derived from a translog cost function. The cost function is given by :

$$\ln C = \beta_0 + \sum_i \beta_i \ln(Z_i) + \frac{1}{2}\sum_i \sum_j \beta_{ij} \ln(Z_i)\ln(Z_j) \tag{7}$$

$$Z = [Q, t, P_1, \ldots, P_n]' \tag{8}$$

where C is costs, Q is the level of output, t is the time trend, and $P_1, \ldots, P_n$ are the prices of the n factors of production. The rate of growth of total factor productivity is defined as the partial derivative of the logarithm of costs with respect to time in equation (1) (Slade, 1989) :

$$\frac{\partial \ln C}{\partial t} = \frac{\partial C}{\partial t}\frac{1}{C} = \beta_t + \beta_{tt} t + \beta_{tQ} \ln Q + \beta_{tB} \ln B + \beta_{tS} \ln S + \sum_i \beta_{tPi} \ln P_i \tag{9}$$

We would estimate this equation with the first difference of the logarithm of an index of total factor productivity as the dependent variable with the addition of an additive error term :

$$\Delta \ln(TFP_t) = \beta_t + \beta_{tt} t + \beta_{tQ} \ln Q + \beta_{tB} \ln B + \beta_{tS} \ln S + \sum_i \beta_{tPi} \ln P_i + \varepsilon_t \tag{10}$$

In practice other alterations may be desirable (see Slade, 1989) but they are irrelevant to the point being made. The true trend in technology is measured by $\beta_t + \beta_{tt} t$. A test for improving technology can be executed by testing whether the mean of $\beta_t + \beta_{tt} t$ is greater than zero. It is therefore crucial that the other variables are entered into the equation in such a way that the coefficients are invariant to the units of measurement. But also the transformation must yield coefficients that are meaningful in terms of a technical progress trend. In this case if the transformation is applied to the independent variables then $\beta_t + \beta_{tt} t$ is the expected value of $\Delta\ln(TFP_t)$ when all the independent variables are at their sample mean. If the variables have a log-normal distribution then the sample mean of the logarithm is an estimate of the mode of the variable. This would then be an intuitively meaningful definition of the trend in underlying technical change.

Blanchflower and Oswald (1990) estimate an equation where the dependent variable is the real wage rate and the independent variables are a constant, log unemployment and the cube of log unemployment. If the transformation is applied to log unemployment then the constant is given by :

$$\alpha_0 = \frac{1}{T}\sum_t W_t - \alpha_2 \frac{1}{T}\sum_t [\ln(U_t) - \frac{1}{T}\sum_t \ln(U_t)]^3 \qquad (11)$$

where U is the level of unemployment and W the real wage rate. If α_2 is negative (positive) then the constant will be below (above) the mean real wage rate. Intuitively α_0 is the real wage rate at the "natural level" of unemployment. If W is lognormally distributed and the relationship implies that the mode of W occurs in association with the mode of U then the second term on the RHS would appear to be an adjustment to take into account that . α_2 is not, however, invariant to the units of measurement of W and therefore it is difficult to give this a precise interpretation. So here it is more difficult to show that the coefficients are economically meaningful than in the previous case - this is partly due to the semi-logarithmic form of the model. Hunt and Lynk's (1993) comments appear to be directed at the interpretation of α_1 and α_2. In this case the transformation does not help as the dependent variable is non-logarithmic.

In the the Evans and Heckman (1984, 1986) case, Hunt and Lynk (1993) comment on the coefficients of the non-multiplicative logarithmic terms in a standard translog cost function. Though the transformation would yield measurement unit invariant coefficients, it is difficult to show that this results in economically meaningful coefficients. In the general multiplicative function the regression coefficients are not partial derivatives in an economic sense as a change in the logarithm of a variable affects the value of the multiplicative terms as well. So it is unlikely that one would propose any hypotheses about these coefficients. Hypotheses are likely to be formulated in terms of functions of the coefficients as in the technical change example above. It is important to note that neither Evans and Heckman (1984, 1986) nor Blanchflower and Oswald (1990) refer explicitly to the individual t-statistics that they include in their papers, or attribute to them any economic meaning.

4. DEMONSTRATION

I use the same translog production function as Hunt and Lynk (1993) but use USA macro data for 1948 to 1990 as a matter of convenience. In the first model I indexed all variables to 1 in 1948. The estimate of the untransformed model was made. Then logarithms are taken of each variable and then the mean subtracted and the transformed model estimated. In the second model all the variables were indexed to 100 in 1948, and in the third model only capital was indexed to 100, the other variables were indexed to 1, before taking logarithms. Table 1 presents the results for each of the three models using the suggested transformation and not-using the suggested transformation. The results clearly demonstrate that the transformation results in a model whose coefficients and standard errors are invariant to the units of measurement. Note that unless the transformation is also be applied to the trend variable and to the dependent variable, as was done in these examples, the constant term will be dependent on the units of measurement.

5. CONCLUSION

Though Hunt and Lynk (1993) raise an important point in their paper, there is a simple method that econometricians can use to circumvent the problem of non-multiplicative coefficients in multiplicative logarithmic models being dependent on the units of measurement which is demonstrated in this comment. This is the only such transformation. This transformation should, therefore, always be used by researchers interested in testing hypotheses relating to the values of the coefficients of the non-multiplicative variables.

Table 1. OLS Estimates of a Translog Production Function for the US Macroeconomy 1948-1990

	I. Untrans-formed	I. Trans-formed	II. Untrans-formed	II. Trans-formed	III. Untrans-formed	III. Trans-formed
Constant	0.0017 (0.1727)	0.0378 (8.3369)	-2.4474 (-0.2228)	0.0378 (8.3369)	-2.8011 (-0.3175)	0.0378 (8.3369)
lnK	0.3086 (2.7519)	-0.0169 (-0.1691)	4.2545 (0.9816)	-0.0169 (-0.1691)	0.9087 (0.2352)	-0.0169 (-0.1691)
lnL	1.0886 (8.8822)	1.0977 (13.7062)	-2.5880 (-0.2853)	1.0977 (13.7062)	4.4344 (0.5469)	1.0977 (13.7062)
$(\ln K)^2$	-0.0652 (-0.1539)	-0.0652 (-0.1539)	-0.0652 (-0.1539)	-0.0652 (-0.1539)	-0.0652 (-0.1539)	-0.0652 (-0.1539)
$(\ln L)^2$	0.7624 (0.4106)	0.7624 (0.4106)	0.7624 (0.4106)	0.7624 (0.4106)	0.7624 (0.4106)	0.7624 (0.4106)
lnKlnL	-0.7265 (-0.4108)	-0.7265 (-0.4108)	-0.7265 (-0.4108)	-0.7265 (-0.4108)	-0.7265 (-0.4108)	-0.7265 (-0.4108)
t	0.0133 (4.2501)	0.0133 (4.2501)	0.0133 (4.2501)	0.0133 (4.2501)	0.0133 (4.2501)	0.0133 (4.2501)

Note :

I Untransformed : Labor (L), capital (K), dependent variable GDP, all indexed to 1948=1.

I Transformed : Indexed to 1948=1 and mean subtracted from the log of each variable.

II Untransformed : All variables indexed to 100 in 1948.

II Transformed : Indexed to 1948=100 and mean subtracted from the log of each variable.

III Untransformed : Only capital indexed to 100 in 1948 others indexed to 1 in 1948

III Transformed : Only capital indexed to 100 in 1948 others indexed to 1 in 1948 and mean subtracted from the log of each variable.

t statistics in parentheses

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