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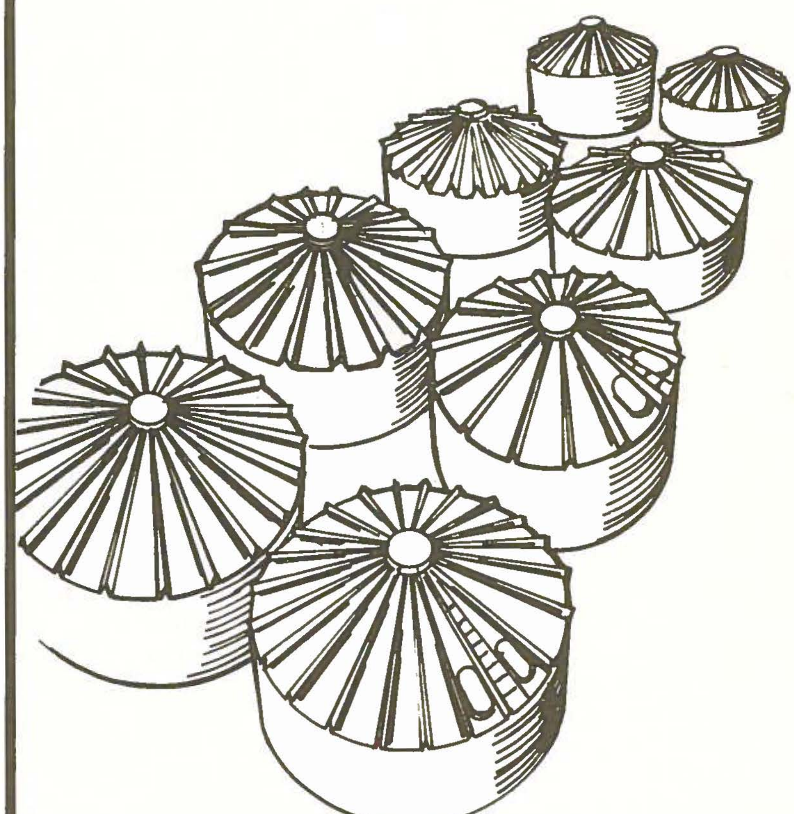

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An Expected Utility Model Of Grain Storage And Hedging By Farmers

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1. Introduction

In this paper a model is developed to indicate how a risk-averse, expected utility-maximizing farmer would decide on the amount of grain to be stored on the farm and on hedging activities through forward sales and/or futures transactions. Assumptions about the farmer's circumstances are indicated in the next section. A specific decision problem is developed. In Section 3, a qualitative solution is presented. The qualitative solution does not depend on detailed knowledge of the farmer's utility function or expectations of future prices. It depends only on the assumptions of risk aversion (decreasing marginal utility of wealth) and on the means of the decision-maker's subjective probability distribution of future prices.

Since we are largely ignorant of details of utilities and expectations, it seems desirable to determine as fully as possible the consequences of general properties believed to prevail. In the storage-hedging context it is encouraging that, under risk aversion, quite a few inferences can be drawn from the means. These are summarized in a "conclusion tree" on page 4.

If a specific utility function and a specific distribution of unknown prices are assumed, then corresponding quantitative optimal choices can be deduced. This is done in Section 4 for several alternative combinations of utility function and probability distribution. It turns out that the qualitative decisions of Section 3 are a very useful first step in calculating more precise optima.

The present model is centered on decisions at harvest time. It is thus incomplete in not treating problems of marketing decisions that may be made during the growing season and in not considering alternative strategies for closing positions taken at harvest time. These and related issues are briefly considered in Section 5. Despite the limited scope of the analysis presented here, it does sharpen some of our insights into considerations affecting storage-hedging decisions and, hopefully, it will prove a useful step in the development of more comprehensive theories.

The detailed derivations of some of the results for the "conclusion tree" are developed in Appendix A. Appendix B gives the result of relaxing one of the independence assumptions maintained throughout the rest of the paper. Some properties of an interesting family of utility functions are viewed in Appendix C.

2. The Model

A grain farmer has recently harvested and has n bushels on hand. He must decide how much to sell now, how much to store, how much to contract for future delivery, and what position, if any, to take on the futures market.

Suppose that some months ahead is the time (τ) of year when this grain usually attains its seasonal peak price and that his opportunity to sell forward would involve delivery at τ.[1] Let a represent his current cash price and c (known) the price at which he can contract for τ-delivery. Let m ($0 \leq m \leq n$) be the amount he decides to store and g ($0 \leq g \leq m$) be the quantity he decides to sell forward.

He can also take a hedging (short) position on the futures market. Let b be the current price (per bushel) for futures contracts maturing at τ. Assume that any physical grain stored and not covered by a forward contract will be sold at τ and any short futures position will be closed at τ. His return will then be:

$$\pi = r\{a(n-m)-fs\} + cg - dm + A(m-g) + (b + f - q - B)s$$

where:

r: cumulation factor converting current dollars to dollars at time τ; $r = (1 + \frac{i}{12} R)$ where R

1. For Minnesota crops, wheat and oats would typically be harvested in August with a peak price in January. The respective months would be November and June for corn; October and June for soybeans. See Houck {8}. In the winter wheat belt, harvest is in June or July with top price in December or January. The assumption that forward sales necessarily involve delivery at τ is relaxed in Sec. 5, page 9.

is annual interest rate and j is number of months until τ.

f: margin requirement per bushel for futures transactions.

d: direct cost of storage.

A: a random variable, unknown price to be realized for local cash grain at τ.

B: a random variable, unknown price of maturing futures contracts at time τ.

q: commission on futures contracts.

s: size (in bushels) of short position in futures market ($0 \leqslant s$).

m, g, s are the decision variables. Rewriting:

$$
\begin{aligned}
(2.1)\quad \pi &= ran + (A-ra-d)m + (b-(r-1)f-q-B)s + (c-A)g \\
&= k_0 + (A-k_1)m + (k_2-B)s + (k_3-A)g.
\end{aligned}
$$

The k_i are known when m, s, g must be decided. In the formal analysis which follows, it is assumed that the farmer acts as though he has a subjective probability distribution of unknown A and B and acts to maximize expected utility of return or gain with respect to that subjective distribution. The restrictions $0 \leqslant s$, $0 \leqslant g \leqslant m \leqslant n$ are imposed to reflect usual circumstances and practices. He could, of course, store more than his supply ($m>n$) by acquiring some grain from neighbors or dealers. This would, however, involve expenses not included in the above equation for his return. In an initial model it seems desirable to keep the analysis as simple as feasible by omitting possibilities that would ordinarily not be seriously considered. Similarly, he could sell forward more than he stores ($g>m$), but this would require him to locate additional grain before delivery causing additional expenses and risks and diverting the decision-maker from his usual activities.

There is also nothing to prevent a farmer from taking a long (speculative) position on the futures market. This possibility could be incorporated in the present model by letting s be positive or negative and substituting $\{-(r-1)f-q\}\ |s|$ for $\{-(r-1)f-q\}s$ in (2.1). However, with the current crop on hand, a farmer is unlikely to take a long speculative position and, again, it seems desirable in the first instance to consider alternatives that are commonly relevant.

With these provisions, the farmer's decision problem is to choose m, s, g to maximize the expected utility function

$$(2.2)\ \eta(m,s,g) = E\psi\{(A-k_1)m + (k_2-B)s + (k_3-A)g\}$$

subject to $0 \leqslant s$, $0 \leqslant g \leqslant m \leqslant n$ where ψ is utility of gain.[2] Gain is measured from (ran) the amount he would realize at time τ if the whole supply were sold now.

3. A Qualitative Solution

The storage-hedging problem indicated above has the mathematical form of the usual portfolio problem. If (1) $\psi' > 0$, (2) $\psi'' < 0$ (risk aversion), and (3) $\lim_{x \to 0} \psi'(x) = 0$; then existence of a solution is guaranteed by a theorem of Leland {9} or its generalization by Bertsekas {1}. $\psi'' < 0$ implies that η is strictly concave. Strictly concavity together with the fact that the admissible set $\{(m,s,g)\colon 0 \leqslant s,\ 0 \leqslant g \leqslant m \leqslant n\}$ is convex implies that the solution is unique. Denote the solution, or optimal choice, by $(\hat{m}, \hat{s}, \hat{g})$.

In many applications an investigator will not know the exact subjective probability distributions of the decision-makers, so it is of interest to see what conclusions can be drawn from partial information. In this section, a number of conclusions are obtained from assumed relations among the known constants k_1, k_2, k_3 and the subjective expected values of the unknown random variables A,B. Properties (1), (2), (3) of the preceding paragraph are all that are assumed about the utility function.

It is convenient to start with a change of variable. Let $w = m-g$; w then represents grain that is stored but not covered by a forward contract. Substituting $(m-w)$ for g in (2.2) and collecting terms, expected utility may be written:

$$
\begin{aligned}
(3.1)\quad \theta(m,s,w) = \eta(m,s,(m-w)) &= E\psi\{(k_3-k_1)m + (k_2-B)s + (A-k_3)w\} \\
& 0 \leqslant s,\ 0 \leqslant w \leqslant m \leqslant n
\end{aligned}
$$

Tentatively suppose $\hat{s}$, $\hat{w}$ were known and consider the optimal value of m. Under mild regularity conditions {4, page 9}, one can differentiate expected utility under the integral, yielding:

$$
\begin{aligned}
(3.2)\quad D_m\theta(m,\hat{s},\hat{w}) &= (k_3-k_1)E\psi'\{(k_3-k_1)m + (k_2-B)\hat{s} + (A-k_3)\hat{w}\} \\
&\overset{s}{=} (k_3-k_1).
\end{aligned}
$$

where "$\overset{s}{=}$" means "agrees in sign with."

That $D_m\theta$ agrees in sign with (k_3-k_1) is justified by recalling that $\psi'>0$ and, therefore, $E\psi'>0$. Suppose $k_3-k_1>0$. Then expected utility increases whenever m increases and is maximized when m is made as large as possible, namely $m=n$. Thus:

$$(3.3)\quad k_3-k_1 > 0 \Rightarrow \hat{m} = n.$$

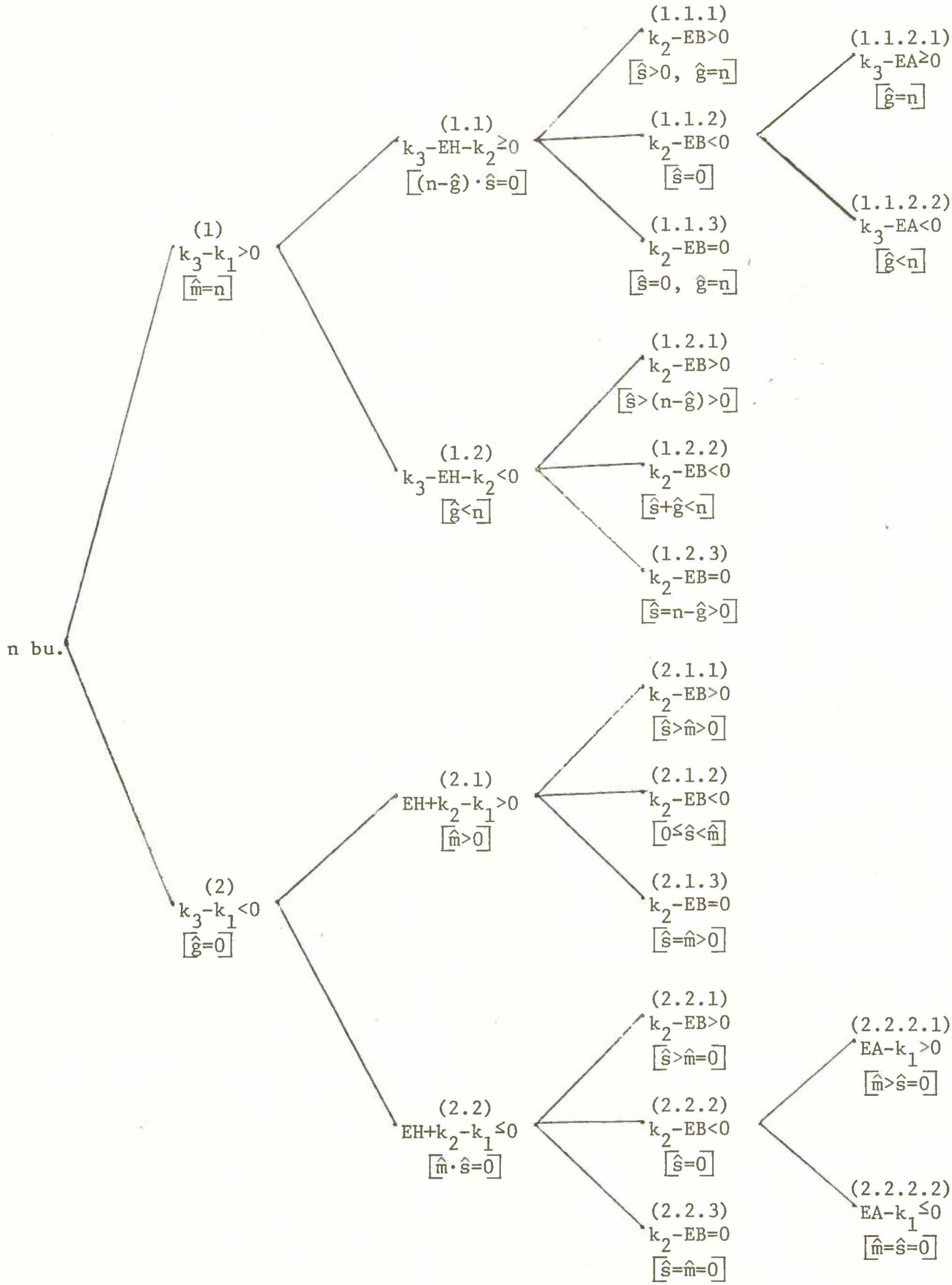
Alternatively, if $k_3-k_1 < 0$, expected utility is maximized by making m as small as possible, namely $\hat{m} = \hat{w}$ (see the restrictions associated with (3.1)), so $k_3-k_1 < 0 \Rightarrow m = \hat{w}$. Since $w=m-g$, this can also be stated:

$$(3.4)\quad k_3-k_1 < 0 \Rightarrow \hat{g} = 0.$$

These and other results are conveniently compiled in the "conclusion tree" on page 4. At each branch or terminal there is an identifying position number in parentheses. Below the position number is a piece of information and below the information, in square brackets, is a conclusion justified by that

2. In almost any applied decision problem, there will be some aspects of the decision-maker's future wealth that are not affected by the current decision-fluctuations in the value of assets he is not considering selling, accidental personal liabilities or windfalls, etc. Elsewhere {6, page 101} I have called these his initial prospect and the aspects that are affected by current decisions a venture. Denoting the initial prospect by a random variable X and the venture by a random variable Y, the decision-maker's problem is to choose a venture to maximize the expected utility $E\phi(X+Y)$ where ϕ is his utility of wealth function. If X and Y are stochastically independent, this is equivalent to choosing a venture to maximize $E\psi(Y)$ where $\psi(y) = E\,\phi\,(X+y)$. In {6}, I called ψ utility *after* gain but utility *of* gain now seems less awkward.

Conclusion Tree



information. H is defined as A−B and EA, EB, EH are the respective means of A, B, H. Reading back from any terminal, one sees the conclusions about the optimal choice $(\hat{m}, \hat{s}, \hat{g})$ that are implied by certain parcels of information. For example, reading back from (1.1.2.2) one sees that if $k_3-k_1 > 0$, $k_3-EH-k_2 > 0$, $k_2-EB < 0$, and $k_3-EA < 0$; then $\hat{m} = n$, $\hat{s} = 0$ and $\hat{g} < n$. To determine $\hat{g}$ exactly one would have to know the decision-maker's utility function and subjective probability distribution of A, B, H more completely.

(3.3) and (3.4) above are restated at positions (1) and (2), respectively. To consider the justification for conclusions at other positions, three propositions are useful. Let:

(3.5) $f(\alpha) = E\psi(X+\alpha Y)$

where X, Y are random variables: α is a real variable; and ψ is increasing and strictly concave. Suppose f can be differentiated under the expectation and attains its unique maximum at $\alpha = \hat{\alpha}$. Then:

(i) $(\hat{\alpha}-\alpha) \overset{s}{=} f'(\alpha) \ \forall \alpha \epsilon R$

(ii) $f'(\alpha) = EY\psi'(X+\alpha Y) = (EY)\,(E\psi'(X+\alpha Y)) + Cov(Y,\psi'(X+\alpha Y))$
= expectation term + covariance term.

Note that since $\psi'>0$, expectation term $\overset{s}{=}$ EY. The third proposition will sometimes help determine the sign of the covariance term.

(iii) Suppose Z, W, V are independent random variables and $R = r(Z,W)$, $Q = q(Z,V)$ where r and q are strictly monotonic in their first arguments. If second moments of R and $\psi'(Q)$ exist, then $cov(R, \psi'(Q))$ is negative if r,q are of the same monotonicity (both increasing or both decreasing) and $cov(R, \psi'(Q))$ is positive if r,q are of opposite monotonicity.

Proofs:

(i) may be seen from the graph of a strictly concave, non-monotonic function as in Figure 1.

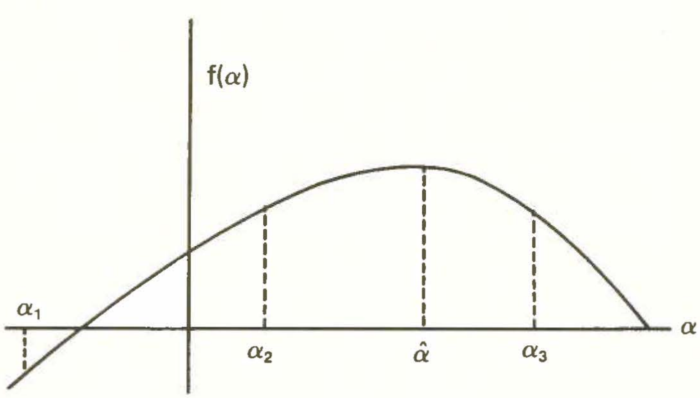


Figure 1.

$f'(\alpha_1) \overset{s}{=} (\hat{\alpha}-\alpha_1)>0$, $f'(\alpha_2) \overset{s}{=} (\hat{\alpha}-\alpha_2)>0$, $f'(\alpha_3) \overset{s}{=} (\hat{\alpha}-\alpha_3)<0$

(ii) By definition the covariance of any two random variables G, H with finite second moments is $cov(G,H) = E[(G-EG)\,(H-EH)] = EGH-(EG)\,(EH)$.

(iii) is proved in {7, page 385}.

Recall, Section 2, page 3, that $k_3 = c$ the price at which a particular farmer can sell for τ-delivery and $k_1 = ra+d$ is the opportunity cost of keeping a bushel in storage until τ. Thus k_3-k_1 is the advantage of concurrently selling forward and storing as opposed to selling now. k_3-k_1 will sometimes be called the return to *forwarded storage.*

Suppose $k_3-k_1 < 0$. This places the decision-maker at position (2) and justifies his setting $\hat{g} = 0$. Expected utility becomes a function of the remaining decision variables, m and s, and can be written:

(3.6) $\mu(m,s)=\eta(m,s,0) = E\psi[(A-k_1)m+(k_2-B)s]$
$0 \leqslant s,\ 0 \leqslant m \leqslant n$

It will again be convenient to make a change of variable. Let $v = m-s$ be the part of stored grain (possibly negative) that is not hedged. Rewrite expected utility:

(3.7) $\nu(m,v)=\mu(m,m-v) = E\psi[(H+k_2-k_1)m + (B-k_2)v]$
$0 \leqslant m \leqslant n,\ v \leqslant m$

where $H = A-B$ is the farmer's basis (see {3} for a general discussion of basis). The rationale for futures hedging is that A and B tend to fluctuate together (because, among other things, of the possibility of arbitrage between cash and futures markets) so fluctuations in H tend to be smaller[3] and H is more predictable. Someone who stores unhedged grain until τ receives a return $(A-k_1)$ which depends on the random variable A, which is relatively unpredictable. Someone who stores grain hedged by a short futures position receives a return $(H + k_2-k_1)$ which depends on the random variable H and is more predictable. Hedgers in futures markets are thus said to be "gambling on the basis."

B, the futures market quotation, is determined by national and international supply and demand. H, the basis, is determined in a particular locality by such factors as transporation and handling costs, quality premiums or discounts, and circumstances of local supply and demand. These seem sufficiently unconnected that one might consider treating H and B as statistically independent in a first approximation. This has been done in deriving the "conclusion tree." However, it is shown in Appendix B that a simple modification of the model which does not change any important formal features may be used to accommodate dependence.

3. For example, for the data on pages 13, the sample variances of H, A, B are respectively .01, .41, .49 for corn and .04, 5.6, 6.4 for soybeans.

Observe that increasing m with v held constant in (3.7) means that m and s are increased together. The farmer is simultaneously putting a bushel in storage and increasing his short position on the futures market by a bushel. Accordingly, the coefficient of m, $H + k_2 - k_1$, is the return to a stored and hedged bushel. It is the sum of the return to storing, $A - k_1$, and the return to futures hedging, $k_2 - B$. $H + k_2 - k_1$ will sometimes be called the return to *futured storage.*

Temporarily suppose that $\hat{v}$ is known in (3.7). Then $\nu(m,\hat{v})$ is a function of a single variable, and has the form of $f(\alpha)$ defined by (3.4), page 3 if we identify α with m, X with $(B-k_2)\hat{v}$ and Y with $(H + k_2 - k_1)$. Then:

$$(3.7) \quad D_m\nu(m,\hat{v}) = E(H+k_2-k_1)\psi'\left[(H+k_2-k_1)m + (B-k_2)\hat{v}\right]$$

$$= \text{expectation term} + \text{covariance term}$$

$$\text{expectation term} \overset{s}{=} EY = EH + k_2 - k_1$$

$$\text{covariance term} = \mathrm{Cov}(R,\psi'(Q)) \text{ where}$$

$$R = H+k_2-k_1,\ Q = (H + k_2 - k_1)m + (B-k_2)\hat{v}$$

First note that if $m = 0$, R and Q are independent (since B, H ind.); therefore, R and $\psi'(Q)$ are independent and covariance term = 0. If $m > 0$ both R and Q are increasing functions of H and, by (iii), page 5, covariance term <0.

Thus, if $EH + k_2 - k_1 \leq 0$, then $D_m \nu \leq 0$ for all admissible values of m (m must be between max $\{0,\hat{v}\}$ and n) and $D_m\nu<0$ for $m>0$. Expected utility is maximized by making m as small as admissible, *i.e.*:

$$(3.8) \quad EH+k_2-k_1 \leq 0 \Rightarrow \hat{m}=\max\{0,\hat{v}\}.$$

In terms of the original decision variables, $\hat{m} = \hat{v} \Rightarrow \hat{s} = 0$, so (3.8) says that either $\hat{m}$ or $\hat{s}$ must be zero if $EH + k_2 - k_1 \leq 0$. This can be stated as:

$$(3.9) \quad EH+k_2-k_1 \leq 0 \Rightarrow \hat{m}\cdot\hat{s}=0.$$

If $EH + k_2 - k_1 > 0$, then $D_m\nu(0,\hat{v})>0$ and there are positive values of m which make expected utility larger than at $m = 0$. We can thus say:

$$(3.10) \quad EH+k_2-k_1 > 0 \Rightarrow \hat{m}>0.$$

(3.9) and (3.10) furnish conclusions for positions (2.2) and (2.1) of the "conclusion tree."

The decision-maker (dm) could continue to look at particular relations among k_1, k_2, and k_3 and EH, EA, EB and draw qualitative conclusions about his optimal choice. The arguments for other conclusions are similar to those involved for (3.9) and (3.10) and are sketched in Appendix A.

One can visualize a decision-maker moving from left to right on the "conclusion tree," across the path that corresponds to his data and beliefs. He draws successive conclusions which, by the end of the path, represent the most that can be said about his optimal choices without using additional information.

The conclusions at the various positions seem to the author to have plausible interpretations. For example, looking at $k_3 - k_1 > 0$ one knows selling forward is better than selling now so selling now can be eliminated — everything is stored. What to do about hedging forward and/or futures? Note that $k_3 - EH - k_2$ is the expected advantage of forwarded storage $(k_3 - k_1)$ over futured storage $(EH + k_2 - k_1)$. If there is a positive advantage one favors selling forward and decides there will be no recourse to the futures market until and unless the whole supply is sold forward $((n-\hat{g})\cdot\hat{s}=0)$. One then looks at expected return to futuring $(k_2 - EB)$ to settle whether $\hat{s} > 0$ or $\hat{s} = 0$. Note that $\hat{s} > 0$, $\hat{g} = n$ implies overhedging or speculation since $\hat{s} + \hat{g} > n$. However this only occurs if the decision-maker expects there will be a positive return to a short futures position between harvest and the seasonal peak, an unusual circumstance.

If $\hat{s} = 0$, position (1.1.2), dm looks at expected return to forwarding to decide whether to sell the entire supply forward, position (1.1.2.1), or to leave some unhedged as in position (1.1.2.2). Other paths can be comparably interpreted.

The parcels of information used to construct the "conclusion tree" do not have to be examined in any particular order. There is, however, a considerable simplification in comparing k_3 with k_1 at the first step. Beyond this, some procedures would clearly involve unnecessary complications but, as far as the author can see, the exact order is partly arbitrary.

To form a very rough and preliminary notion of which positions and paths on the "conclusion tree" might be empirically relevant, some price data for crop years 1963-76 were obtained from the Farmers' Elevator Company, Stewartville, Minnesota. These were compiled with Board of Trade futures quotations and used to estimate other variables of the model. Terminal positions corresponding to the estimated data for each crop year were then obtained.

The results are listed in Table 1 (corn) and 2 (soybeans).

Table 1 shows average price of corn at Stewartville (based on Wednesday closings) during the harvest month, November, and the month of the usual seasonal peak price, June, for crop years 1963-76 (columns headed a and A). Also shown are the corresponding Board of Trade quotations for July corn at harvest and in June (columns headed b and B). k_1, the opportunity cost of holding corn from November to June, is approximated as $1.05a + .07$. This allows for seven months' interest at a 9 percent annual rate and 7¢ storage costs.[4] k_2 is the value to the short seller of a bushel of July corn. It is the Board of Trade quotation less 1.5¢ allowance for interest on a 30¢ margin and .5¢ commission.

In the absence of specific information on expectations, the tabulated (realized) values of A, B, H were used for EA, EB, EH in determining terminals. In effect, this would say the mean of the farmer's distribution is a perfect forecast — clearly false.

4. The typical rate for commercial grain storage is 2¢/bushel/month. On farm should be substantially less.

Table 1. Terminals corresponding to Stewartville data—corn.
(prices in $/bushel)

Crop year	a	b	A	B	k_1	k_2	H	$H+k_2-k_1$	k_2-B	$A-k_1$	Terminal	k_3	k_3-H-k_2	k_3-A	Terminal*
1963	1.01	1.27	1.13	1.23	1.13	1.25	−.10	.02	.02	0	2.1.1	1.18	.03	.05	1.1.1
1964	1.09	1.29	1.23	1.32	1.21	1.27	−.09	−.03	−.05	−.02	2.2.2.1	1.26	.08	.03	1.1.2.1
1965	1.05	1.23	1.21	1.29	1.17	1.21	−.08	−.04	−.08	.04	2.2.2.1	1.22	.09	.01	1.1.2.1
1966	1.25	1.50	1.26	1.32	1.38	1.48	−.06	.04	.16	−.12	2.1.1	1.44	.02	.18	1.1.1
1967	.98	1.23	1.07	1.14	1.10	1.21	−.07	.04	.07	−.03	2.1.1	1.14	0	.07	1.1.1
1968	1.02	1.24	1.15	1.28	1.14	1.22	−.13	−.05	−.06	−.01	2.2.2.1	1.19	.10	.04	1.1.2.1
1969	1.00	1.18	1.13	1.32	1.12	1.16	−.19	−.15	−.16	−.01	2.2.2.1	1.16	.19	.03	1.1.2.1
1970	1.23	1.47	1.33	1.54	1.36	1.45	−.21	−.12	−.09	−.03	2.2.2.2	1.41	.17	.08	1.1.2.1
1971	.94	1.15	1.06	1.21	1.06	1.13	−.15	−.08	−.08	0	2.2.2.2	1.10	.12	.04	1.1.2.1
1972	1.12	1.37	1.93	2.30	1.25	1.35	−.37	−.27	−.95	.68	2.2.2.1	1.30	.32	−.63	1.1.2.2
1973	2.14	2.59	2.62	2.86	2.32	2.57	−.24	−.01	−.29	.30	2.1.2	2.41	.08	−.21	1.1.2.2
1974	3.27	3.67	2.69	2.81	3.50	3.65	−.12	−.08	.84	−.81	2.1.1	3.64	.11	.95	1.1.1
1975	2.34	2.69	2.66	2.95	2.53	2.67	−.29	−.15	−.28	.13	2.2.2.1	2.63	.25	−.03	1.1.2.2
1976	2.11	2.62	2.06	2.35	2.29	2.60	−.29	−.02	.25	−.23	2.1.1	2.38	.07	.32	1.1.1

a: Average of Wednesday closing cash prices of corn at Stewartville during November of crop year (1963 crop year is November, 1963, to October, 1964, etc.).
b: Average of Wednesday closing quotations for July corn on Chicago Board of Trade during November of crop year.
A: Average of Wednesday closing cash prices of corn at Stewartville during June of crop year.
B: Average of Wednesday closing quotations for July corn on Chicago Board of Trade during June of crop year.
$k_1 = 1.05a + .07$
$k_2 = b - .02$
$H = A - B$
$k_3 = 1.04k_1$

Table 2. Terminals corresponding to Stewartville data—soybeans.
(prices in $/bushel)

Crop year	a	b	A	B	k_1	k_2	H	$H+k_2-k_1$	k_2-B	$A-k_1$	Terminal	k_3	k_3-H-k_2	k_3-A	Terminal*
1963	2.60	2.89	2.34	2.50	2.80	2.86	−.16	−.10	.36	−.46	2.2.1	2.91	.21	.57	1.1.1
1964	2.51	2.78	2.75	2.90	2.71	2.75	−.15	−.11	−.15	−.04	2.2.2.1	2.82	.22	.07	1.1.2.1
1965	2.29	2.58	3.14	3.34	2.47	2.55	−.20	−.12	−.79	−.67	2.2.2.1	2.57	.22	−.57	1.1.2.2
1966	2.75	3.04	2.70	2.85	2.96	3.01	−.15	−.10	.16	−.26	2.2.1	3.08	.22	.38	1.1.1
1967	2.43	2.75	2.53	2.68	2.62	2.72	−.15	−.05	.04	−.09	2.2.1	2.72	.15	.19	1.1.1
1968	2.38	2.63	2.47	2.65	2.56	2.60	−.18	−.14	−.05	−.09	2.2.2.2	2.66	.24	.19	1.1.2.1
1969	2.19	2.57	2.59	2.77	2.37	2.54	−.18	−.01	−.23	.22	2.2.2.1	2.46	.10	−.13	1.1.2.2
1970	2.72	3.11	2.97	3.19	2.93	3.08	−.22	−.07	−.11	.04	2.2.2.1	3.05	.19	.08	1.1.2.1
1971	2.91	3.29	3.31	3.48	3.13	3.26	−.17	−.04	−.22	.18	2.2.2.1	3.26	.17	−.05	1.1.2.2
1972	3.09	3.52	9.93	10.08	3.41	3.49	−.90	−.82	−7.34	6.52	2.2.2.1	3.55	.96	−6.38	1.1.2.1
1973	5.27	6.00	5.17	5.56	5.60	5.97	−.39	−.02	.41	−.43	2.2.1	5.82	.24	.65	1.1.2.1
1974	8.01	8.93	4.90	5.10	8.48	8.90	−.20	.22	3.80	−3.58	2.1.1	8.82	.12	3.92	1.1.1
1975	4.90	5.64	6.01	6.32	5.61	5.61	−.31	−.31	−.71	.40	2.2.2.1	5.83	.53	−.18	1.1.2.2
1976	6.04	6.65	8.09	8.35	6.62	6.62	−.26	−.26	−1.73	1.47	2.2.2.1	6.88	.52	−1.21	1.1.2.2

a: Average of Wednesday closing cash prices of soybeans at Stewartville during October of crop year (1963 crop year is October, 1963, to September, 1974, etc.).
b: Average of Wednesday closing quotations for July soybeans on Chicago Board of Trade during October of crop year.
A: Average of Wednesday closing cash prices of soybeans at Stewartville during June of crop year.
B: Average of Wednesday closing quotations for July soybeans on Chicago Board of Trade during June of crop year.
$k_1 = 1.05a + .07$
$k_2 = b - .03$
$H = A - B$
$k_3 = 1.04k_1$

However, it could still be true that frequencies of terminals indicated by this extreme assumption would crudely resemble frequencies based on actual expectations.

For each crop year two terminals were determined — one based on the assumption that the farmer in question did not have a favorable opportunity to sell forward and the other (marked by an asterisk) assuming that there was a favorable opportunity. For present purposes, a favorable opportunity is taken to be opportunity cost plus 4 percent, $1.04k_1$ in the present notation. This is just a guess at the price at which a buyer might typically be able to arrange a substantial volume. These amounts are shown in the column headed k_3.

Table 2 presents comparable data and terminals for soybeans based on Stewartville figures. Harvest is taken to be in October and the usual 50¢ margin requirement for soybeans makes $k_2 = b-.03$. Otherwise, the tabulation is similar.

Taking the corn and soybean data together, the frequencies of various terminals are:

2.2.2.1 −14
2.1.1 −6
2.2.1 −4
2.2.2.2 −3
2.1.2 −1

when a favorable forward sale is presumed not to exist and:

1.1.2.1 −10
1.1.1 −9
1.1.2.2 −9

when a favorable forward sale does exist.

This is, to be sure, a very qualified indication of possible empirical relevance. We need to carefully study expectations and expectation formation and we need to consult data for a variety of crops and locations.

4. Exact Choices Under Alternative Assumptions

If an investigator assumes an exact utility function and a specific distribution for unknown prices (in addition to data normally known at the time a decision is made), then corresponding optimal choices can be exactly calculated. Even though we do not yet know much about empirical utilities and expectations, it seems desirable to acquire experience with such calculations. If highly implausible results are encountered, the assumptions are reexamined. Sometimes the nature of the implausibilities furnish hints as to probable sources. We can begin to form some judgments as to how well particular families of utility functions and probabilities may permit us to approximate actual decision patterns.

Results for several combinations of utility functions and probability distributions are given in this section. Three utility functions have been considered. One is a linear utility function which has been included for casual comparisons; results using this function will not be examined in detail. The others are chosen to illustrate constant absolute risk aversion and decreasing absolute risk aversion.

If $\psi(y)$ is a utility of gain function, then:

$$(4.1)\quad \rho(y) = -\frac{\psi''(y)}{\psi'(y)}$$

is called absolute risk aversion. Pratt {10} showed that constant absolute risk aversion (CAR) implies a function of essentially (except for positive linear transformations) the form:

$$(4.2)\quad \psi(y) = -e^{-\lambda y} \qquad \lambda > 0$$

where $\rho(y) = \lambda$. He also showed that a sum of constant absolute risk functions (denoted here as SCAR) has the property that absolute risk aversion decreases with y. Economists have generally considered decreasing absolute risk aversion a plausible hypothesis.

The CAR used in Table 5 and 6 is:

$$(4.3)\quad \psi(y) = -e^{-.00006y}$$

For this function: $\rho(y) = .00006 \qquad -\infty < y < \infty.$

The SCAR employed is:

$$(4.4)\quad \psi(y) = -e^{-.00001y} - .018e^{-.0002y}$$

It can readily be verified that $\rho(y)$ is decreasing and:

$$(4.5)\quad \lim_{y\to\infty} \rho(y) = .00001,\ \lim_{y\to-\infty} \rho(y) = .0002,\ \rho(0) = .00006.$$

Some interesting properties of SCAR are examined in Appendix C. The chosen levels of risk aversion correspond roughly to levels indicated by recent responses of a panel of Minnesota farmers to hypothetical problems involving risk.[5]

Optimal choices were calculated for eight situations involving corn crops and eight situations involving soybean crops. The Stewartville data from Tables 1 and 2 of Section 3 were used as a guide in constructing situations. Assumed data for corn marketing situations are shown in Table 3. Variables are defined in Section 2, pages 2 and 3. Entries for situation I were obtained as follows:

a = (average price of corn at Stewartville during November) × 1.77

b = (average price in November of a Chicago corn futures contract maturing in July) × 1.77

EA = (average price of corn at Stewartville during June) × 1.77

EB = (average June price in June of a Chicago corn futures contract maturing in July) × 1.77

EH = EA − EB

5. The responses were highly variable and subject to some interpretation so this preliminary impression should not be taken very seriously. Interviews with the panel were supported by the Economics, Statistics, and Cooperatives Service. Analysis of the interviews will be prepared.

$k_1 = 1.05a + 7$

$k_2 = b - .05 \times 30 - \frac{1}{2}$

$k_3 = 1.04k_1$ for I-III, $1.08k_1$ for IV.

The 1963-76 average prices were multiplied by 1.77 to obtain numbers closer to recent levels. In using adjusted historical averages for EA, EG, EH in situation I we are supposing that a farmer's expectations might sometimes reflect recent historical experience with regard to price relations and current experience with regard to the general level. In situation II, lower future prices are expected; EA and EB have been lowered 4 percent, other entries are unchanged. Situation III reflects 4 percent higher expected future prices than I and, in situation IV, current local price has been lowered 4 percent (which also lowers k_1). The latter might, for instance, represent a situation in which transportation shortages at harvest lower local price below its usual relation to the central market.

As in Section 3, it seemed worthwhile to consider a farmer who has a favorable opportunity for forward sales and one who does not. Therefore, two versions of each of the four initial situations were explored. For the first version, it was assumed that $k_3 < k_1$. For the second version, the listed value of k_3 was used. This is 4 percent above the original value of k_1 and was judged to be a price at which a local buyer should be able to obtain a substantial supply. The second version ($k_3 > k_1$) of each situation is denoted by an asterisk in Table 5.

Table 4 indicates four comparable situations for a soybean grower. Situation I is based on average 1963-76 prices increased by a factor of 2.15. Situation II is obtained from I by lowering EA, EB 4 percent, situation III by raising EA, EB 4 percent, and situation IV by lowering a and k_1 4 percent below their levels in situation I. Optimal decisions for soybeans are listed in the lower half of Table 5. As with corn, an asterisk indicates a situation that includes a favorable opportunity to sell forward.

To compute a specific set of optimal decisions, it is also necessary to assume a specific distribution to represent the decision-maker's subjective beliefs about probable values of the unknown prices A, B. In the absence of good information on expectations, two conventional forms were tried — normal and uniform (the latter has density $\frac{1}{x_2 - x_1}$ in the range (x_1, x_2) and 0 elsewhere). In the univariate case, both are two parameter families so the distribution is completely specified if we have the mean and variance of the random variable.

The means used in different calculations have already been listed in Tables 3 and 4. To obtain roughly corresponding variances, the Stewartville data were again consulted. It seems reasonable to suppose that known values of a and b are combined with other information to form expectations about A and B. This suggested the following procedure which the author recognizes as largely arbitrary and in need of revision after further study. For corn, the variance of residuals in the regression of B on a, b for 1963-76 is about .12. Allowing for other information, this was rounded down to .08. The sample variance of H was .0093 which was rounded to .01. Continuing to treat B, H as statistically independent[6] implies a variance of .009 for A and a covariance of .008 for A and B. The comparable figures for soybeans are Var B = .20, Var H = .04, Var A = .24, Cov AB = .20. All of the calculations reported in this paper involved the above variances and covariances.

In Table 5, the total supply of grain (n) is taken to be 50,000 bushels. In more than half of the 64 cases examined, at least one of the decision variables equals this upper limit. To provide some additional comparisons of the effects of varying the underlying assumptions, some calculations were made with n = 400,000. The results appear in Table 6. Only the eight situations in which one or more decision variables are changed by the increased supply are shown.[7]

Several comparisons of results in Tables 5 and 6 are of interest. A number of patterns are expected and realized. Amount stored changes in the same direction as expected future prices and opposite to changes in current prices. Changing from normally distributed future prices to uniformly distributed future prices increases willingness to take changes, *i.e.,* to hold unhedged inventories in the typical cases where hedging involves a loss of expected return and to overhedge more in the few cases (situations CII, CII*) when hedging increases expected return.

This conservative implication of normality is not surprising because the combination of a utility function (either CAR or SCAR) that is unbounded below and an unbounded probability distribution (normal) holds the possibility that extremely large losses which a farmer would regard as impossible (because he regards prices as nonnegative) will enter the calculations with sufficient weight to influence the calculated decision.

The fact that changing from normal to uniform, with other assumptions unchanged, changes the calculated value of the decision variable by more than 50 percent in 14 of the 44 cases in which comparisons can be made suggests that this is a nonnegligible consideration. It has sometimes been suggested that bias due to unboundedness of normal variables will be small if the coefficients of variation (ratios of means to standard deviations) are large. The coefficients of variation of A and B in the situations examined vary from 9 to 18, which would ordinarily be considered large. This problem should be explored further in other contexts, both theoretically and with more adequate data.[8]

6. Note that when B, H are assumed independent and uniformly distributed, A is not exactly uniform but has a density of the shape ⏢.
7. Of course, changes under linear utility change with n.
8. One could alternatively bound the utility function and this may sometimes seem more plausible. In the present context, price and return variables bounded from below are natural and SCAR utility fns have potentially useful properties (see Appendix C).

Table 3. Assumed situations—corn. ($/bushel)

Variables	I Adjusted averages	II Lower expected prices	III Higher expected prices	IV Lower local prices
a	2.60	2.60	2.60	2.50
b	3.10	3.10	3.10	3.10
EA	2.83	2.72	2.94	2.83
EB	3.15	3.04	3.26	3.15
EH	−.32	−.32	−.32	−.32
k_1	2.80	2.80	2.80	2.70
k_2	3.08	3.08	3.08	3.08
k_3	(2.91)	(2.91)	(2.91)	(2.91)

Table 4. Assumed situations—soybeans. (¢/bushel)

Variables	I Adjusted averages	II Lower expected prices	III Higher expected prices	IV Lower current price
a	7.00	7.00	7.00	6.72
b	7.88	7.88	7.88	7.88
EA	7.89	7.57	8.21	7.89
EB	8.34	8.02	8.66	8.34
EH	−.45	−.45	−.45	−.45
k_1	7.42	7.42	7.42	7.13
k_2	7.85	7.85	7.85	7.85
k_3	(7.72)	(7.72)	(7.72)	(7.72)

Table 5. Optimal decisions. (entries in 000 bushel; n=50)

Crop and situation	Terminal	Linear utility ($\hat{m}$, $\hat{s}$, $\hat{g}$)	Normally distributed prices						Uniformly distributed prices					
			CAR			SCAR			CAR			SCAR		
			$\hat{m}$	$\hat{s}$	$\hat{g}$	$\hat{m}$	$\hat{s}$	$\hat{g}$	$\hat{m}$	$\hat{s}$	$\hat{g}$	$\hat{m}$	$\hat{s}$	$\hat{g}$
CI	2.2.2.1	(n,0,0)	5.56	0	0	5.45	0	0	5.57	0	0	5.56	0	0
II	2.2.1	(0,∞,0)	0	8.33	0	0	8.08	0	0	8.37	0	0	8.38	0
III	2.2.2.1	(n,0,0)	25.9	0	0	21.9	0	0	27.1	0	0	28.6	0	0
IV	2.1.2	(n,0,0)	50.0	35.4	0	50.0	36.0	0	50.0	35.0	0	50.0	31.8	0
I*	1.1.2.1	(n,0,0)						same as linear						
II*	1.1.1	(n,∞,n)	50.0	8.33	50.0	50.0	14.0	0	50.0	8.37	50.0	50.0	15.3	50.0
III*	1.1.2.2	(n,0,0)	50.0	0	44.4	50.0	0	40.1	50.0	0	44.4	50.0	0	39.6
IV*	1.1.2	(n,0,n)						same as linear						
SI	2.2.2.1	(n,0,0)	32.6	0	0	23.6	0	0	41.2	0	0	50.0	0	0
II	2.2.2.1	(n,0,0)	10.4	0	0	9.43	0	0	10.6	0	0	10.9	0	0
III	2.2.2.1	(n,0,0)	50.0	0	0	35.7	0	0	50.0	0	0	50.0	0	0
IV	2.1.2	(n,0,0)	50.0	9.17	0	50.0	37.7	0	50.0	0	0	50.0	0	0
I*	1.1.2.2	(n,0,0)	50.0	0	38.2	50.0	0	27.1	50.0	0	37.9	50.0	0	16.9
II*	1.1.2.1	(n,0,n)						same as linear						
III*	1.1.2.2	(n,0,n)	50.0	0	16.0	50.0	0	14.3	50.0	0	5.88	50.0	0	0
IV*	1.1.2.2	(n,0,n)	50.0	0	38.2	50.0	0	17.8	50.0	0	37.9	50.0	0	0

Table 6. Optimal decisions. (entries in 000 bushel; n=400)

Crop and situation	Normally distributed prices						Uniformly distributed prices					
	CAR			SCAR			CAR			SCAR		
	$\hat{m}$	$\hat{s}$	$\hat{g}$	$\hat{m}$	$\hat{s}$	$\hat{g}$	$\hat{m}$	$\hat{s}$	$\hat{g}$	$\hat{m}$	$\hat{s}$	$\hat{g}$
CIV	100.0	85.4	0	78.7	67.2	0	109.7	93.7	0	121.6	103.9	0
II*	400.0	8.33	400.0	400.0	47.9	400.0	400.0	8.37	400.0	400.0	49.9	400.0
III*	400.0	0	394.4	400.0	0	366.7	400.0	0	394.4	400.0	0	366.7
SIII	54.9	0	0	35.7	0	0	284.8	0	0	400.0	0	0
IV	112.5	71.7	0	72.3	46.0	0	400.0	228.2	0	400.0	97.4	0
I*	400.0	0	388.2	400.0	0	334.6	400.0	0	387.9	400.0	0	327.4
III*	400.0	0	366.0	400.0	0	322.6	400.0	0	355.9	400.0	0	135.3
IV*	400.0	0	388.2	400.0	0	329.2	400.0	0	387.9	400.0	0	327.4

In forming impressions of changes in values of decision variables due to changes in utility functions and distribution functions, it should be recognized that, in the present context, most of the determinations can be made independently of these considerations. Of the 288 values of decision variables appearing in Tables 5 and 6, 194 are completely determined by the "conclusion tree."

When the utility function is changed from CAR to SCAR one again gets a variety of small and large changes in values of decision variables with changes as large as 50 percent occurring in about one third of the cases not determined by the "conclusion tree." In this context, there is a suggestion that constant absolute risk aversion may not provide a satisfactory approximation.

5. Some Comments

As developed in Section 2, the model studied in this paper is too narrow to cover some important grain marketing decisions. There are some natural reinterpretations that can extend its relevance a little, but it also needs substantial formal extensions. Hopefully the analysis so far will aid in their construction.

An operating farmer has more options than appear in the model. It is not uncommon for some marketing decisions and contracts to be made during the growing period. Decisions may also be made after harvest. In particular, futures positions may be modified or closed at any time and uncommitted grain may be sold from storage. Various things happen to make harvest an inevitable time of decision — the grain must be physically handled, financial obligations sometimes mature, the size and quality of the farmer's crop becomes definite, estimates for the region and nation become more precise, crop information gets incorporated into quoted prices, etc. But decisions at other times are also important and a multi-period model is indicated to take them into account.

Some flexibility can be achieved in the present model by relaxing the assumption that all contracts are closed at time τ. This was used in including interest charges and in getting some data to make preliminary calculations. However, we could interpret A and B as prices realized by the farmer whenever he sells grain (A) or repurchases futures (B) regardless of the timing. This would implicitly assume that he has a strategy for post-harvest decision and a subjective probability distribution of possible outcomes (effective prices to be realized) of applying the strategy. The model presented would not tell us what this post-harvest strategy should be, but would indicate how harvest decisions would fit in, given such a strategy. Final dispositions at some predetermined time would not be required.

Interest calculations would become less straightforward and would, in principle, have to be treated as an additional uncertainty. However, within the marketing season, interest uncertainty may often be of sufficiently secondary importance that an estimated interest charge treated as certain would not make the model seriously misleading. It is partly a question of purposes a model is to serve. The above treatment of interest might provide a crude first approximation, but allowance for uncertainty in the availability and cost of credit should certainly be part of any program to produce successively more comprehensive models.

Another aspect of the farmer's situation that has been neglected so far is tax considerations. If we assume that utility is a function of after-tax wealth and that tax liability is a smooth (differential) function of prospective gains and losses regardless of their nature, then we could either convert the utility function to a pre-tax basis or we could convert prospective gains and losses to an after-tax basis without changing basic assumptions. Alas, life is not this simple and taxes will have to be studied in some detail in the model improvement program.

Further trials with different combinations of utility functions and subjective probability distributions are to be encouraged. SCAR utility functions have some interesting properties (see Appendix C). Investigations of nonnormal distributions should continue. The gamma distribution seems a likely candidate when boundedness from below, but not necessarily from above, is desired. In cases where boundedness on both sides is desired one could approximate the subjective density by a simple function or consider a Beta distribution.

Such investigations should be accompanied by continuous empirical testing based on interviews, experiments, and analyses of actual decisions.

One aspect of farmers' decisions that will have to be studied more intensively as empirical testing progresses is the relation between the farmer's prospect at the time he is observed (his general exposure to financial contingencies) and the ventures, real or contrived, that are under consideration when he is observed. The convenient relation between utility of wealth and utility of gain when SCAR functions are employed will be helpful if this type of function proves sufficiently flexible to represent actual behavior accurately.[9] The present paper contains the implicit assumption that the ventures considered (ways of marketing the grain) are statistically independent of the farmer's other prospects. This seems more likely to be true for a specialized grain farmer than for a grain-livestock farmer, but the matter should be carefully investigated in both cases.

Finally, nonfinancial considerations should be studied to see to what extent and how they may influence even those decisions that are primarily

9. A preliminary examination of this question will be undertaken using the panel data cited on page 8.

financial (see {5}, page 899). When the solutions of Section 4 were calculated, it was noted that there is a substantial region of near flatness of the expected utility function near the optimal choice. This would make it possible for a nonfinancial consideration to substantially change the optimum if combined with the utility function for gain. This is reminiscent of results in mathematical programming of farm decisions where it frequently happens that, in the vicinity of the optimum, a fairly substantial change in some inputs affects net revenue modestly.

Appendix A: Completion of the Conclusion Tree

In Section 3, conclusions for positions (1), (2), (2.1), (2.2) of the "conclusion tree" on page 4 were proved. To continue, recall that if $k_3 < k_1$, then $\hat{g} = 0$ and expected utility may be written as on page 5.

(3.5) $\mu(m,s) = E\psi[(A-k_1)m+(k_2-B)s]$ so,
$0 \leqslant s,\ 0 \leqslant m \leqslant n$
temporarily treating $\hat{m}$ as given,

(I.1) $D_s\mu = E(k_2-B)\psi'[B(\hat{m}-s)+H\hat{m}-k_1\hat{m}+k_2 s]$
$=$ expectation $+$ covariance
expectation $\overset{s}{=} k_2-EB$ and, by proposition (iii), page 5,
covariance $\overset{s}{=} (\hat{m}-s)$ Thus:

(I.2) $(k_2-EB)>0 \Rightarrow (D_s\mu>0 \text{ for } s\leqslant\hat{m}) \Rightarrow \hat{s}>\hat{m}$
$(k_2-EB)<0 \Rightarrow (D_s\mu<0 \text{ for } s\geqslant\hat{m}) \Rightarrow \hat{s}\leqslant\hat{m}$ and $\hat{s}<\hat{m}$ for $\hat{m}>0$
$(k_2-EB)=0 \Rightarrow D_s\mu \overset{s}{=} (\hat{m}-s) \Rightarrow \hat{s}=\hat{m}$

(I.2) justifies the conclusions at positions (2.1.1), (2.1.2), (2.1.3), (2.2.1), (2.2.2), (2.2.3).

Now suppose dm is at (2.2.2) where $\hat{g} = \hat{s} = 0$.

(I.3) $D_m\mu(m,0) = E(A-k_1)\psi'[(A-k_1)m]$
$=$ expectation $+$ covariance
expectation $\overset{s}{=} EA-k_1$
covariance $\overset{s}{=} -m$ so

(I.4) $EA-k_1 >0 \Rightarrow \hat{m}>0$
$EA-k_1 \leqslant 0 \Rightarrow \hat{m}=0$

These cover (2.2.2.1) and (2.2.2.2).

Let $k_3 > k_1$ and write:

$\eta(n,s,g) = E\psi[(A-k_1)n+(k_2-B)s+(k_3-A)g]$
$0 \leqslant s,\ 0 \leqslant g \leqslant n$

Let $v = g + s$, the total hedge, and let:

$\zeta(v,g) = \eta(n,v-g,g) = E\psi[(A-k_1)n+(k_2-B)v+(k_3-H-k_2)g]$
$0 \leqslant g \leqslant n,\ g \leqslant v$

Temporarily suppose $\hat{v}$ known:

(I.5) $D_g\zeta = E(k_3-H-k_2)\psi'[B(n-\hat{v})+H(n-g)-k_1 n+k_2\hat{v}+(k_3-k_2)g]$
$=$ expectation $=$ covariance
expectation $\overset{s}{=} k_3-EG-k_2$
covariance $\overset{s}{=} n-g$ so

(I.6) $k_3-EH-k_2 \geqslant 0 \Rightarrow \hat{g} = \min\{n,\hat{v}\}$
$k_3-EH-k_2 < 0 \Rightarrow n-\hat{g} > 0$

These justify (1.1) and (1.2) when we note that $\hat{g} = \hat{v} = \hat{s} = 0$. Temporarily setting $g = \hat{g}$,

(I.7) $D_s\eta(n,s,\hat{g}) = E(k_2-B)\psi'[B(n-s-\hat{g})+H(n-\hat{g})-kn+k_2 s+k_3\hat{g}]$
$=$ expectation $+$ covariance
expectation $\overset{s}{=} k_2-EB$
covariance $\overset{s}{=} n-s-\hat{g}$ so

(I.8) $k_2-EB > 0 \Rightarrow \hat{s} > n-\hat{g}$
$k_2-EB < 0 \Rightarrow \hat{s} \Rightarrow n-\hat{g}$
$k_2-EB = 0 \Rightarrow \hat{s} = n-\hat{g}$

(I.8) accounts for (1.1.1), (1.1.2), (1.1.3), (1.2.1), (1.2.2) and (1.2.3).

Now suppose dm is at (1.1.2). Observe that $\hat{s} = 0$.

(I.9) $D_g\eta(n,o,g) = E(k_3-A)\psi'[A(n-g)-k_1 n+k_3 g]$
$=$ expectation $+$ covariance
expectation $\overset{s}{=} k_3-EA$
covariance $\overset{s}{=} n-g$, therefore:

(I.10) $k_3-EA\geqslant 0 \Rightarrow \hat{g} = n$
$k_3-EA<0 = \hat{g}\ n$
as in (1.1.2.1) and (1.1.2.2).

One contingency that has not yet been covered is the remote possibility that $k_3 = k_1$, selling forward promises exactly the same return as selling now. In practice this might frequently be settled by some secondary consideration that did not enter our return function. dm might sell forward to maintain goodwill with the buyer hoping to be contacted early in years when forward opportunities are limited. Or, if he has any qualms at all about the security of his own storage or about the forward buyer's eventual performance, he might prefer to sell cash. Such secondary considerations could, of course, be explicitly entered into an extended model, but it seems that the extensions discussed in Section 5 are more important for the near future. Meanwhile, it is interesting to take a brief look at the situation created by $k_3=k_1$ in the present framework.

Write expected utility as:

(I.11) $\theta(m,s,w) = E\psi[(k_3-k_1)m + (k_2-B)s + (A-k_3)w]$
$= E\psi[(k_2-B)s + (A-k_3)w] = \Upsilon(s,w)$
$0 \leqslant s\ 0 \leqslant w \leqslant n$

Varying m while holding s and w constant does not affect the argument of ψ and, therefore, does not affect expected utility. Clearly, optimal choice is not generally unique if $k_3 = k_1$. Recall that $w = m-g$ and $0 \leqslant g \leqslant m$, so if $\hat{w} = n$, then $\hat{m} = n$, $\hat{g} = 0$. If $\hat{w} = 0$, then $\hat{m} = \hat{g} = 0$. However, if $0 < \hat{w} < n$, there is a range of variation for m, g (specifically $\hat{w} \leqslant m \leqslant n$ with $g = m - \hat{w}$) that corresponds to maximum expected utility.

Setting $m = n$ does not restrict the range of variation of s, w. Since, in this case, expected utility may be stated as a function of s, w; this means that no expected utility is lost if the decision-maker sets $m = n$ and then proceeds as from position (1). Alternatively he could set $g = 0$ and proceed as from position (2). Thus the "conclusion tree" is also relevant to $k_3 = k_1$ in that the decision-maker can proceed from either (1) or (2) without loss of expected utility.

Appendix B: Effects of Interdependence Between Price and Basis

Recall that the version of basis relevant to this analysis is the difference between the price a farmer can realize in his local market and the contemporary quotation for maturing or soon-to-mature futures contracts. At the time storage and hedging decisions are made these prices and, therefore, their difference, the basis, are random variables.

Factors that finally determine a particular producer's basis will vary from one farm situation to another, but cost of transportation to the relevant terminal market; quality of the farmer's grain; quality premiums and discounts in that particular year; and current circumstances of local demand and supply are likely to be dominant factors in many cases. These seem sufficiently independent of the general national and often international supply and demand forces determining the futures quotation that the statistical independence assumed so far may often be a good first approximation.

However, it also seems plausible that some dependence might occasionally exist—quality premiums or discounts might well have some tendency to increase when price is higher, local demand may be to some extent associated with total world demand although the association may typically be weak.

Pending careful empirical studies to determine the kind and importance of dependence, it seems useful to consider possible effects in a preliminary fashion. Recall that A is local cash price, B the concurrent futures quotation, and $H = A - B$, the basis. Instead of assuming H is independent of B, assume $H = \delta B + V$ with V independent of B.[10] δ is then the regression coefficient of H on B($\delta = \frac{\text{Cov} BH}{\text{Var} B}$). Then $A = \nu B + V$ with $\nu = (1 + \delta)$ the regression coefficient of A on B.

The observed fact that year to year fluctuations in a particular basis tend to be small relative to price fluctuations suggests $|\delta|$ should be quite a lot smaller than one and this is reinforced if, as is presumed, H and B are not highly correlated. ν is then close to one. The special case $\nu = 1$ corresponds to independence between B and H. Using the Stewartville data from Tables 1 and 2 yields estimates of $\delta = -.09$ for corn and $-.07$ for soybeans.

These revised assumptions require only modest revisions in the "conclusion tree." Recall (page 3):

$$(2.2)\quad \eta(m,s,g) = E\psi[(A-k_1)m + (k_2-B)s + (k_3-A)g]$$
$$0 \leq s,\ 0 \leq g \leq m \leq n.$$

Rewrite expected utility:

$$(II.1)\quad \bar{\eta}(m,\bar{s},g) = E\psi[(A-k_1)m + (\bar{k}_2-\bar{B})\bar{s} + (k_3-A)g]$$
$$0 \leq s,\ 0 \leq g \leq m \leq n$$

where: $\bar{k}_2 = \nu k_2$, $\bar{B} = \nu B$, $\bar{s} = \nu^{-1} s$.

10. If, as seems plausible, EV = 0 we have regression through the origin. For both corn and soybeans, regressions using the Stewartville data yielded negligible constant terms.

$\bar{\eta}$ satisfies all the qualitative assumptions of η and the same restrictions apply to the new variables. Therefore, all the conclusions stated for η also apply for $\bar{\eta}$ with the substitution of $\bar{k}_2$, $\bar{B}$, $\bar{s}$, V for, respectively, k_2, B, s, H in both conditions and conclusions.

For example, the following information and conclusions would apply to the indicated positions in a "conclusion tree" revised to take account of dependence between B and H.

(1.1) $k_3 - EV - \nu k_2 \geq 0 \Rightarrow (n-\hat{g}) \cdot \hat{s} = 0.$

(1.2) $k_3 - EV - \nu k_2 < 0 \Rightarrow \hat{g} < n$

(2.1) $EV + \nu k_2 - k_1 > 0 \Rightarrow \hat{m} > 0$

(2.2) $EV + \nu k_2 - k_1 \leq 0 \Rightarrow \hat{m} \cdot \hat{s} = 0$

(1.2.1) $\nu k_2 - EB > 0 \Rightarrow \nu^{-1}\hat{s} > (n-\hat{g}) > 0$

(1.2.2) $\nu k_2 - EB < 0 \Rightarrow \nu^{-1}\hat{s} + \hat{g} < n.$

V might reasonably be called the adjusted basis.

The relation between inferences that can be drawn under dependence and independence can be partly rationalized by noting that the unhedged storer is exposed to uncertainty represented by the random variable A. Forward selling (if he regards the buyer as completely reliable) lets him substitute a constant k_3 for A. Futures hedging lets him substitute a random variable $(A - \alpha B + \alpha k_2)$ for A where α is chosen by dm and represents the ratio of bushels sold short to bushels stored. The variance of the new random variable is minimized if:

$\alpha = \nu = \frac{\text{Cov } AB}{\text{Var } A}$ in which case $A - \nu B + \nu k_2 = V + \nu k_2$ (see Heifner {2}, page 28).

Appendix C: Some Useful Properties of SCAR

Flexibility and Differential Properties

Sums of constant absolute risk aversion utility functions have some interesting properties which will prove useful if it is found that actual decision-making can satisfactorily be represented by maximizing expectations of such functions. The chance that such representations can be found is enhanced by the fact that the system of SCAR functions is flexible. The number of free parameters available to approximate empirical utility functions can be increased indefinitely without changing basic properties of the function. Let:

$$(C.1)\quad \phi(x) = \sum_1^n a_i e^{-\lambda_1 x} \qquad \lambda_i > 0,\ a_i > 0\ i = 1 \cdots n$$

be such a function. The effective number of parameters is $2n-1$ since any a_i could be set equal to unity by multiplying ϕ by $\frac{1}{a_i}$ and $\frac{1}{a_i}\phi$ would be essentially the same utility function.

As n is increased, a number of desired properties — smoothness; positive, but decreasing slope; positive, but decreasing absolute risk aversion — remain. An investigator can use a function with as many terms as seem needed in his particular context. Thus it appears worthwhile to look at some other features of this family of functions and to experiment with it in practice.

Functions of Wealth and Gain

Let $\phi(x)$ represent a decision-maker's utility of wealth function (not necessarily SCAR) and let the random variable X represent his current prospect. His utility of gain function is then:

(C.2) $\psi(y) = E\phi(X + y)$.

If ϕ and X were known and well behaved, one could typically infer ψ. Reasoning from ψ to ϕ is more troublesome, particularly if X is not completely known. One context in which the latter problem arises is in studying utilities of real people by interviewing or by confronting them with experimental choices. These typically give direct evidence related to ψ while, for many purposes, ϕ is needed (for example, if X changes).

If ϕ is SCAR, as given by (C.1), things are simpler. Then:

$$\text{(C.3)}\quad \psi(y) = -E\sum_1^n a_i e^{-\lambda_i(X+y)} = -\sum_1^n (Ee^{-\lambda_i X}) a_i e^{-\lambda_i y}$$

$$= -\sum_1^n \theta_i a_i e^{-\lambda_i y} = -\sum_1^n c_i e^{-\lambda_i y}$$

which is again SCAR and is equal to ϕ with new weights attached to the exponential components. To infer ϕ from ψ, one needs the factors:

(C.4) $\theta_i = Ee^{-\lambda_i X}$.

It may sometimes be possible to approximate these even if X is not completely known. For example, if it is believed that X can be approximated by a random variable with moment generating function m, then θ_i should be approximately $m(-\lambda_i)$.

Bounds on Risk Aversion

Proposition C.1. Let: $\phi(x) = -\sum_1^n a_i e^{-\lambda_i x}$ $a_i > 0$, $\lambda_i > 0$ $i = 1 \cdot\cdot n$. Define: $r(x) = -\dfrac{\phi''(x)}{\phi'(x)}$.

Then: (a) $\min\{\lambda_i \mid i = 1 \cdot\cdot n\} < r(x) < \max\{\lambda_i \mid i = 1 \cdot\cdot n\}$ $\quad -\infty < x < \infty$

(b) $\lim_{x\to\infty} r(x) = \min\{\lambda_i \mid i = 1 \cdot\cdot n\}$

(c) $\lim_{x\to-\infty} r(x) = \max\{\lambda_i \mid i = 1 \cdot\cdot n\}$

Proof:

$$r(x) = -\frac{\sum_1^n a_i \lambda_i^2 e^{-\lambda_i x}}{\sum_1^n a_i \lambda_i e^{-\lambda_i x}} = \sum_1^n w_i \lambda_i$$

where: $w_i = \dfrac{a_i \lambda_i e^{-\lambda_i x}}{\sum_1^n a_i \lambda_i e^{-\lambda_i x}} > 0$

and: $\sum_1^n w_i = 1$. This proves (a).

$$\frac{w_i}{w_j} = \frac{a_i \lambda_i e^{-\lambda_i x}}{a_j \lambda_j e^{-\lambda_j x}} = \frac{a_i \lambda_i}{a_j \lambda_j} e^{(\lambda_j - \lambda_i)x}, \text{ hence:}$$

$$\lim_{x\to\infty} \frac{w_i}{w_j} \begin{array}{l} = \infty \text{ if } \lambda_j > \lambda_i \\ = 0 \text{ if } \lambda_j > \lambda_i \end{array}.$$

Thus if: $\lambda_{i^*} < \lambda_j$ for: $j \neq i^*$, $\lim_{x\to\infty} w_j = 0$, $\lim_{x\to\infty} w_{i^*} = 1$ and $\lim_{x\to\infty} r(x) = \lambda_{i^*}$.

This proves (b) . (c) is similar.

Proposition C.1 was stated in terms of the utility of wealth function $\phi(x)$, but clearly the mathematical result is independent of the interpretation. Thus, if absolute risk aversion for gain is defined by:

$$\rho(y) = -\frac{\psi''(y)}{\psi'(y)}$$

where: $\psi(y) = E\phi(X+y) = -\sum_1^n c_i e^{-\lambda_i y}$,

the conclusions stated for r(x) also hold for $\phi(y)$. This justifies the assertion on page 8 that for the function

$$\psi(y) = -e^{-.00001y} - .0183^{-.002y}$$

we have: $.00001 \leq \rho(y) \leq .0002$.

Bounds on Optimal Choice

Proposition C.2. Suppose:

$f(\alpha) = \sum_{i=1}^n c_i f_i(\alpha)$ where: $c_i > 0$ and f_i is twice differentiable and strictly concave: $i = 1 \cdot\cdot n$. Let $\hat{\alpha}_i$ be the unique maximizing argument of f_i (assumed finite) and $\hat{\alpha}$ be the unique maximizer of f. Suppose $\hat{\alpha}_i \neq \hat{\alpha}_j$ for some i, j. Then:

$\min\{\hat{\alpha}_i \mid i=1 \cdot\cdot n\} < \hat{\alpha} < \max\{\hat{\alpha}_i \mid i=1 \cdot\cdot n\}$.

Proof:

$$(*) Df(\hat{\alpha}) = \sum_1^n c_i Df_i(\hat{\alpha}) = 0$$

By (i), page 5, $(\hat{\alpha}_i - \hat{a}) \doteq Df_i(\hat{\alpha})$. Since not all $\hat{\alpha}_i = \hat{\alpha}$, not all $Df_i(\hat{\alpha})$ equal zero. For (*) to hold $\exists j, k \in Df_j(\hat{\alpha}) > 0$, $Df_k(\hat{\alpha}) < 0$. By: (i) $\hat{\alpha}_j > \hat{\alpha}$, $\hat{\alpha}_k < \hat{\alpha}$.

When results from the "conclusion tree" are taken into account and restrictions are temporarily waived, all of the examples of Section 4 except those for situation 4 are of the form:

$$\text{(C.5)}\quad \max_{\alpha\epsilon R} f(\alpha) = E\psi(X+\alpha Y)$$

$$= Ee^{-.00001(X+\alpha Y)} - .018e^{-.0002(X+\alpha Y)}$$

which satisfies the conditions of proposition C.2. Therefore, one can calculate $\hat{\alpha}_1$, which maximizes $-e^{-.00001(X+\alpha Y)}$ and $\hat{\alpha}_2$ which maximizes the second term and iterate for $\hat{\alpha}$ in the interval $(\hat{\alpha}_2, \hat{\alpha}_1)$. Restrictions are then easily imposed (see {4}, page 10).

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