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A DYNAMIC INVESTIGATION INTO LOGICALLY CONSISTENT MARKET SHARE MODELS

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A Dynamic Investigation into Logically Consistent Market Share Models

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Abstract

This paper introduces a dynamic zero-sum game framework inspired by the Lanchester combat model to resolve issues that relate to logically consistent market share models. Our new theorem shows that three of four axioms in Bell, Keeney and Little's (1975) "market share theorem" are either incomplete or unnecessary; concerns about the nonlinearity and asymmetry aftermath of the market share system also are incorporated into in our analysis. Implications of this new theorem resolve some of the confusion surrounding the logically consistent market share models proposed by Naert and Bultez (1973) and McGuire and Weiss (1976), and the recent empirical paper by Brodie and de Kluyver (1987). More specifically, this paper explains the ways to incorporate both range and sum constraints to complete the logical consistency problem, and demonstrates the feasibility of adopting model robustness and descriptive validity which previously have been thought to be incompatible in construction of market share response models.

1. INTRODUCTION

This paper reexamines the logically consistent issue of the market share model. The paper begins with a review of the debate between the view that the elasticities of marketing variables such as advertising and pricing can be estimated by econometric models in a nonrestrictive description manner (Beckwith 1972, 1973), and the view that explanatory variables and associated coefficients should be constrained by certain logically consistent rules if market share is specified as the dependent variable for a linear model (Naert and Bultez 1973; McGuire and Weiss 1976). Recently this debate has been revitalized by the forecasting accuracy of econometric market share models. Brodie and de Kluyver (1987) argue that the econometric market share model performs no better than a "naive" model (i.e., $m_t = m_{t-1}$), and question the usefulness of the more sophisticated econometric models for prediction.

Discussion papers by Bass (1987), Wittink (1987), Erickson (1987), Aaker and Jacobson (1987), and Hagerty (1987) replying to Brodie and de Kluyver's enigma, have focussed on the following questions:

(1) What is the "true" structure of a causal market share system?

(2) What is the best approximation of parameters for estimation?

(3) Will the quality and quantity of observed data confound with that of empirical models?

With regard to the first issue, papers by Naert and Bultez (1973) and McGuire and Weiss (1976) have advanced theoretical development of restrictive rules or axioms for a causal market share system. The remaining papers, Gaver, Horsky and Narasimhan (1988) and Naert and Weverbergh (1985) as well as many

other papers related to estimation of the market share attraction model, have addressed one or more of the latter two issues.

In this paper, we propose a new way to approach these problems. We start from the naive model, also called the random walk model or the autoregressive model with a unit root (Dickey and Fuller 1976), which has a long history of describing phenomena of the financial and commodity markets and yet best characterizes aggregate time series data (Hall 1978). Both Wittink (1987) and Aaker and Jacobson (1987) have emphasized that the naive model is not expected to do well on disaggregate data, since time-related factors such as cyclical movement and seasonality may be involved, and the observed system is nonstationary.

To get away from a stationary assumption, we construct a dynamic market share system from which naive and econometric models can be derived in the stationary state. This requires strict specification of the model, and we use the generalized n-firms Lanchester combat model (Little 1979) as a benchmark. The reasoning behind this approach, as shown by Little (1979) and Erickson (1985), is that market share can be specified by attraction formulation from the Lanchester model at equilibrium state. If the Lanchester model is true, the partitioned brands are then reacting dynamically such that the proportions of the rivals' market shares gained (own market shares lost) are directly proportional to their own (rivals') attraction powers.

There are several advantages to applying this dynamic market share attraction idea. First, the simplest implication of the naive model is that market share lagged more than one period has no predictive power for current value, and market share is independent of any marketing decision variable observed in the same and earlier periods. In other words, if changes in market

share are permanent -- that is, market share is a random walk variable -- any exogenous effect on the current market share should differ only slightly from any exogenous effect on the permanent market share. Therefore, if the observed system is stationary, it should be no surprise that the naive model and the "true" causal market share system perform equally well for model fitting and forecasting. Yet both the time series and the causal market share are dynamic in nature. As will be shown later, the naive model captures only the permanent, not the time-variant, component of the market share system.

Second, the attraction formulation stemming from the utility theory of the consumer discrete-choice model (Luce 1959; McFadden 1973) is the basis for many aggregate-level consumer choice models in the marketing literature (e.g., Guadagni and Little 1983; Nakanishi and Cooper 1974; Russell and Bolton 1988). Bell, Keeney and Little (1975) have derived a general market share theorem based on this static attraction formulation from four plausible axioms. This work has heavily influenced theoretical frameworks for purchase probability and market share models. Their theorem is problematic, however, and its ambiguities and restrictive assumptions have engendered numerous challenges such as those of Barnett (1976) and Chatfield (1976). Under our dynamic investigation, it can be shown that three of the four axioms are indeed incomplete or unnecessary. Axioms 2 and 3, for instance, are valid if and only if the market share system is stationary.

Third, in reply to Brodie and de Kluyver's enigma, Wittink (1987) suggested that using a relative variable such as price ratio imposes restrictions on the regressor coefficients in the multiplicative market share model. This argument embodies the classic debate between Beckwith (1973) and Naert and Bultez (1973) on the logical consistency of the market share model

and between McGuire and Weiss (1976) and Naert and Bultez (1973) in response to McGuire et al.'s (1968) early study. This restriction is also related to the symmetry assumption of the market share theorem discussed in the Bell, Keeney and Little (1975) and Barnett (1976) papers. None of the studies seem to offer a complete and satisfactory answer.

Solution of the puzzle lies at the heart of the sum-constrained market share system which implies: (1) the system is singular and (2) competition follows an n-players-zero-sum game rule in the dynamic system (Tang 1989). We use this observation plus range constraint and nonlinearity, which are ignored by most prior studies, to help construct the new market share theorem from our dynamic market share attraction model. We set up for our new theorem three lemmata, all of them counter to the Bell, Keeney and Little (hereafter BKL) static axioms.

Our aim in this research is to shed some light on the true structure of the market share system, with possible extensions to the link with the aggregate consumer brand choice process. It should be noted that there are several possible formulations other than the Lanchester model that can also capture the dynamics of the causal market share system. Since the system discussed in this paper is composed of n first-order deterministic differential equations, stochastic variations such as random errors of the observed system are not included in the model. More specifically, we are more concerned with the first issue -- the "true" structure of the causal market share system -- and use the result as guidelines for empirical issues and future studies.

Presentation of the new theorem is as follows. A preliminary market share theorem adapted from the n-firm Lanchester model is illustrated, with focus on clarification of misunderstandings about the Lanchester model. In the third

section, BKL's axioms on the "market share theorem" are reviewed and examined under the dynamic market share attraction model. Proof is provided of lemmata derived from the new theorem. In section 4, issues in the logically consistent model are briefly described, with a focus on McGuire and Weiss'(1976) four types of restrictions of explanatory variables and coefficients. Range constraint and non-linearity can be shown to be adapted into our new theorem. A counterexample to Naert and Bultez, and McGuire and Weiss' logically consistent rules derived from our new theorem is provided. This section concludes with a summary and suggestions of directions for future research.

2. MODEL FORMULATION

The Lanchester (1916) model was first introduced and modified by Kimball (1957), who described the effects of competitive advertising, and Morse and Kimball (1951), who showed that equilibrium exists in a finite time. It is described in Little's (1979) review as one of the "*a priori* aggregate advertising models" and recently has been reexamined by Jorgensen (1982) Erickson (1985) and Rao (1988). Little (1979) pointed out two interesting aspects of this type of modified Lanchester model: (1) It is a competitive generalization of the Vidale-Wolfe model (1957), with the competitor's response function modeled as a decay constant; (2) In steady state, the model reduces to an attraction-type market share model of the general form us/(us + other). For an n-firm market share model, a generalized form can be written as follows (see also Little (1979), p. 650):

$$\frac{dm_i}{dt} = -\sum_{j \neq i}^{N} a_j m_i + a_i \left(\sum_{j \neq i}^{N} m_j\right) \qquad (1)$$

$$m_i(0) = c_i, \quad c_i \geq 0, \quad \sum_{j=1}^{N} c_i = 1,$$

$$i, j = 1, 2, \ldots, N,$$

where $m_i(t)$ are market shares of firm i at time t;

a_i are the associated attraction functions of firm i;

c_i are associated initial market shares of firm i;

N is the total number of firms (brands) with an *a priori* basis for the market structure;

t is defined in a finite time range $t \in [0, T]$.

In steady state ($dm_i/dt = 0$), the system equations (1) reduce to

$$m_i = \frac{a_i}{\sum_{j=1}^{N} a_j}, \tag{2}$$

a general form of market share attraction model cited in most literature.

In applying the modified Lanchester combat model to competitive advertising, Kimball interprets m_i as the number of customers, a direct translation from military operation terms such as fighting units or number of missiles. Little (1979) interprets m_i as levels of sales that can be normalized into market share if the total market sale is constant. Suppose consumers are homogeneous and each individual contributes the same amount of sales. This is equivalent to saying that consumers are homogeneous in buying power; thus, the treatment of m_i by Kimball and Little is the same. Erickson

(1985) has made it even clearer that equation (1) can be stated as the market share model, while for a general sales response function, a sales expansion function must be added. This sales expansion function, which captures exogenous factors outside the system such as growth in the economy and population, allows for industrial growth or decline. After adding the initial market share constraint, we treat the system equations (1) the same as those in Erickson's (1985) description of the dynamic competitive market share model.

According to Little (1979), a good advertising response model should be able to display or capture the following phenomena (p. 644):

1. dynamic interaction of sales and advertising;
2. a sales response function that can be concave or s-shaped at steady state, and allows for positive sales at zero levels of advertising;
3. competitive interactions of advertising with sales;
4. changes in advertising dollar efficiencies over time; and
5. erosion of increasing sales response under constant advertising.

As Little shows, the modified Lanchester model in equation (1) captures only the first four phenomena, not non-zero sales at zero advertising in phenomenon 2; it also does not accommodate phenomenon 5. This observation about the modified Lanchester model in Little's work is incomplete because he only allows attraction function a_i to be a linear function of the sole marketing instrument -- advertising, such that:

$$a_i = \rho_i \, AD_i \ ,$$

where ρ_i are advertising effectiveness constants of firm i; AD_i are defined as advertising expenditures of firm i. This linear function assumption is the same as that in Kimball's (1957) original paper.

Erickson (1985) suggests that attraction functions a_i can be generalized so a_i are general functions of ADi, such that the first derivatives, a_i', are positive and second derivatives, a_i'', are negative, which exhibits diminishing returns to advertising effectiveness. Rao (1988) has specifically defined the attraction function in multiplicative form as:

$$a_i = \theta_i \, AD_i^{\alpha i} e \quad ,$$

where α_i is the advertising elasticity, θ_i and e are copy effectiveness and market growth factor, respectively; and both parameters are positive. However, the attraction function modeled this way can only accommodate phenomenon 5, not "non-zero sales at zero advertising." A better functional form of attraction power to capture all phenomena addressed by Little is achieved by following the tradition of logit modeling (e.g., Guadagni and Little 1983; Russell and Bolton 1988) such that

$$a_i = \text{Exp} \, (\beta_{i,o} + \beta_{i,1} \, AD_i), \tag{3a}$$

or in a general form, by combining with all other marketing instruments as:

$$a_i = \text{Exp} \, (\beta_{i,o} + \sum_{k=1}^{P} \beta_{i,k} \, Z_{ik}), \tag{3b}$$

$$i = 1, 2, \ldots, N,$$

where Z_{ik} denotes Kth decision variable for firm i;

$\beta_{i,k}$ are parameters to be estimated;

P is the number of decision variables.

To show Little is wrong about the Lanchester model, first, set $AD_i = 0$ in equation (3a), which leaves only the constant term; we get non-zero sales at

zero advertising condition. Second, phenomenon 5--erosion of increasing sales response under constant advertising -- holds in (3a) because attraction is a non-linear function of advertising. Therefore, phenomena 1-5 can be satisfied simultaneously in the Lanchester model as defined in equations (1) and (3b). This non-linear functional relationship between attraction power and advertising can also be shown (for instance, see Russell and Bolton (1988)) in exhibits of diminishing returns of advertising effectiveness suggested by Erickson (1985).

This non-linear formulation in (3b) is of particular importance in developing the market share theorem. First, it demonstrates the linkage between the market share model and the brand choice model under a dynamic framework. Past studies (e.g., Cooper and Nakanishi 1983, Karnami 1985) have limited the theoretical integration between a firm's attraction function and the aggregate consumer's utility function to a static assumption. Second, attraction function (and utility function) defined in this way always yields a non-negative value. This non-negative property has a very important implication in equation (2), i.e., that the range constraint of a logically consistent market share is guaranteed. (Note that the multiplicative form of the attraction model used in most of prior studies satisfies this non-negative property, but Little's "non-zero sales at zero advertising (marketing decision variables)" phenomenon does not.)

Third, many other non-linear forms besides that of equation (3b) are possible. For instance, there is the double exponential function of marketing instruments with respect to attraction:

$$a_i = e^{-e^{-(\beta_{i,o} + \sum_{k=1}^{P} \beta_{i,k} Z_{ik})}}$$

that satisfies both Little's five conditions and range constraint. We will leave this and other forms for future investigation. As long as the issue of logical consistency is the main concern, any non-negative definition of attraction guarantees that the share range constraint has the following general function:

$$a_i(Z_{i1}, Z_{i2}, \ldots Z_{ik}) \geq 0 \qquad i = 1, 2, \ldots, N \tag{3c}$$

This function is used for the discussion that follows.

Similarities between military and marketing operations, however, do not imply that existing combat theory perfectly matches marketing theory. For example, suppose the stock of a single weapon -- say missiles -- is homogeneous in fighting power. In an arms race, if two sides attack each other, the original number of missiles each possesses will have decreased after a period of combat. There is no analogy to this situation in marketing. Let us assume instead that the total market is composed of a fixed number of customers with homogeneous buying power and only two competing firms, X and Y. Firm X and firm Y claim x and y as their exclusive customer territories, respectively, and the sum of x plus and y equals a fixed market size. After an advertising campaign initiated by one firm, say firm Y, X loses θ_x customers to Y ($\theta_x \leq x$). Here the quantity θ_x does not "disappear" as in the arms race model but represents the segment switching firms (from X to Y). This switching proportion, the change rate of market share for the firms, is captured by another logically consistent property of the market share system under dynamics -- the n-player-zero-sum differential game property, such that any player's gain in market share is always at the expense of other players, and the summation of the system equation equals zero through all time periods.

In summary, to equate the Lanchester combat model with the market share model, we have derived the following three invariant rules in equation (1) when N firms investigated are exhaustive and mutually exclusive in one product class:

(1) Share Range-constraint: $m_i \geq 0,$

(2) Share Sum-constraint: $\sum_{i=1}^{N} m_i = 1,$

(3) Share change rates Zero-sum-constraint: $\sum_{i=1}^{N} \frac{dm_i}{dt} = 0.$

Range constraint is the direct consequence of condition (3c), which requires the attraction function to be non-negative. Condition (3c) is also equivalent to BKL's axiom 1, and is the only explicit condition needed to construct the market share theorem. This will be discussed in the next section. Sum constraint is derived from the initial state of the system. As far as the dynamic system is concerned, the physical basis for this initial state is irrelevant to the system and the process is ergodic. Change rates zero-sum-constrained is a dynamic generalization by share sum-constrained. This is obtained by summing up N differential equations from equation (1).

(Note here that general treatment of the range constraint, $0 \leq m_i \leq 1$, is not necessary. The non-negative market share combined with the sum constraint is sufficient to lead to logical consistency.)

Equations (1) and (3c), therefore, imply the following theorem

Market Share Theorem:

The system of differential equations, as defined in equations (1) and (3c), produces state variable m_i, which satisfies sum and range constraints.

(Proof in Appendix I.)

This new theorem is fairly common to other market share "theorems." We have mentioned that past studies tried to determine what rules govern the market share system in order to yield logical consistency. This theorem suggests that for a given set of initial conditions, any state variable $m_i(t)$ with the linearly additive functional relationship in system equations (1) and with non-negative a_i can lead to logical consistency.

This preliminary theorem is a step ahead of BKL's "theorem" in that: (1) attraction functions a_i are measurable and estimable, (2) market shares m_i are functions of all firms' attraction powers and are measurable and estimable, and most important of all, (3) market shares m_i are functions of time; this is offered as a reply to Chatfield's (1976) concern about BKL's theorem that "... we are not told how these attraction values actually can be found!"

In what follows, we will use the dynamic property of the theorem to show that BKL's Axioms 2 and 3 are unnecessary and the linear dependence properties of the ai coefficients in (1) to show that Axiom is incomplete.

3. MARKET SHARE THEOREM: A REVISIT

According to Bell, Keeney and Little (1975), the market share theorem is stated as follows: "If a market share is assigned to each seller based only on the attraction vector and in such a way that assumptions A1 - A4 are satisfied, then market share is given by" equation (2). The assumptions A1 - A4, which are stated as axioms in their paper, are:

$$\text{A1:} \quad \mathbf{a} \geq 0, \text{ and } \sum_{i=1}^{N} a_i > 0,$$

A2: $a_i = 0 \rightarrow m_i = 0$,

A3: $a_i = a_j \rightarrow m_i = m_j$,

A4: $f_i(\mathbf{a} + \Delta e_j) - f_i(\mathbf{a})$, for $j \neq i$

is independent of j, where e_j is the jth unit vector;

where

$\mathbf{a} = [a_1, a_2, \ldots, a_n]'$, vector of attractions,

$m_i = f_i(\mathbf{a})$, market share of firm i.

Note here the "If ..., then ..." logic rather than the "If and only if ..., then..." logic of BKL's theorem. While the necessary conditions A1 - A4 are satisfied and the market share variable can be proven and the attraction formulation is true, the theorem does not show what sufficient conditions are required for the market share variable with an attraction form. One plausible counterexample from our theorem for the sufficient condition can be given as "If market share is obtained in attraction at steady state, then the market share variable is governed by system equations (1)." Therefore, the statement and the proof of BKL's theorem was only half done. But why are dynamics important to the theorem? We will briefly review the concept of a dynamic system and demonstrate this with a two-firm example to show that Axioms 2 and 3 are indeed unnecessary.

3.1 Market Share as a Function of Time

The distinction between statics and dynamics originated in early works in mathematical mechanics. In mechanics, "statics" denotes the resting state of an object, whereas dynamics embraces the motion and time path of an object. In marketing, however, there is no concept comparable to a mechanical system at

rest because the system is presumed to be always in motion. Despite this, many marketing models are based on static assumptions.

Let us look for a moment at BKL's market share theorem. One might ask, "Why do two sellers with equal attraction power have equal market shares?" This implication is a restatement of Axiom 3 in BKL's theorem. Thus, the axiom refers to states of competition, not to any processes of change. Accordingly, in a static market system, certain key variables such as carryover effects of attraction power are ignored and have been operating in the same ways toward their own market shares over time, which, in turn, leads to the above static conclusion.

The distinction between dynamics and statics leads us to another important concept, that of equilibrium, also borrowed from the physical sciences. A simple definition of equilibrium is that it is a state in which forces are "in balance." In terms of market share attraction power, a state of equilibrium exists when all sellers in the market choose the attraction power level that, of several alternatives available, is the most preferred. When the market share system is said to be in equilibrium, it is implied that a balance of power occurs and each seller reacts *optimally* over time.

It is common for a marketing model to be in a steady state, since the time factor is purposely removed to make the system appear stationary and to invoke comparative statics. The removal of time, except where strictly defined, will automatically bring the system into equilibrium. This is the principal concept that is missing from past theorem proposals.

Deriving the entire system is somewhat involved, however. According to a Chinese proverb, "every journey of ten thousand miles must begin somewhere." Let us examine the simplest case by assuming that there are only two firms in

the market and that attraction powers, a_1 and a_2, are non-negative constant functions. Setting N = 2 in equation (1) and using logical consistency $m_1 + m_2 = 1$ yields

$$\frac{dm_1}{dt} = a_1 - (a_1 + a_2)\, m_1$$

$$\frac{dm_2}{dt} = a_2 - (a_1 + a_2)\, m_2$$

where $m_1(0) = c_1$, $m_2(0) = c_2$;

$c_1, c_2 \geq 0$; $c_1 + c_2 = 1$; $t \geq 0$.

The solutions for m_1, and m_2 above are:

$$m_1(t) = \frac{a_1}{a_1+a_2} + \left(c_1 - \frac{a_1}{a_1+a_2}\right) \exp\left[-(a_1+a_2)t\right]$$

$$m_2(t) = \frac{a_2}{a_1+a_2} + \left(c_2 - \frac{a_2}{a_1+a_2}\right) \exp\left[-(a_1+a_2)t\right]$$

For a general N-firm solution, this yields:

$$m_i = \frac{ai}{\sum_{j=1}^{N} a_j} + \left(c_i - \frac{ai}{\sum_{j=1}^{N} a_j}\right) \exp\left[-\left(\sum_{j=1}^{N} a_j\right)t\right] \qquad (5)$$

Permanent Component | **Time-variant Component**

where $m_i(0) = c_i$, $c_i \geq 0$, $\sum_{j=1}^{N} c_j = 1$,

$a_i \geq 0$, $i = 1, 2, \ldots N$, $t \geq 0$.

Market share, which is treated as a continuous time function, can be decomposed into two parts: a permanent attraction component and a time-variant component. The permanent attraction component, as discussed above, is the ratio of a firm's attraction power to the summation of all firms' attraction powers. The time-variant component, which is a function of time, decreases exponentially with the parameters sum $(a_1 + a_2 + \ldots + a_N)$. If time goes to infinity, that is, if we set the time frame to long-term equilibrium, the time variant component will vanish and leave only the equilibrium attraction term. Another mathematical check is to set $t = 0$; this will bring the system into the initial state, $m_i(0) = c_i$.

Now we are ready to examine BKL's axioms. Let us look first at the most controversial one - Axiom 3, which says "two sellers with equal attraction have equal market shares." For the simple two-firm example in equation (5), equal attraction $a_1 = a_2 = k$ implies:

$$m_1(t) = 0.5 + (c_1 - 0.5) \exp(-2kt) \qquad (6)$$
$$m_2(t) = 0.5 + (c_2 - 0.5) \exp(-2kt)$$

Therefore $a_1 = a_2$ implies $m_1 = m_2$ if, and only if, the following conditions hold:

(1) equal initial market shares, $c_1 = c_2$, or, (7.a)

(2) time goes to infinity, $t \dashrightarrow \infty$. (7.b)

In dynamic research, the initial state is a very important condition in that it makes an impact throughout the system in the process over time. (In

meteorology, this is sometimes referred to as the **"butterfly effect"** -- a butterfly fluttering its wings in Peking will cause a storm in New York City one month later (Gleick 1987, pp. 9-33)). Consider the advertising carryover effect in marketing as an example. Even if RC Cola matches its advertising expenditures (attraction) with those of Coca Cola at certain time periods, both firms will have the same market share level if, and only if, Coca Cola's goodwill effect advantages die down ($t \dashrightarrow \infty$) or both firms' starting points are at the same level ($c_1 = c_2$). Therefore, BKL's Axiom 3 holds only for the long-term equilibrium state, not for any periods before reaching equilibrium.

Let us now look at Axiom 2, which states that "A seller with zero attraction has no market share." Using the same two-firm example, and setting firm 1's attraction equal to zero such that $a_1 = 0$ yields:

$$m_1 = c_1 \exp (-a_2 t) \qquad (8)$$
$$m_2 = 1 - (1 - c_2) \exp (-a_2 t).$$

Again, this is another "butterfly effect" example. Firm 1 with zero attraction does not necessarily lose its market share to Firm 2 immediately and completely. The initial state c_1 can keep firm 1 "alive" for a little while, but the firm will lose market share exponentially because of its rival's attraction power a_2. Firm 1's market share will be equal to zero in the long run if it maintains its zero attraction policy. Therefore, $a_1 = 0$ implies $m_1 = 0$ if, and only if, the conditions in (7.a) or (7.b) hold.

Therefore BKL's Axioms 2 and 3 are very demanding for a market share theorem under dynamics. This is because these two axioms capture only the permanent component of the market share variable rather than treating it as a complete unit. We summarize the dynamic competition phenomena in equations (6

- 8) in the following two lemmata

Lemma 1:

For a given set of firms, zero attraction function implies zero market share for any particular firm at a certain point in time, if, and only if, (1) firms have equal initial market shares, and (2) the system is observed under equilibrium conditions.

Lemma 2:

For a given set of firms, equal attraction functions imply equal market shares at a certain point in time, if and only if, (1) equal initial market share conditions hold, and (2) the system is observed under equilibrium conditions.

For the general formulation that attraction is a function of marketing instruments, lemmata 1 and 2 can equivalently apply.

3.2 The Issue of Symmetry Assumption

The symmetry assumption in BKL's market share theorem states that if changes in the attraction of one seller are effectively the same across all sellers, it follows that regardless of which competitor sustains the attraction change, the market share of any given seller will be affected in the same manner (Axiom 4 in the theorem). Barnett (1976) claims that eliminating BKL's Axiom 3 and defining m_i as differentiable functions of attractions removes the symmetry assumption. The three firm examples he provides (p. 106) rewritten in the form of equation (1) are as follows:

$$\frac{dm_1}{dt} = 2a_1 m_2 + 2a_1 m_3 - (2a_2 + a_3) m_1$$

$$\frac{dm_2}{dt} = 2a_2\, m_1 + 2a_2\, m_3 - (2a_1 + a_3)\, m_2$$

$$\frac{dm_3}{dt} = a_3\, m_1 + a_3\, m_2 - (2a_1 + 2a_2)\, m_3$$

In steady state, the solution for the above model has attraction forms as:

$$m_1 = \frac{2a_1}{2a_1+2a_2+a_3}$$

$$m_2 = \frac{2a_2}{2a_1+2a_2+a_3}$$

$$m_3 = \frac{a_1}{2a_1+2a_2+a_3}$$

Removal of the symmetry assumption, as in Barnett's demonstration, allows firm 1 and firm 2 to obtain double amounts of market share from their rivals, while firm 3 can only gain unity. We call this kind of symmetry **between symmetry**, meaning symmetry happening between firms. Between symmetry can be removed from the system as demonstrated by Barnett such that the attraction coefficients are relaxed from the unity constant function and are non-linear functions of marketing instruments.

There are two hidden symmetries, however, that Barnett has missed. The first is called **within symmetry**, or row symmetry in which symmetry happens "within" a specified market. This is the symmetry discussed in BKL's study. Take the beverage industry, for instance. Firms 1, 2, and 3 are Coke, Pepsi, and RC Cola respectively, and their market shares are 45%, 40%, 15%. Suppose

Pepsi's attraction power is 0.2; after a series of periodic encounters, Pepsi's market share gains will be 0.09 and 0.03 from Coke and RC Cola, respectively. If Pepsi and RC Cola are not in the same market partition (Hendry Corporation 1970; Kalwani and Morrison 1977), then Pepsi might not be able to gain any market share from RC Cola. Therefore, the within symmetry assumption requires that firms in the system be "directly competing brands." For this reason, the market share model in equation (1) is a dynamic Hendry system, since we require that firms compete directly.

The second hidden symmetry is called **diagonal symmetry**, a situation in which each diagonal element is equal to the sum of the off-diagonal elements in the same column. Both within symmetry and diagonal symmetry cannot be removed from the market share system as long as we study the system as a whole because the sum constraint always applies to the market share system. These three symmetry conditions, introduced here for the first time, are called differently in the marketing literature -- the logically consistent market share model (Naert and Bultez 1973). (Logical consistency is discussed separately in the next section.) Symmetry conditions can be revealed and interpreted easily in matrix form as follows.

Rewriting the market share system in equation (1) in compact form, and assuming attraction powers a_i to be constant functions, the matrix form can be expressed as

$$\frac{d\mathbf{m}}{dt} = \mathbf{K}\,\mathbf{m} \qquad \mathbf{m}(0) = \mathbf{c} \tag{9}$$

where $\mathbf{m} = [m_1, m_2, \ldots, m_N]'$, an N x 1 market share vector;

$\mathbf{c} = [m_1(0), m_2(t), \ldots, m_N(0)]'$, an N x 1 initial market share vector,

the elements satisfy sum and range constraints;

$\mathbf{K} = [k_{ij}]$, a singular N x N matrix;

and k_{ij} are <u>linear dependent</u> such that:

$$k_{ij} = a_i, \quad i \neq j \tag{9a}$$

$$k_{ii} = -\sum_{j \neq i}^{N} a_j, \tag{9b}$$

for any $a_i \geq 0$, $i = 1, 2, \ldots, n$,

and $\sum_{i=1}^{N} a_i \neq 0$.

Matrix **K** can be written as:

Diagonal Symmetry **Within Symmetry**

$$\mathbf{K} = \begin{bmatrix} -(a_2+a_3+\ldots+a_N) & a_1 & \ldots & a_1 \\ a_2 & -(a_1+a_3+\ldots+a_N) & \ldots & a_2 \\ \vdots & \vdots & & \vdots \\ a_N & a_N & \ldots & -(a_1+a_2+\ldots+a_{N-1}) \end{bmatrix} \tag{9c}$$

Between Symmetry

Matrix **K** has the following properties: first, the off-diagonal elements are equal across each column in the same row; this is the within symmetry discussed in BKL and Wittink's (1987) papers. Second, each diagonal element is the sum of off-diagonal elements in the same column; this is the property of the continuous-time Markov process where market share vector **m** is treated as a probability vector and attraction matrix **K** is a Markov matrix (Bellman 1970,

pp. 263-280). This is also the major evidence that shows that both market share and purchase probability follow the same stochastic process. Third, matrix **K** is a singular matrix. This is because elements in the same column are linearly related. System equations of this nature in which linear relationships exist among elements of state vector **m** also hold among the corresponding multiple matrix **K** and are said to be consistent (e.g., Searle 1982. p. 229). This consistent property of the market share system is summarized as follows

Lemma 3:

The linear dependence of the market share vector implies that linear relationships exist among corresponding attraction functions.

In summary, we have demonstrated that three of four of BKL's conditions, Axioms 2, 3, and 4 can be removed under the dynamic market share system. What we really need, except for the non-negative property of the attraction function, is some form of linear relationship -- the "within symmetry" and continuous Markov chain property among the attraction functions. Although past studies such as those of Horsky (1977) and Little (1979) have used the related Lanchester model in studying the market share system, because of the duopoly-equation approach they adopt, they are unable to reveal the linearly dependent relationship among the attraction functions from the market share system. This linearly dependent relationship, termed logical consistency, is the central argument between Naert-Bultez (1973) and McGuire-Weiss (1976). We will clarify these linear dependencies in next section.

4. IMPLICATIONS FOR A LOGICALLY CONSISTENT MARKET SHARE MODELS

4.1 Nature of the Problem

As a matter of theory, the logical consistency of market shares is widely accepted as a proper constraint in the problems of market share response models at both the consumer and firm levels. In their classic work on the desirability of sum constraint, Naert and Bultez (1973) (hereafter N-B) propose a market share theorem that includes three necessary and sufficient conditions for a sum constraint to be held in estimating market share variables. Their theorem emphasizes explanatory variables, and constant coefficients are constrained to particular relationships if logical consistency is to follow. Subsequent study by McGuire and Weiss (1976) (hereafter M-W) has followed the methodology of McGuire et al. (1968) and N-B. M-W's main concern about N-B's study was that the generalization of varying the sum constraint over time was unnecessary unless there were an *a priori* basis for the market structure. They claim that the constrained explanatory variables and coefficients should be one of the four types specified in their study.

A major problem in empirical research based on restrictive rules of the sum-constrained model has arisen in the fitting of the symmetrical competition assumption of the model that yields robust estimated results but loses the descriptive validity of market behavior. Beckwith (1973), for instance, provides a counterexample of the N-B theorem by withholding one equation from the estimation process and using the traditionally unconstrained methodology for the remaining equations. This adjustment, Beckwith claims, can be shown to contradict N-B's theorem. Also, controversy about empirical implications of the tradeoff between model robustness and descriptive validity of market share

response models has raged openly in marketing literature for the past several years (Brodie and de Kluyver 1984; Ghosh, Neslin and Shoemaker 1984; Leeflang and Reuyl 1984; Naert and Weverbergh 1981) and recently in a special issue of the International Journal of Forecasting (1987).

Our aim of this section is to resolve such an argument. In doing so, we provide a dynamic market share model that allows general formulation for linear, multiplicative and attraction market share models in past studies. This proceeds from a theoretical examination of range and sum constraints of market share models. Traditionally, discussion of the issue of the logically consistent market share model has been confined to the sum constraint; this is because requiring a range constraint makes for additional problems such as quadratic programming and parameter approximation (Theil and Rey 1966, Lee, Judge, and Zellner 1968). When a the range constraint is not considered, M-W asserts that "... a model thus restricted has only limited logical consistency" (p. 296), and "... there is no guarantee that predicted market share will never be negative or exceed unity" (p. 301). For these reasons, it becomes necessary to develop generalized rules of logical consistency that adopt non-linearity and employ both a sum constraint and a range constraint.

Application of the theorem begins with investigation of the issue of the range constraint, which is equivalent to BKL's Axiom 1 and equation (3c) in this paper, and then exploits the descriptive validity and nonlinearity for logically consistent market share models. Since the major errors in N-B's and Beckwith's counter-examples are clarified in M-W's article, we will focus our discussion of the sum constraint mainly on M-W's work, especially, those parts dealing with the asymmetry and nonlinearity issues.

4.2 The Issue of Range Constraint

The linear sum-constrained market share model in most studies including those in papers by N-B, M-W, and McGuire et al. (1968) is of the form:

$$m_{it} = Z_{i,t}\,\beta_{i1,} + Z_{i2t}\,\beta_{i2} + \cdots + Z_{ikt}\beta_{ik} + \cdots + Z_{ipt}\,\beta_{ip} + \epsilon_{it}, \qquad i = 1, 2, \cdots, n. \tag{10a}$$

and

$$\sum_{i=1}^{n} m_{it} = 1, \qquad t = 1, 2, \cdots, T, \tag{10b}$$

where m_{it} is the market share of firm i at time t;

Z_{ikt}, K = 1,2, • • •P, are explanatory variables k of firm i at time t;

β_{ik} are associated coefficients of Z_{ikt};

ϵ_{it} are random disturbance with zero means;

n is total number of firms;

P is total number of explanatory variables.

Two major assumptions in equation (10a) are the following. First, this linear regression model is a static model. It is easily seen from the model that none of the time-lagged variables are included. The t notations for variables m_{it} and Z_{ikt} are not identified as past and future movements in the system, but instead represent the index of data observations in time sequential orders randomly drawn from a stationary observed system. Hicks (1985, pp. 1-28) calls this a "static methodology," since time factor is not an argument and the model "... can be treated as if it were in equilibrium." (Therefore, notation t can be disregarded in equations in (10a).) Second, n equations in (10a) are independent with regard to marketing mix decisions. This is because the dependent variable, i.e., market share of one seller, is determined solely by its own explanatory variables.

The concern about the range constraint of the market share variable is immediately apparent if the right-hand-side of equation (10a) is not well bounded. Both N-B and M-W have noted this problem and have suggested using the attraction-type model as a possible solution. However, this is only a partial solution, since development of the nonlinear attraction model is separated from their discussion of restrictive rules. It is not clear whether the constrained rules should also apply to explanatory variables and coefficients of the nonlinear model they develop.

Following N-B and M-W's suggestion of the attraction modeling approach but defining the explanatory variables with associated coefficients that are linearly additive as in (10a) rather than in the nonlinear logit form, the attraction function of firm i is given by

$$a_i = \mathbf{Z}_i \boldsymbol{\beta}_i, \quad i = 1,2, \cdots n \tag{11a}$$

where $\boldsymbol{\beta}_i = (\beta_{i1}, \beta_{i2}, \cdots \beta_{ik}, \cdots \beta_{ip})$, the coefficient vector of firm i,

$\mathbf{Z}_i = (Z_{i1}, Z_{i2}, \cdots Z_{ik}, \cdots Z_{ip})$, the explanatory of variables vector of firm i.

Now from (10a) and (11a) it is easy to see that the market share variable of firm i is equivalent to its own attraction function in an equilibrium state.

$$m_i = a_i \ , \tag{11b}$$

since both functions are equal to the sum of the firm's marketing mix variables with coefficients. Substituting (11b) into (10b), we have

$$\sum_{i=1}^{n} a_i = 1 \ . \tag{11c}$$

It is now apparent that under static and independent assumptions in linear model (10a), attraction functions can be interpreted directly as the market shares, and the sum constraint is applied equally to the attraction functions.

To further check that our results are well-founded, consider the static model in (10a) as a special case of the dynamic market share attraction model in equation (1). This can be shown by making the continuous first-order differential term in the left-hand-side of equation (1) approximate to the discrete first-order differences term and rearranging the right-hand-side of the equation following attraction formulation; we have

$$m_{i,t+1} - m_{i,t} = a_i - (\sum_{j=1}^{n} a_j)\, m_{i,t}, \qquad (12a)$$

$$i=1,2, \cdots ,n$$

In steady state, the change rate of the market share variable is approaching zero; the following condition should hold for any market share variable $m_{i,t}$ such that

$$m_{i,t+1} - m_{i,t} = 0 \quad \text{all } t, \qquad (12b)$$

which is a random walk model (or, a so-called naive model). From (12b) and (11c), it is apparent that $m_i = a_i$. These results reflect the fact that the linear model (10a) contains no competition effects and the system under study is always stationary.

It is easy to show that other models that are not necessarily linear but are imbued with static and competition independent assumptions have the same restrictions as those in (11b) and (11c). For instance, the multiplicative model

$$m_i = \prod_{k=1}^{P} Z_{ik}^{\beta_{ik}} e^{\epsilon_i}$$

in Brodie and de Kluyver (1987) is nonlinear but falls into the "one seller's market share is equal to its own attraction function" constraint as the linear model (10a).

In addition, it appears that BKL's two static axioms -- Axioms 2 and 3 -- can be applied to the stationary condition in (11b). In particular, if the attraction function is equal to zero, then the market share is equal to zero; and if two firms' attraction functions are equal, then their market shares are equal. Both axioms support the stationary assumption in (10a).

To summarize, we have found that a simple way to insure range constraint in a market share model is to require that attraction functions be non-negative. This is the same assumption as in BKL's only valid axiom -- Axiom 1 and in equation (3c) of our paper. N-B and M-W fail to incorporate range constraint into their studies simply because the market share regression model they use is linear which makes the range constraint difficult to satisfy. Gaver, Horsky and Narasimhan (1988), however, demonstrated the possibility of considering both a linear model and range constraint. This is possible because only advertising ratio is used and its associated coefficient is assumed to be non-negative in their model, which is a special case in equation (10a).

4.3 The Issue of Sum Constraint

Beckwith's main concern about the logically consistent market share model,like most econometricians, is that "... the coefficients of explanatory variables are not necessarily equal across the equations, and the explanatory variables need not be sum-constrained since any finite values can be used."

The counterexample he provides can be expressed in the generalized condition as follows (Beckwith 1973, pp. 341-42):

$$Z_i\beta_i = 1 - \sum_{j \neq i}^{n} Z_j \beta_j \quad , \qquad i,j = 1, 2, \cdots, n.$$

Although restrictions need not apply to explanatory variables and coefficients, for the market share system to be logically consistent, we require some linear-dependent relationships among attraction functions. This has been proven in Lemma 3. Beckwith's concern, expressed in terms of matrix terminology in equation (9), is that if state vector **m** is sum-constrained (and therefore **m** is singular), is attraction matrix **K** also sum-constrained? It is true that if we only look at N-1 equations and decompose matrix K into the form

$$\mathbf{K} = \left[\begin{array}{c|c} K_1 & 0 \\ \hline 0' & 0 \end{array}\right],$$

where K_1 is a (N-1) x (N-1) matrix in full rank;

$0'$ = [0, 0, ..., 0], and 1 x (N-1) zero vector;

the sum-constraint for explanatory variables and coefficients is not necessary. Yet, if we posit that all N firms are competing directly and the dependent variable is sum- and range-constrained, the element k_{ij} should follow the linear dependent relationship in equation (9c).

Therefore, Beckwith is right about non-constrained modelling of explanatory variables and associated coefficients, since the restrictions apply only to attraction functions. However, the counterexample he provides explicitly admits that the restriction rule applies if we consider all

partitioned firms as a whole. His counterexample above can be rewritten in the general formulation as:

$$\sum_{i=1}^{n} Z_i \beta_i = 1$$

This results from (11a) and (11c), which are derived from the linear model (10a). In addition, we allow non-linearity of explanatory variables and coefficients; this is not included in his comments.

M-W's work on the sum constraint issue is more detailed. Following McGuire et al.'s (1968) development and Beckwith's counterexample, M-W claim that "... to be logically consistent, explanatory variables must be one of the four types and the coefficients of each type of variable must satisfy the condition specified." These four categories are: (1) constant (intercepts) (2) brand dummies (differential intercepts) (3) homogeneous variables and (4) others. M-W's four categories, however, unfortunately are inconsistent and incomplete. Our main concern is whether these four categories should be unified as one category or be separate. We agree with M-W's comments that N-B's logical consistency of three necessary and sufficient conditions is deficient; however, N-B's assertion that these three conditions should hold simultaneously for the theorem is correct.

The integration of M-W's four categories into one proceeds as follows. First, categories 1,2 and 4 can be satisfied by the following four conditions:

$$\text{(i)} \sum_{i=1}^{n} Z_i \beta_i = 1, \tag{13a}$$

$$\text{(ii)} \ \beta_{ik} = \beta_k \ , \ i = 1,2, \ \ldots \ n \tag{13b}$$

$$\text{(iii)} \quad \sum_{i=1}^{n} Z_{ik} = 1, \quad k = 1,2, \ldots P \tag{13c}$$

$$\text{(iv)} \quad \sum_{k=1}^{P} \beta_k = 1. \tag{13d}$$

Restriction (13a) is almost identical to Beckwith's counterexample, and is the general condition for category four in equation (13b) of M-W (p. 299). Setting k=1, the constant intercepts model (category 1) and differential intercepts model (category 2) can be implied. Restriction (13b) is the second condition in N-B's market share theorem, which is also the "symmetry" issue discussed in BKL's market share theorem. M-W's paper shows that this condition holds for categories 1, 2, and 4 in their equations (1.4) , (1.6) and (1.40). Restrictions (13c) and (13d) apply to the special case of (10a) when the explanatory variables Z_{ik} are normalized (see conditions (2.7) and (2.8) of McGuire et al.). These two conditions also hold for categories 1 and 2 when k=1. Therefore, categories 1, 2, and 4 are in the same category.

Second, category 3 is a special case of equation (13a), which is Beckwith's counterexample. According to M-W, the marketing mix variable k is a homogeneous variable if the ratio of two variables is a constant:

$$\frac{Z_{ik}}{Z_{jk}} = h_{ijk} \qquad i,j = 1, 2, \ldots, n, \text{ all } t \tag{14a}$$

The associated coefficients of homogeneous variables have the following property:

$$\beta_{jk} = \sum_{i \neq j}^{n} h_{ijk} \beta_{ik} \;, \; j = 1, 2, \ldots, n, \; k = 2, 3, \ldots, p; \tag{14b}$$

this is a general formulation in equation (1.13) of M-W.

From condition (14a), by substituting h_{ijk} into equation (14b), this can be rewritten as:

$$\sum_{i=1}^{n} Z_{ik}\,\beta_{ik} = 0 \qquad K = 2,\ 3,\ \ldots,\ P. \tag{14c}$$

After adding the intercept as demonstrated in M-W's paper, equation (14c) yields Beckwith's counterexample as follows:

$$\sum_{i=1}^{n} 1\beta_{i1} + \sum_{i=1}^{n}\sum_{k=2}^{P} Z_{ik}\,\beta_{ik} = \sum_{i=1}^{n} Z_i\,\beta_i = 1.$$

To summarize, we found that M-W's work can be summarized by two general restrictive rules if linear regression model (10a) and sum-constrained model (10b) are to follow. The first restriction is that the explanatory variables with associated coefficients are summed to unity as in (13a), the second is that the coefficients are equal across equations as in (13b). Since it is attraction functions rather than explanatory variables and coefficients that are sum-constrained according to our Lemma 3, M-W's restrictions can be interpreted in terms of symmetry issues as discussed in section 3.2: restriction (13a) is equivalent to diagonal symmetry in (9b) in discrete form, while restriction (13b) is equivalent to within symmetry in (9a). This is demonstrated below.

Rewrite the discrete version of the dynamic attraction model in (12a) in matrix form as

$$\mathbf{m}_{t+1} = \mathbf{K}\mathbf{m}_t \tag{15a}$$

where $\mathbf{m}_t = (m_{,t}\ m_{2t},\ \ldots\ m_{nt})$, is the market share vector;

$$\mathbf{K} = \begin{bmatrix} 1-(a_2+a_3+\ldots+a_N) & a_1 & \ldots & a_1 \\ a_2 & 1-(a_1+a_3+\ldots+a_N) & \ldots & a_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ aN & a_N & \ldots & 1-(a_1+a_2+\ldots+a_{N-1}) \end{bmatrix} \qquad (15b)$$

The off-diagonal attraction elements k_{ij} are equal across each column in the same row $k_{ij} = a_i$, while the diagonal elements are the sum of off-diagonal elements in the same column as

$$k_{ii} = 1 - \left(\sum_{j \neq 1}^{n} k_{ji}\right) \qquad (15c)$$

and can be written as:

$$\sum_{j=1}^{n} k_{ji} = 1 \qquad i = 1, 2, \ldots, n \qquad (15d)$$

This is Beckwith's counterexample as well as M-W's general restriction that attraction functions sum to one.

The following sum-constrained example conflicts with N-B and M-W's restrictive rules after adopting asymmetry and nonlinearity. Suppose there are three firms that are exhaustive and mutually exclusive in one product class; their attraction functions are asymmetric and possibly non-linear as follows:

$$a_i = Z_1 \beta_1 ,$$

$$a_2 = \prod_{k=1}^{P} Z_{2k}^{\beta_{2k}} ,$$

$$a_3 = e^{Z_3 \beta_3} ,$$

From our discrete version of the market share theorem in (15b), matrix **k** can be

expressed as

$$K = \begin{bmatrix} 1-\left(\prod_{k=1}^{P} Z_{2k}^{\beta_{2k}} + e^{Z_3\beta_3}\right) & Z_1\beta_1 & Z_1\beta_1 \\ \prod_{k=1}^{P} Z_{2k}^{\beta_{2k}} & 1-\left(Z_1\beta_1 + e^{Z_3\beta_3}\right) & \prod_{k=1}^{P} Z_{2k}^{\beta_{2k}} \\ e^{Z_3\beta_3} & e^{Z_3\beta_3} & 1-\left(Z_1\beta_1 + e^{\prod_{k=1}^{P} Z_{2k}^{\beta_{2k}}}\right) \end{bmatrix}$$

In steady state, the market shares of three firms reduced to:

$$m_1 = \frac{Z_1\beta_1}{Z_1\beta_1 + \prod_{k=1}^{P} Z_{2k}^{\beta_{2k}} + e^{Z_3\beta_3}},$$

$$m_2 = \frac{\prod_{k=1}^{P} Z_{2k}^{\beta_{2k}}}{Z_1\beta_1 + \prod_{k=1}^{P} Z_{2k}^{\beta_{2k}} + e^{Z_3\beta_3}},$$

$$m_3 = \frac{e^{Z_3\beta_3}}{Z_1\beta_1 + \prod_{k=1}^{P} Z_{2k}^{\beta_{2k}} + e^{Z_3\beta_3}},$$

and $m_1 + m_2 + m_3 = 1$.

Therefore, it is not explanatory variables and associated coefficients, as

claimed by N-B and N-W, that are sum-constrained by restrictive rules to yield summation of dependent variables to be unity. *As long as within symmetry and diagonal symmetry conditions hold in attraction matrix **K**, the market shares are always summed to one*. Furthermore, the market share models are not necessarily linear. It is apparent that linear, multiplicative and logarithmic linear models are special cases of (17a), (17b) and (17c), respectively, if the denominator is equal to one (related to conditions (11c) and (13a) in steady state). It is also easy to show that the so-called MCl (multiplicative competitive interaction) model (Nakanishi and Cooper 1974) and multinomial logic model (e.g., Theil 1969) are special cases of equations (17b) and (17c), respectively, if the between symmetry assumption of attraction function is applied.

In summary, we show the compatibility between model robustness and descriptive validity of market share response models. This is based on the premise that the seller's market share is completely determined by the relative power of its own attraction. Since a seller's explanatory variable is defined straightforwardly as its own attraction function rather than as a market share function, we can avoid the linear functional relationship between market share and explanatory variables. In other words, we permit market share and attraction to be linearly related but not necessarily market share and explanatory variables. Therefore, restrictions of logical consistency applying directly to the attraction function and the robustness of the model can be achieved. Furthermore, the descriptive validity of the model can be accommodated, since nonlinearity and (between) asymmetry of the attraction functions are not affected by the restrictive rules.

5. SUMMARY AND CONCLUSIONS

In this paper we attempt to integrate two alternative tenets about the nature of modeling the market share system. One is the widely held view that such a system can be unbindingly described by marketing decision variables with any functional forms; the other is that if logical consistency of the estimated results is to be obtained, the system must be restricted by certain rules. We have introduced a dynamic market share attraction model inspired by the Lanchester combat model to resolve the issue. Our analyses are consistent with the latter tenet in that certain restrictive rules are required but with the difference that the restrictions are applied to attraction functions rather than to explanatory variables and coefficients. Our analyses also are consistent with the first tenet, since our model can be shown to be a general formulation for various models such as the naive, linear, multiplicative, MCI, logarithmic linear, and multinomial logit models.

Integration of model robustness and the descriptive validity of the market share response models through dynamic investigation has broadened our understanding of the nature of competitive effects. On a theoretical level, we show that two axioms in BKL's (1975) controversial work are insufficient and one axiom is incomplete. Decomposition of their symmetry structure of the market share system into within, between and diagonal symmetries forms the basis for our analysis of logical consistency, which marketing researchers have misunderstood and been puzzled by for a long time. Furthermore, a natural extension of the market share theorem is into the arena of consumer behavior. We have mentioned several times in this study that the Lanchester combat model is related to the continuous Markov chain. The prototypical duopoly example is found in Horsky (1977), where attraction function is defined as transition

probability, which, in the consumer choice behavior literature, is interpreted as repeat purchase probability.

On a practical level, it is common practice in applied regression to assume that market structure is stationary. It follows that the observed market share system must be treated as if it were stationary. Brodie and de Kluyver's study provides the consequences of such an assumption. Since all of the naive and econometric models they adopt are static, the performance of the predictive power, which relies on the stability of the observed data, can be equally good (or bad) in generating estimates. A better approach to obtaining consistent methodology for a dynamic market structure is to separate the observed market share system into permanent and time-variant components as suggested in this paper; this method captures the dynamics of the data. The idea that cyclical movements of an economic mechanism can be separated from its permanent component is a very old one (for example, see Fellner 1956; and Friedman 1957), and recently has been applied to decomposition of the market share variable in a study by Crawford and Geurts (1988). We view Crawford and Geurts' decomposition as an alternative approximation of our dynamic system.

APPENDIX I

Proof:

Summing N equations in equation (1), we have:

$$\frac{d}{dt}\left(\sum_{i=1}^{N} m_i\right) = \sum_{i=1}^{N}\left[a_i\left(\sum_{j\neq i} m_j\right) - \left(\sum_{j\neq i} a_j\right) m_i\right] = 0.$$

The condition of change rate zero-sum-constrained is satisfied.

From the given initial state for all t, $t \geq 0$, we have:

$$\sum_{i=1}^{N} m_i(t) = \sum_{i=1}^{N} m_i(0) = 1,$$

the sum-constraint is satisfied.

To show that $m_i \geq 0$, rewrite $dm_i(t)/dt$ by moving $\left(\sum_{j=1}^{N} a_j\right)m_i$ to the left-handside and multiply the whole equation by $\exp\left(\sum_{j\neq i}^{N} a_j\right)t$, such that:

$$e^{\left(\sum_{j\neq i}^{N} a_j\right)t}\left[m_i\left(\sum_{j\neq i}^{N} a_j\right) + \frac{dm_i}{dt}\right] = e^{\left(\sum_{j\neq i}^{N} a_j\right)t}\left[a_i\left(\sum_{j\neq i}^{N} m_j\right)\right]$$

This can be written as:

$$\frac{d}{dt}\left(m_i\, e^{\left(\sum_{j\neq i}^{N} a_j\right)t}\right) = e^{\left(\sum_{j\neq i}^{N} a_j\right)t}\left[a_j\left(\sum_{j\neq i}^{N} m_j\right)\right].$$

Since $m_j \geq 0$ at $t = 0$, the right handside is positive. This shows that $\left[m_i \exp\left(\sum_{j\neq i}^{N} a_j\right)t\right]$ are monotonically increasing, therefore $m_i \geq 0$ for all i Q.E.D.

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