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WP 118

FACULTY WORKING PAPERS

Beyond Risk Aversion: Eccentricity
In Weighted Expected Utility

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Faculty Working Paper No. 118a. April, 1988

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BEYOND RISK AVERSION: ECCENTRICITY IN WEIGHTED EXPECTED UTILITY

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ABSTRACT: In recent survey articles Bell and Farquhar and Machina have brought to the attention of decision scientists an alternative to the expected utility model called weighted expected utility. Developed by MacCrimmon, Chew and Fishburn, this is the simplest alternative to expected utility that permits an interpretation and rationalization of Allais' famous paradox. This paper identifies the two crucial parameters of weighted expected utility -- risk aversion and eccentricity -- by studying demand for insurance. The first of these is a measure that generalizes the Arrow-Pratt risk aversion measure of expected utility. The other, which we call a measure of eccentricity, has no counterpart in expected utility theory. Risk aversion is a concept with which decision analysts are quite comfortable, but eccentricity is simultaneously a new concept and one whose predictions are more subtle than those of risk aversion. In essence, eccentricity is a measure of how susceptible the decision maker is to Allais' paradox or how much he differs from expected utility maximization. Together risk aversion and eccentricity completely determine weighted expected utility and provide insight into the behavior of decision makers under uncertainty, behavior that can be quite different from that predicted by expected utility. Explanations are given of the impact of increases in risk aversion and eccentricity on demand for insurance. Perhaps the most striking difference is that when the decision maker is eccentric, the standard analysis of decision trees by "averaging out and folding back" provides suboptimal decisions and the losses are magnified as the degree of eccentricity grows.

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1. Introduction

The most widely used model of choice under uncertainty is expected utility (Bernoulli (1954), von Neumann and Morgenstern (1944)). Expected utility theory has great appeal because it provides a framework for explaining behavior such as insurance purchasing, hedging, diversification and demand for information. Underlying preferences are summarized by the von Neumann-Morgenstern utility function which can be assessed by relatively simple interrogation procedures (Winkler (1972)). In addition, the resulting mathematical form of the decision maker's objective is linear in probabilities, which makes it especially amenable to study both analytically and computationally.

Over the past few decades, however, a growing list of anomalous experimental and case study results have led many to reject expected utility as the general theory of choice under uncertainty and to search for alternative theories. The list of alternative models of preferences is growing rapidly as decision theorists try to incorporate stylized facts about the deviations of behavior from the classical expected utility model (for example, Becker and Sarin (1987) list fifteen alternatives models).

One particular alternative that Bell and Farquhar (1986) and Machina (1987) have recently brought to the attention of decision scientists is called weighted expected utility. Weighted expected utility preserves much of the analytic structure of the expected utility paradigm while allowing preferences consistent with a wider range of observed behavior. Moreover, it will be argued that weighted expected utility is the simplest alternative to expected utility that permits an interpretation and rationalization of Allais' famous paradox.

However, weighted expected utility is not simple. For example, there are two parameters that determine the behavior of weighted expected utility decision makers. One of these is a measure of risk aversion that generalizes the risk aversion measure of expected utility. The other, which we call a measure of eccentricity, has no counterpart in expected utility theory.

Risk aversion is a concept with which decision scientists are quite comfortable now that there are standard methods for evaluating its magnitude and theories that predict its impact on observable behavior. Eccentricity, on the other hand, is simultaneously a new concept and one whose predictions are more subtle than those of risk aversion. In essence, eccentricity is a measure of how susceptible the decision maker is to Allais' paradox or how much he differs from expected utility maximization. Only in certain situations will the predicted choice of an eccentric decision maker differ from one who is not eccentric, but experimental evidence indicates that these situations are not particularly extraordinary.

Together risk aversion and eccentricity provide insight into the behavior of decision makers under uncertainty, and that behavior can be quite different from the behavior predicted by expected utility. Perhaps the most striking difference was first pointed out by LaValle and Wapman (1986) who noted that when the decision maker is not an expected utility maximizer the standard analysis of decision trees by "averaging out and folding back" provides suboptimal decisions. Using the eccentricity measure, it is possible to predict conditions under which averaging out and folding back leads to serious problems.

In this paper we give geometric expositions to both expected utility and weighted expected utility. The motivation for this focus on the geometry of

preferences is that it provides a convincing case for weighted expected utility as the simplest generalization of expected utility. It also provides suggestions for the axiomatic development of other alternatives to expected utility. Geometric interpretations of important concepts such as risk aversion, risk premium, probability premium and mean preserving spread are developed. We identify the crucial characteristics of weighted expected utility, risk aversion and eccentricity, develop parameters that measure them, and suggest methods for assessing the two parameters in decision analysis. Finally, we explore, in a geometric framework, the impact of these characteristics on behavior in uncertain environments.

2. The Independence Axiom and Weighted Expected Utility

The expected utility axioms of von Neumann-Morgenstern can be summarized by three major assumptions: preferences are transitive, preferences are continuous, and preference rankings are independent of convex mixtures of identical probability distributions. Let a lottery be denoted $\langle X,P\rangle$, where X is a vector of outcomes and P is the corresponding vector of probabilities of these outcomes. The first two axioms imply that there is a continuous utility function over lotteries, $V(\langle X,P\rangle)$, but does not impose structure on this function. The last assumption, the Independence Axiom, states that (for identical outcomes X) if P is preferred to Q, then $\lambda P+(1-\lambda)R$ is preferred to $\lambda Q+(1-\lambda)R$ for any complication R, $\lambda \epsilon (0,1]$. This has the very important implication that the utility function is linear in probabilities and forms an expected utility. The decision maker acts as though he ranks alternative lotteries by comparing the numerical values of their expected utilities,

$V(\langle X,P\rangle) = EU = \Sigma\, p_i\, U(x_i)$. The Independence Axiom is very attractive because of this structure it imposes on the form of the decision maker's objective function.

On the other hand, there is a long history of behavior of decision makers that conflicts with the Independence Axiom, the most famous called the Allais Paradox (Allais and Hagen (1979), but see also Kahneman, Slovic and Tversky (1982), Slovic and Lichtenstein (1983), and Schoemaker (1982)). The paradox involves preference rankings over pairs of lotteries such as the following:

L_1: { \$3,000 for certain

L_2: $\begin{cases} \$4{,}000 & \text{with probability } .75 \\ \$0 & \text{with probability } .25 \end{cases}$

L_3: $\begin{cases} \$4{,}000 & \text{with probability } .15 \\ \$0 & \text{with probability } .85 \end{cases}$

L_4: $\begin{cases} \$3{,}000 & \text{with probability } .20 \\ \$0 & \text{with probability } .80 \end{cases}$

Experimenters repeatedly have found that a large proportion of individuals tested (sometimes more than half) prefer L_1 to L_2 and L_3 to L_4. In the first case (L_1 versus L_2), the individual acts risk averse, desiring the certainty of \$3,000 to a gamble with the same expected value. In the second (L_3 versus L_4), the person typically prefers greater risk, thinking perhaps that since it is unlikely he will get either positive outcome, he might as well go for the bigger one.

This pattern of preferences violates the Independence Axiom. To see why, write the four lotteries using our earlier notation:

$L_1 = \langle(0,3000,4000),\ (0,1,0)\rangle$, $\quad$ $L2 = \langle(0,3000,4000),\ (.25,0,.75)\rangle$

$L_3 = \langle(0,3000,4000),\ (.85,0,.15)\rangle$, $\quad$ $L_4 = \langle(0,3000,4000),\ (.8,.2,0)\rangle$.

Now define another lottery $L_5 = \langle(0,3000,4000), (1,0,0)\rangle$. Then by the Independence Axiom, since L_1 is preferred to L_2, it must be that lottery $\langle X, .2P^1+.8P^5\rangle$ is preferred to lottery $\langle X, .2P^2+.8P^5\rangle$, where P^i is the probability distribution for lottery L_i. But $\langle X, .2P^1+.8P^5\rangle = L_4$ and $\langle X, .2P^2+.8P^5\rangle = L_3$, so the Independence Axiom implies that L_4 is preferable to L_3, contrary to typical preferences.

Chew and MacCrimmon (1979), Chew (1983) and Fishburn (1983) have shown that the Allais Paradox may be explained by splitting the Independence Axiom and replacing it with two related axioms, the Mixture-Dominance Axiom and the Symmetry Axiom (this is Fishburn's terminology although Chew and MacCrimmon have similar axioms). The utility function that follows from these alternative axioms has a form

$$V(\langle X,P\rangle) = \frac{\Sigma\, U(x_i)p_i}{\Sigma\, W(x_i)p_i}.$$

This has been alternatively called alpha-nu utility, ratio form utility, weighted linear utility, and weighted expected utility. We choose to use this last name since it accurately reflects the fact that the utility can be written as a lottery dependent expected utility (Becker and Sarin (1987)) where the lottery dependent utility function is adjusted by a weighting function of the probabilities:

$$V(\langle X,P\rangle) = \sum_i \frac{U(x_i)}{\sum_j W(x_j)p_j}\, p_i.$$

3. The Geometry of Expected Utility

To study the utility for lotteries of the form $\langle X,P\rangle$, we fix X and consider utility as a function of only P. For simplicity suppose there are three possible outcomes, $X=(x_1,x_2,x_3)$ where $x_1 < x_2 < x_3$. This allows the representation of preferences in the triangle diagram in Figure 1 where the probabilities of outcomes 1 and 3 are plotted. Implicitly the probability of outcome 2 is determined by the fact that probabilities add to one. When (p_1,p_3) is on the hypotenuse in Figure 1, p_2 must be zero; but for points on the interior of the triangle, p_2 is positive. At the origin both p_1 and p_3 are zero, so p_2 takes on the value 1.

In Figure 1 there are several positively sloped parallel dashed lines drawn in the simplex. These graph the combinations of probabilities that hold constant the expected value of the lottery: $p_1x_1+(1-p_1-p_3)x_2+p_3x_3 = \bar{x}$. That is, they are graphs of

$$p_3 = \frac{\bar{x} - x_2}{x_3 - x_2} + \frac{x_2 - x_1}{x_3 - x_2} p_1. \tag{1}$$

As the mean of the lottery increases, these iso-mean lines shift toward the upper left corner of the simplex, since x_3 is the largest outcome and x_1 is the smallest. Moving along an iso-mean line from small values of (p_1,p_3) to larger values (i.e., moving in a northeasterly direction) implies an increase in the riskiness of the lottery (a mean preserving spread in the terminology of Rothschild and Stiglitz (1970)).

An indifference curve in the probability triangle is made up of combinations of (p_1,p_3) such that expected utility is constant, $p_1U(x_1)+(1-p_1-$

$p_3)U(x_2)+p_3U(x_3) = \bar{U}$. Solving for p_3 in terms of p_1 and using U_i to denote $U(x_i)$, the indifference curves for expected utility are given by

$$p_3 = \frac{\bar{U} - U_2}{U_3 - U_2} + \frac{U_2 - U_1}{U_3 - U_2} p_1. \tag{2}$$

The slopes of these indifference curves are constant and independent of $\bar{U}$ and thus are parallel straight lines. The magnitude of the slope is directly related to the degree of risk aversion as measured by the Arrow-Pratt measure of absolute risk aversion, $r = -U''/U'$ (Arrow (1970), Pratt (1964)). To see this, manipulate the slope of the indifference curve and write it as

$$\left[- \frac{\dfrac{U_3 - U_2}{x_3 - x_2} - \dfrac{U_2 - U_1}{x_2 - x_1}}{\dfrac{U_3 - U_2}{x_3 - x_2}} + 1 \right] \frac{x_2 - x_1}{x_3 - x_2} = [\bar{r}+1] \frac{x_2 - x_1}{x_3 - x_2}, \tag{3}$$

which is positively related to the discrete version of the Arrow-Pratt measure, denoted $\bar{r}$. The more risk averse an expected utility maximizer, the steeper are the indifference curves in the probability triangle.

Figure 2 plots both iso-mean lines (dashed lines) and indifference curves (solid lines) for an expected utility maximizer. If U(x) is risk averse, the indifference curves are steeper than the iso-mean lines as depicted in Figure 2. If U(x) is risk loving, the indifference curves are flatter than the iso-mean lines.

The concepts of risk premium and certainty equivalent are central to choice under uncertainty. For a given lottery and decision maker, the risk premium π is the amount the decision maker would pay to receive the expected value of the lottery in lieu of the lottery. The certainty equivalent π_a (also the asking price for the lottery) is the amount that makes the decision maker indifferent between having that amount for certain or the lottery. It follows that $\pi_a = \mu - \pi$, where μ is the expected value of the lottery. Consider the lottery at point A in Figure 2. The certainty equivalent for this lottery is x_2 because the indifference curve at A runs through the origin. The risk premium for lottery A is the difference between the mean of the lottery and x_2. Unfortunately, the magnitude of the risk premium cannot be easily measured in the triangle diagram. In fact, for most lotteries the risk premium cannot be shown at all because the certainty equivalent is not one of the three outcomes at the corners of the triangle diagram. Because of this geometric difficulty, we turn instead to a related measure called the probability premium of the lottery.

Consider lottery B in Figure 2. A risk neutral decision maker would be indifferent between lotteries B and C because they fall on the same iso-mean line. The risk averse decision maker whose indifference curves are plotted in Figure 2 would be indifferent between lotteries B and D because they fall on the same indifference curve. We define the <u>probability premium</u> to be the difference between the probabilities of x_3 in lotteries D and C. Using p for probability premium, $p = p_3^D - p_3^C$. This is a generalization of Pratt's probability premium and is, like Pratt's, an increasing function of risk aversion.[1] Unlike the risk premium, the probability premium can always be found for any

lottery in the triangle diagram, and thus we will make use of it rather than the risk premium in what follows.

The concept of increasing risk can be shown in the triangle diagram. For an expected utility maximizer, if the Arrow-Pratt local measure of risk aversion is positive for all levels of wealth, then an increase in risk lowers expected utility (Rothschild and Stiglitz (1970)).[2] Consider again the decision maker faced with lottery B in Figure 2. A movement along the iso-mean line to lottery C would represent an increase in risk (it is a mean preserving spread in the sense of Rothschild and Stiglitz) and would put the decision maker on a lower indifference curve. It follows that risk averse decision makers dislike mean preserving spreads. The lotteries along line segment BC become increasingly less attractive. Conversely, a move from lottery B to lottery E would correspond to a mean preserving decrease in risk which would put the decision maker on a higher indifference curve.

4. The Geometry of Weighted Expected Utility

The fact that the indifference curves of an expected utility maximizer are parallel might be interpreted as saying that they intersect each other at infinity. The insight of Chew, MacCrimmon and Fishburn was to maintain the linearity of the indifference curves but to allow them to intersect at a point other than infinity.

The Mixture-Dominance Axiom states that when two points in the probability triangle are indifferent, then all the points on the line segment between them are indifferent; this implies that indifference curves in the simplex figure are straight lines but not necessarily with the same slope. The Symmetry Axiom

implies that the indifference curves fan out from a unique hub point. See point (α,β) in Figure 3. Notice that the indifference lines get progressively steeper as one moves from the lower right corner of the probability triangle toward the upper left corner. From the discussion of the previous section this might be interpreted as increases in the realized risk aversion of the decision maker.

The geometry of these indifference curves -- spokes fanning out from a hub point -- is not the only alternative to the parallel straight lines of expected utility. One possible alternative would be non-linear and non-intersecting indifference curves over the entire Euclidian n-space. Another would have linear indifference curves emanating from a distinct line segment instead of a single hub point. These indifference curves would not intersect at a single point, but the linearity of the indifference curves and the their progressive steepening in the upper left corner would be preserved. A kaleidoscope of alternatives suggest themselves, but the fanning out from a single hub point clearly is the least complicated alternative to the expected utility model.

If preferences are transitive (indifference curves do not cross for legitimate probability distributions), then the hub point must be located outside the probability simplex, but the Symmetry Axiom does not require the hub point be at infinity. Preferences consistent with the Allais Paradox require the hub point to be located in the third quadrant as shown in Figure 3. Chew and Waller's (1986) experimental evidence also implies that the hub point for most individuals is located in the third quadrant. For these reasons we limit our discussion to third quadrant hub points. However, our results are general and can be extended to hub points lying in the first quadrant. Hub points located in the second and fourth quadrants are ruled out because

preferences implied by those locations are not monotonic. (See Fishburn (1982) for a generalization that permits intransitivities.)

Taken together, the Mixture-Dominance Axiom and the Symmetry Axiom imply that the decision maker will choose among lotteries so as to maximize a utility function of the form

$$V(\langle X,P\rangle) = \frac{\Sigma\, U(x_i)p_i}{\Sigma\, W(x_i)p_i}. \tag{4}$$

This <u>weighted expected utility</u> is nonlinear in the probabilities and thus is not an expected utility. The numerator function U is called a <u>valuation</u> function and the denominator function W is called a <u>weighting</u> function. If the weighting function is a constant, then the numerator is independent of the probabilities, and weighted expected utility reduces to expected utility.

The valuation and weighting functions are not uniquely defined for a given preference for lotteries. (Recall that the von Neumann-Morgenstern utility function may be modified by a positive affine transformation.) Fishburn (1983) has shown that if valuation function U and weighting function W represent a person's preferences, then so do any linear transformations of these, $U^* = aU + bW$, $W^* = cU + dW$, as long as $ad-bc > 0$. One must be careful in making statements about the valuation or weighting functions given this ability to transform them without making changes in the essential preferences.

5. Risk Aversion and Eccentricity

Suppose an individual with initial wealth x is faced with a stochastic additional source of wealth, ϵ, which has expected value μ and small variance

σ. How much would the individual pay to eliminate this additional uncertainty? In weighted expected utility, the risk premium π that makes the decision maker indifferent between paying the premium or bearing the entire risk is defined implicitly by:

$$\frac{U(x+\mu-\pi)}{W(x+\mu-\pi)} = \frac{E[U(x+\epsilon)]}{E[W(x+\epsilon)]}. \tag{5}$$

Following standard practice, take Taylor series approximations of the U and W functions on the left-hand side of first order in $\mu-\pi$ and on the right-hand side of second order in ϵ. Solving for π gives

$$\pi = \frac{R + \mu E}{2/(\sigma^2+\mu^2) + E}, \tag{6}$$

where

$$R = \left[\frac{UW'' - U''W}{U'W - W'U}\right], \tag{7}$$

and

$$E = \left[\frac{U'W'' - U''W'}{U'W - W'U}\right]. \tag{8}$$

The parameter R will be called the <u>local measure of risk aversion</u> for weighted expected utility. For a constant weighting function, W(x) = constant, R reduces to the Arrow-Pratt measure of absolute risk aversion, $-U''/U'$. It is easy to check that R is also independent of any equivalent representations of the U and W functions $U^* = aU + bW$, $W^* = cU + dW$, $ad>bc$. As in Pratt's (1964) derivation of the risk premium, the local measure of risk aversion for weighted expected utility, R, is analytically useful. A person with a large measure of

risk aversion **R** would pay more to avoid small risks than a person with a small **R**, as can be seen from Equation (6).

We call the parameter **E** the <u>local measure of eccentricity</u>. Like **R**, it is independent of allowable transformations of U and W. A decision maker with large eccentricity, **E**, may or may not value insurance more than a decision maker with small eccentricity.

To explore the behavior associated with risk aversion and eccentricity in weighted expected utility, it is useful to express the coordinates of the hub point in terms of **R** and **E**. In Figure 3 the indifference curves radiate from the hub point labelled (α,β). They are graphs of the relationship

$$V = \frac{p_1U_1 + (1-p_1-p_3)U_2 + p_3U_3}{p_1W_1 + (1-p_1-p_3)W_2 + p_3W_3}, \tag{9}$$

for a fixed V.

To identify the hub point, set V in Equation (9) equal to two arbitrary values (V=0 and V=1, for example) and solve the resulting two equations for the two unknown values of p_1 and p_3. The hub point, $(p_1,p_3) = (\alpha,\beta)$, can be expressed in terms of the local measures of risk aversion and eccentricity:

$$(\alpha,\beta) = \left[\frac{-1}{\mathbf{E}\ (x_2-x_1)}, \frac{-(\mathbf{R}+1)}{\mathbf{E}\ (x_3-x_2)} \right]. \tag{10}$$

Here the values of **R** and **E** are the discrete versions of Equations (7) and (8):

$$R = \frac{U_2 \left[\frac{W_3-W_2}{x_3-x_2} - \frac{W_2-W_1}{x_2-x_1}\right] - W_2 \left[\frac{U_3-U_2}{x_3-x_2} - \frac{U_2-U_1}{x_2-x_1}\right]}{\frac{U_3-U_2}{x_3-x_2} W_2 - \frac{W_3-W_2}{x_3-x_2} U_2}, \tag{11}$$

and

$$E = \frac{\frac{U_3-U_2}{x_3-x_2} \left[\frac{W_3-W_2}{x_3-x_2} - \frac{W_2-W_1}{x_2-x_1}\right] - \frac{W_3-W_2}{x_3-x_2} \left[\frac{U_3-U_2}{x_3-x_2} - \frac{U_2-U_1}{x_2-x_1}\right]}{\frac{U_3-U_2}{x_3-x_2} W_2 - \frac{W_3-W_2}{x_3-x_2} U_2}. \tag{12}$$

Notice that β increases with **E** and decreases with **R**, while α increases with **E** but is independent of **R**.

The hub point could be represented in polar coordinates (ρ,θ) that give the radius, ρ, and angle, θ, of the hub point as shown in Figure 3.

$$\rho = 1/E \sqrt{\frac{1}{(x_2-x_1)^2} + \frac{(R+1)^2}{(x_3-x_2)^2}} \tag{13}$$

$$\theta = \cos^{-1}\left\{\left[\sqrt{\frac{1}{(x_2-x_1)^2} + \frac{(R+1)^2}{(x_3-x_2)^2}}\right]^{-1}\right\} \tag{14}$$

It is easy to see that as local risk aversion, **R**, increases, the hub point rotates counterclockwise and its radius, ρ, increases. The slopes of all the indifference curves in the unit triangle increase, indicating greater risk aversion. As eccentricity, **E**, falls toward zero, the hub point moves out along a ray from the origin to minus infinity and the indifference curves become parallel; zero eccentricity implies the utility function is an expected

utility. As eccentricity increases, the hub point moves toward the origin and indifference curves in the lower right corner of the unit triangle become flatter, indicating less risk aversion, while those in the upper left corner become steeper, indicating greater risk aversion. In fact, for fixed monetary outcomes, a completely rational weighted expected utility maximizer may act risk averse or risk seeking depending only on the probabilities. This slightly schizophrenic behavior is consistent with the Allais Paradox and is our motivation for calling **E** a measure of eccentricity.

The dependence of the hub point on the magnitude of the measures of local risk aversion and eccentricity suggests a simple method of simultaneously assessing risk aversion and eccentricity. Suppose that a decision maker was asked to identify the probability of best and worst outcomes that would leave him indifferent between that lottery and some reference lottery (like lottery A or B in Figure 3). That is, if the reference lottery was point A in Figure 1 assess the location of A′ on the hypotenuse of the probability triangle. Repeat the assessment for lottery B. By then tracing the two lines connecting A to A′ and B to B′, the hub point (α,β) can be located and then equation (10) can be solved for the discrete versions of the local measures of risk aversion and eccentricity.

6. Probability Premium

How does the probability premium vary in response to changes in **R** and **E**? Suppose that the decision maker faces the probability distribution given by point A in Figure 4. By drawing the indifference curve that goes through this probability vector and the hub point and also drawing the iso-mean line through

point A, we can identify the probability premium $p_3^B - p_3^C$ on the p_3 axis. It is not difficult to see that the counterclockwise rotation of the hub point that follows from an increase in risk aversion implies that the probability premium is increased. The same conclusion is reached if we start at point A′ instead of point A. Thus the general result is that a more risk averse individual will demand a higher probability premium for any risk.

On the other hand, an increase in eccentricity may or may not increase the probability premium. The hub point moves toward the origin along a ray as eccentricity increases, so the probability premium of a lottery like A will decrease with **E**, whereas the probability premium of lottery A′ will increase as **E** increases. The line from the hub point through the origin divides the probability triangle into two regions - one in which the probability premium increases with eccentricity and the other in which it decreases.

These observations establish our first proposition.

<u>Proposition 1</u>: The probability premium is increasing in R, E held constant, and may increase or decrease with changes in E, R held constant.

7. The Impact of Increasing Risk

Risk averse eccentrics do not always dislike increases in risk. At first blush this may seem to contradict Equation (6), which shows the risk premium π as a monotonic increasing function of variance σ^2 even when the measure of eccentricity is positive. It would be incorrect to generalize from this result, however, because it was derived under the assumption that the magnitude of the risk was small enough that a Taylor series approximation was reasonable.

The Allais Paradox shows that not all mean preserving spreads reduce

weighted expected utility. The particular risk (or probability distribution) faced by the decision maker influences his reaction to an increase in risk, even when the measure of local risk aversion is globally positive. This feature of behavior is captured by the degree of eccentricity.

There are regions in the probability triangle of a weighted expected utility maximizer in which increases in risk hurt the decision maker, help, or leave the decision maker indifferent. To identify these regions, we again superimpose iso-mean lines and the indifference curves of a weighted expected utility maximizer (Figure 5).

There is exactly one iso-mean line that, if extended, passes through the hub point. This is identified as line PQ in Figure 5. All the probability distributions along PQ have the same mean but different risks, yet the weighted expected utility maximizer with positive local risk aversion and hub point (α,β) treats them all as equally desirable. For probability distributions in the shaded region above this iso-mean line, the decision maker acts risk averse, disliking all mean preserving spreads. In the region below PQ, the person acts like a risk seeker, since mean preserving increases in risk increase weighted expected utility.

From this we conclude that with weighted expected utility it is insufficient to check only a change in probability distribution to see if it represents a mean preserving increase in risk (satisfies integral conditions like (1n) and (2n)). In addition, one must also check to see if the mean of the distribution is sufficiently large that risk aversion dominates eccentricity. Specifically, the crucial mean, μ^*, of the lotteries that lie along line PQ in Figure 5 can be expressed in terms of the discrete measures of risk aversion and eccentricity:

$$\mu^* = x_2 - R/E. \tag{15}$$

Any increase in risk that begins with a mean less than μ^* will increase the weighted expected utility of a locally risk averse but eccentric decision maker. The larger the local measure of risk aversion relative to the local measure of eccentricity, the smaller the set of probability distributions for which increases in risk are utility increasing. This result establishes the second proposition.

<u>Proposition 2</u>: In the case of three outcomes, $x_1 < x_2 < x_3$, a mean preserving spread decreases (leaves constant) (increases) weighted expected utility if and only if the mean of the probability distribution, μ, is greater than (equal to) (less than) $x_2 - R/E$.

This proposition is the first illustration of a prediction of behavior that depends on the magnitude of eccentricity. Even though an individual is basically risk averse, if the choices all involve high probabilities of low payoffs and the eccentricity is large relative to risk aversion, the observed behavior will appear to be risk seeking. The greater the eccentricity, the larger is the set of probability distributions for which this can occur. When low payoffs have high probabilities, the Allais paradox seems to imply that eccentric decision makers discount probability differences and concentrate instead on the magnitude of rewards.

8. Constant Risk Aversion and Eccentricity

An interesting class of functions is that which has constant measures of local risk aversion and eccentricity. The simplest forms for the U and W functions that will generate positive and constant **R** and **E** are $U(x) = -e^{-\lambda x}$ and $W(x) = e^{\gamma x}$, $\lambda > \gamma > 0$. Of course, allowable transformations of these U and W functions can also be used. These are the unique functions with constant local measure of risk aversion, $\mathbf{R} = \lambda - \gamma$, and constant local eccentricity, $\mathbf{E} = \lambda\gamma$.

Suppose in the constant **R** and **E** case that two decision makers are equally eccentric, but one has a larger measure of local risk aversion. It follows that the decision maker with larger **R** will have the larger risk premium for any risk ϵ. This is a stronger result than that presented in Section 5 because we do not restrict the size of the risk to "small." It is less general than that result, however, because it applies only to the constant **R** and **E** class of functions.

<u>Proposition 3</u>: Let $U_i(x) = -\exp(-\lambda_i x)$ and $W_i(x) = \exp(\gamma_i x)$, $\lambda_i > \gamma_i > 0$ for $i = 1,2$. Then $\mathbf{R}_i = \lambda_i - \gamma_i$, and $\mathbf{E}_i = \lambda_i\gamma_i$ for all x. If $\mathbf{R}_1 > \mathbf{R}_2$ and $\mathbf{E}_1 = \mathbf{E}_2$, then $\pi_1 > \pi_2$.

Proof: First solve Equation (5) explicitly for the risk premium,

$$\pi = (\log(E[e^{-\lambda\epsilon}]) - \log(E[e^{\gamma\epsilon}]))/(\lambda+\gamma). \tag{16}$$

If γ is adjusted to hold **E** constant, then **R** increases with increases in λ. Suppose that $\lambda_1 > \lambda_2$; we want to show that $\pi_1 - \pi_2$ is positive. To do this, define $\varphi = \exp(-\lambda_2\epsilon)$ and $\psi = \exp(\gamma_2\epsilon)$. Invert these to get $\epsilon = -1/\lambda_2 \log(\varphi)$ and $\epsilon = 1/\gamma_2 \log(\psi)$. We can write $E[\exp(-\lambda_1\epsilon)] = E[\varphi^{\lambda 1/\lambda 2}]$. Applying Jensen's inequality, noting that the assumptions on λ's imply the function $\varphi^{\lambda 1/\lambda 2}$ is

convex in φ, we get $\log(E[\varphi^{\lambda 1/\lambda 2}]) > \log(E(\varphi)^{\lambda 1/\lambda 2}) = \lambda_1/\lambda_2 \log E[\varphi]$. The analogue for the gamma term is $\log(E[\exp(\gamma_1\epsilon)]) < \lambda_2/\lambda_1 \log E[\psi]$. Using these to express the difference in risk premiums,

$$\pi_1 - \pi_2 > \frac{((\lambda_1/\lambda_2)^2-1)(\lambda_2/\lambda_1)\ \{\gamma_2\log(E[\exp(-\lambda_2\epsilon)]) + \lambda_2\log(E[\exp(\gamma_2\epsilon)])\}}{(\lambda_1+\gamma_1)\ (\lambda_2+\gamma_2)}. \quad (17)$$

The first term in the numerator is positive by assumption that $\lambda_1 > \lambda_2$, and the right-hand side will be positive if the term in curly brackets is positive. To show this, note that $\exp(-\lambda_2\epsilon)$ and $\exp(\gamma_2\epsilon)$ are convex functions, so applying Jensen's inequality gives

$$\gamma_2\log(E[\exp(-\lambda_2\epsilon)])+\lambda_2\log(E[\exp(\gamma_2\epsilon)]) > \gamma_2\log(\exp(-\lambda_2E[\epsilon]))+\lambda_2\log\exp(\gamma_2E[\epsilon]))$$
$$= -\gamma_2\lambda_2E[\epsilon] + \gamma_2\lambda_2E[\epsilon] = 0. \quad (18)$$

9. Increasing Global Risk Aversion

Suppose two decision makers are equally eccentric at each level of wealth, but one has a larger measure of local risk aversion at every level of wealth. If eccentricity is zero, Pratt (1964) has shown that the globally more risk averse decision maker will be willing to pay more for insurance. Does the same result hold for weighted expected utility?

In Section 8, we showed that an increase in constant local risk aversion, constant eccentricity unchanged, increases the risk premium. This is not a general result, however, because it depends on particular U and W functions. The general result must consider the case where R(x) and E(x) vary with wealth, and $E_1(x) = E_2(x)$ and $R_1(x) > R_2(x)$ for all x. This is a more difficult

proposition to verify because the equations that define R and E are nonlinear second-order differential equations in the unknown valuation and weighting functions U and W. In fact, by simple change of variables they can be transformed into Ricatti differential equations[3] with general forcing functions that are linear combinations of R(x) and E(x). Since such equations have no closed form solutions, one cannot integrate them the same way one can the linear differential equation for the Arrow-Pratt measure of risk aversion. A proof of the following proposition is constructed in the appendix using discrete approximations for the U and W functions. A more general proof is left to those whose mathematical tools are more elegant then ours.

Proposition 4: Consider two weighted expected utility functions, V_1 and V_2, with $E_1(x) = E_2(x)$ and $R_1(x) > R_2(x)$ for all x. Then $\pi_1 > \pi_2$.

10. Decisions with Information

Suppose now that consequences depend on the action, a, of the decision maker and the state, s, according to a reward function x=x(a,s). The action is selected after learning the value of an information variable, y, and the decision rule is denoted $\alpha(y)$. The joint probability density of the two random variables s and y will be denoted p(s,y). If the decision maker is eccentric, we can write the objective function

$$V[\alpha] = \frac{\iint U(x(\alpha(y),s))\ p(s,y)dsdy}{\iint W(x(\alpha(y),s))\ p(s,y)dsdy}. \tag{19}$$

It will be useful to write the valuation/weighting and reward functions as: $U(x(a,s)) = U(a,s)$ and $W(x(a,s)) = W(a,s)$.

What is the optimal decision rule for an individual with weighted expected utility (19) when the information may be incorporated in the choice of action? Suppose that $\alpha(y)$ is the optimal decision rule and consider adding a multiple ϵ of an arbitrary decision function $Z(y)$ to it. This gives weighted expected utility

$$V[\alpha + \epsilon Z]. \tag{20}$$

Since $\alpha(y)$ is optimal, $\partial V/\partial\epsilon = 0$ at $\epsilon=0$, or else the weighted expected utility could be increased by a small adjustment of the decision rule. That is,

$$\left.\frac{\partial V}{\partial \epsilon}\right|_{\epsilon=0} = 0 = \frac{\iint U_a(\alpha(y),s)\ Z(y)\ p(s,y)dsdy}{\iint W(\alpha(y),s)\ p(s,y)dsdy} - \frac{\iint U(\alpha(y),s)\ p(s,y)dsdy}{(\iint W(\alpha(y),s)\ p(s,y)dsdy)^2}\ (\iint W_a(\alpha(y),s)\ Z(y)p(s,y)dsdy) \tag{21}$$

for all functions, Z, where U_a and W_a denote partial derivatives with respect to the action. Rearrangement gives

$$\iint \{U_a(\alpha(y),s)/E[U] - W_a(\alpha(y),s)/E[W]\}\ Z(y)p(s,y)dsdy = 0, \tag{22}$$

for all functions, Z. Applying the fundamental theorem of the calculus of variations and dividing by the prior probability density of the information $p(y)$, we can write the optimality condition as

$$\int \{U_a(\alpha(y),s)/E[U] - W_a(\alpha(y),s)/E[W]\}\ p(s|y)ds = 0, \tag{23}$$

for all possible values of the information variable, y. The most compact expression of this is

$$\frac{E[U_a|y]}{E[U]} = \frac{E[W_a|y]}{E[W]}, \quad \text{for all } y. \tag{24}$$

Actions are optimal only when their marginal impact on the rate of growth of numerator of the weighted expected utility equals the rate of growth of the denominator. If the decision maker is an expected utility maximizer, then the right hand side of (24) vanishes and the criteria becomes the standard $E[U_a|y]=0$: conditional expected marginal utility must be zero.

The denominators of (24) are <u>not</u> conditional expectations. That is, the rate of growth is calculated upon a base that is the average over all possible signals, $E[E[U|y]] = E[U]$. This does not matter if eccentricity is zero since the right hand side of (24) is then zero, but it is crucial when the decision maker is eccentric, as will be shown below.

The traditional Bayesian analysis of decision problems with information begins by supposing the decision maker has observed the random variable y, modified his probability distribution using Bayes rule, and evaluated the resulting prospect by

$$V[a|y] = \frac{\int U(a,s)\ p(s|y)\ ds}{\int W(a,s)\ p(s|y)\ ds}. \tag{25}$$

What is the optimal action, given the above posterior evaluation of the weighted expected utility? The simplest form of the first-order conditions is

$$\frac{E[U_a|y]}{E[U|y]} = \frac{E[W_a|y]}{E[W|y]}, \quad \text{for all } y. \tag{26}$$

Contrasting these conditions (26) with the optimality conditions (24) derived above, we now see that the expectation in the denominator is conditional upon the information. There is no reason for the optimal decision with information y to maximize the posterior weighted expected utility and <u>vice versa</u>. Only in the case of expected utility (eccentricity equals zero) will the two criteria, (24) and (26), generate identical decision rules.

The traditional process of analyzing a decision problem in stages and then averaging out and folding back corresponds to the condition (26). Positive eccentricity is a sign that such "backward induction" will lead to suboptimal decisions. LaValle and Wapman (1986) first noticed that averaging out and folding back decision trees leads to incorrect choices when the Independence Axiom is violated (see also Hazen (1987)). In general decision makers should not use the standard extensive form analysis of decision trees, as the following example illustrates.

<u>Example</u>: Suppose the state takes on one of two values, 0 and 1, and the information variable y also takes on values 0 or 1, where y=0 predicts s=0 and y=1 predicts s=1. The joint probability distribution for the discrete random variables s and y are given in Table 1 along with the conditional posterior probabilities. The information is not a perfect predictor of state.

TABLE 1

	Joint Probabilities				Conditional Probabilities	
	y=0	y=1	Marginals		y=0 observed,	y=1 observed
s = 0	0.42	0.18	0.6	P(s=0|y)	0.84	0.36
s = 1	0.08	0.32	0.4	P(s=1|y)	0.16	0.64
Marginals	0.5	0.5				

For simplicity it will be assumed that the monetary consequence is jointly determined by action and state according to the reward function $x = 1 - 2a + 4as$. In this example the decision maker is limited to one of three action values, -1, 0 or +1. The decision problem is described in extensive form in the decision tree of Figure 6.

The decision problem cannot be analyzed without specifying the degree of risk aversion and the measure of eccentricity. Suppose that risk aversion and eccentricity measures are constants. We will consider only one value for risk aversion, but will explore how the problem varies as the decision maker's eccentricity increases. In Table 2 three scenarios are analyzed. The first corresponds to an expected utility maximizer since the measure of eccentricity is zero. The other two involve identical measures of risk aversion but progressively larger positive eccentricity. In all situations the optimal decision rule is found by evaluating all possible rules and selecting the one that maximizes weighted expected utility. The Bayesian decision rule is found by the traditional technique of working backward through the decision tree in Figure 6.

TABLE 2

R	E	Decision Rules* Bayesian a_0	Bayesian a_1	Optimal a_0	Optimal a_1	WEU of Bayesian	WEU of Optimal	Percent Loss
0.1	0.00	-1	+1	-1	+1	-.83555	-.83555	0.0%
0.1	0.42	-1	+1	-1	0	-.13296	-.13182	0.9
0.1	4.20	-1	+1	-1	0	-.00712	-.00413	72.4

* $a_y = \alpha(y)$

As can be seen in Table 2, when eccentricity is zero there is no loss at all from using the traditional Bayesian approach to analyzing the decision problem. Moreover, when eccentricity is small the loss from the traditional procedure is less than one percent of the maximum possible weighted expected utility value. Very eccentric decision makers, on the other hand, would see a loss of almost 75% if they were to follow the decision analysis procedures described in most textbooks (see Raiffa (1968)).

11. Conclusions

We have shown how weighted expected utility can be used to analyze decisions under uncertainty. In particular, we have developed the two important parameters that characterize the behavior of weighted expected utility maximizers. These parameters are the local measure of risk aversion and the local measure of eccentricity. Geometric interpretations of these parameters were explored, and predictions of behavior developed. The primary conclusions are these. First, even if a decision maker is eccentric, greater risk aversion

implies that the decision maker places more value on insurance. Second, even if the measure of local risk aversion is positive for all levels of wealth, it is possible for an eccentric decision maker to prefer some increases in risk. Third, the more significant risk aversion is relative to eccentricity, the smaller is the set of circumstances which produces this "paradoxical" behavior. Fourth, as a decision maker becomes more eccentric, he acts more risk averse toward lotteries that offer large payoffs with high probabilities and less risk averse toward lotteries offering small payoffs with high probabilities. Fifth, the traditional backward induction strategy of "averaging out and folding back" for choosing decision rules is suboptimal for eccentric decision makers and the magnitude of the error is positively related to the degree of eccentricity.

Much is left undone. Clearly all applications of decision theory previously modelled with expected utility are now open to reinvestigation with weighted expected utility. It will be interesting to see if standard results in areas such as portfolio theory and the theory of the firm carry over under weighted expected utility. We expect not. In addition, consider the following issues. The weighting function that explains the Allais Paradox puts greater relative emphasis on large outcomes. Is there a statistic of the probability distribution that corresponds to the eccentric's emphasis on large outcomes similar to the way that variance corresponds to the risk averter's dislike of risk? Finally, a closely related issue is whether there is a simple relationship between two probability distributions that establishes dominance for eccentrics the way stochastic dominance does for risk averters. These and many other issues remain to be explored.

Appendix - Proof of Proposition 4

In this appendix we will show that when the lottery has discrete outcomes and eccentricity is unchanged, an increase in local risk aversion will increase the risk premium. As mentioned in the text, a completely general proof is difficult to construct since the equations defining the valuation function, $U(x)$, and weighting function, $W(x)$, in terms of the measures of local risk aversion, $R(x)$, and eccentricity, $E(x)$, are nonlinear, second-order differential equations. In this appendix attention will be limited to a special case that is nonetheless general enough that it can be inductively extended.

In particular, suppose that the lottery takes on one of five values $\{x_1, x_2, x_3, x_4, x_5\}$, where for simplicity the difference between successive values is one unit, $x_{i+1} - x_i = 1$. The allowable transformations of the U and W functions will be used to set the values at x_1 and x_5 to $U(x_1)=0$, $U(x_5)=1=W(x_1)=W(x_5)$. The values of the two functions at the intermediate outcomes will be determined by the local measures of risk aversion and eccentricity. For example, at $x=x_2$

$$R_2 = \frac{U_2\,[(W_3-W_2) - (W_2-W_1)] - W_2\,[(U_3-U_2) - (U_2-U_1)]}{W_2\,(U_3-U_2) - U_2\,(W_3-W_2)}, \qquad \text{(i)}$$

and

$$E_2 = \frac{(U_3-U_2)\,[(W_3-W_2) - (W_2-W_1)] - (W_3-W_2)\,[(U_3-U_2) - (U_2-U_1)]}{W_2\,(U_3-U_2) - U_2\,(W_3-W_2)}. \qquad \text{(ii)}$$

Compared with equations (11) and (12), these equations do not include differences in outcome values such as x_3-x_2, since we have set them to unity for simplicity. It is also important to notice that although R_2 and E_2 depend on U_1 and W_1 which will be normalized to 0 and 1 respectively, and R_4 and E_4 depend on U_5 and W_5 which will also be normalized, R_3 and E_3 are completely

free of this normalization. This is the reason for using five outcomes in the lottery. Five is the smallest set that has the property that at least one pair of R and E values do not depend directly on normalized values of U and W.

The valuation and weighting functions will be treated as piece-wise linear for values of the outcome that fall between the discrete values $\{x_1, x_2, x_3, x_4, x_5\}$.

The objective of the first step in the proof is to express the valuation and weighting functions explicitly in terms of the measures of risk aversion and eccentricity. Treating (i) and (ii) as equations defining U_2 and W_2, straightforward algebra allows us to solve them to get

$$U_2 = \frac{0 + U_3(1+R_2)}{1+R_2 + 1+E_2} \tag{iii}$$

$$W_2 = \frac{1 + W_3(1+R_2)}{1+R_2 + 1+E_2}, \tag{iv}$$

where the normalized values of U_1 and W_1 have been used.

Notice that discrete measures of risk aversion and eccentricity always appear added to 1; for notational simplicity from this point forward, write $1+R_i$ as r_i (not to be confused with the Arrow-Pratt measure of risk aversion) and $1+E_i$ as e_i . By similar derivations we can express U_3, U_4, W_3 and W_4.

$$U_2 = \frac{0 + U_3 r_2}{r_2 + e_2}, \tag{v}$$

$$U_3 = \frac{U_2 + U_4 r_3}{r_3 + e_3}, \tag{vi}$$

$$U_4 = \frac{U_3 + 1 r_4}{r_4 + e_4}, \tag{vii}$$

$$W_2 = \frac{1 + W_3 r_2}{r_2 + e_2}, \tag{viii}$$

$$W_3 = \frac{W_2 + W_4 r_3}{r_3 + e_3}, \tag{ix}$$

$$W_4 = \frac{W_3 + 1 r_4}{r_4 + e_4}. \tag{x}$$

After algebraic manipulation one can express U_2, U_3, U_4 and W_2, W_3, W_4 entirely in terms of risk aversion and eccentricity measures.

$$U_2 = r_2 r_3 r_4/\Theta, \tag{xi}$$

$$U_3 = (r_2+e_2) r_3 r_4/\Theta, \tag{xii}$$

$$U_4 = ((r_2+e_2)(r_3+e_3)-r_2) r_4/\Theta, \tag{xiii}$$

$$W_2 = [((r_3+e_3)(r_4+e_4)-r_3) + r_2 r_3 r_4]/\Theta, \tag{xiv}$$

$$W_3 = [(r_4+e_4) + (r_2+e_2) r_3 r_4]/\Theta, \tag{xv}$$

$$W_4 = [1 + ((r_2+e_2)(r_3+e_3)-r_2) r_4]/\Theta, \tag{xvi}$$

where

$$\Theta = (r_2+e_2)(r_3+e_3)(r_4+e_4) - r_2(r_4+e_4) - r_3(r_2+e_2). \tag{xvii}$$

These are the equations that explicitly determine the valuation and weighting function in terms of the measures of risk aversion and eccentricity.

Next, consider the equation defining the risk premium, π,

$$\frac{U(x+\mu-\pi)}{W(x+\mu-\pi)} = \frac{E[U(x+\epsilon)]}{E[W(x+\epsilon)]}. \tag{xviii}$$

Suppose that initially the solution of this equation puts $x+\mu-\pi$ in the interval $[x_2,x_3]$. If it fell in any other interval the details of the proof would change, but not the results. Since the functions U(x) and W(x) are piece-wise linear for values of x that fall between successive x_i's, the equation defining the risk premium in this situation is

$$\frac{U_2+(U_3-U_2)(x+\mu-\pi-x_2)}{W_2+(W_3-W_2)(x+\mu-\pi-x_2)} = \frac{\Sigma\ U_i p_i}{\Sigma\ W_i p_i}. \tag{xix}$$

Solving for the risk premium gives

$$\pi = x+\mu-x_2 + \frac{W_2\ (\Sigma\ U_i p_i)\ -\ U_2\ (\Sigma\ W_i p_i)}{(W_3-W_2)(\Sigma\ U_i p_i)\ -\ (U_3-U_2)(\Sigma\ W_i p_i)}, \tag{xx}$$

$$= x+\mu-x_2 + 1/(Z-1),$$

where

$$Z = \frac{W_3\ (\Sigma\ U_i p_i)\ -\ U_3\ (\Sigma\ W_i p_i)}{W_2\ (\Sigma\ U_i p_i)\ -\ U_2\ (\Sigma\ W_i p_i)}. \tag{xxi}$$

We have now written the risk premium in terms of the expression Z. This is not an approximation that depends on an assumption that the risk is small as was true in Equation (6). Moreover, the value of Z depends on the valuation and weighting functions which we have already expressed in terms of the measures of risk aversion and eccentricity. What remains to be shown is that if discrete risk aversion measures r_2,r_3 or r_4 increase, holding e_2,e_3 and e_4 constant, Z will decrease causing the risk premium, π, to increase.

Substituting from (xi)-(xvi) into (xxi) and simplifying gives Z as a function of risk aversion and eccentricity

$$Z = \frac{-p_1(r_2+e_2)r_3r_4 - p_2r_3r_4 + p_4r_4 + p_5(r_4+e_4)}{-p_1r_2r_3r_4 - p_3r_3r_4 + p_4r_4(r_3+e_3) + p_5((r_3+e_3)(r_4+e_4)-e_3)}. \quad \text{(xxii)}$$

At this point, the proof becomes an exercise in calculus; one must show that *υθε εωπεγυεδ παθοζζ ιτ τναμμ*$\partial Z/\partial r_i < 0$, i=2,3,4. These derivatives are given below.

$$\frac{\partial Z}{\partial r_2} = \frac{-p_1r_3r_4[(p_3+p_2)r_3r_4+p_4r_4(r_3+e_3-1)+p_5\{(r_4+e_4)(r_3+e_3-1)-r_3)\}+p_1r_3r_4e_2]}{[-p_1r_2r_3r_4 - p_3r_3r_4 + p_4r_4(r_3+e_3) + p_5((r_3+e_3)(r_4+e_4)-e_3)]^2}. \text{(xxiii)}$$

$$\frac{\partial Z}{\partial r_3} = \frac{\begin{array}{l}-[p_1r_4\{p_3(r_2+e_2)r_3r_4+p_4r_4((r_2+e_2)(r_3+e_3)-r_2)+p_5((r_2+e_2)(r_3+e_3)(r_4+e_4) - \\ r_2(r_4+e_4) -r_3(r_2+e_2))\} + (p_4r_4+p_5(r_4+e_4)) (p_2r_4e_3+(p_3+p_4)r_4+p_5(r_4+e_4-1))]\end{array}}{[-p_1r_2r_3r_4 - p_3r_3r_4 + p_4r_4(r_3+e_3) + p_5((r_3+e_3)(r_4+e_4)-e_3)]^2}. \text{(xxiv)}$$

$$\frac{\partial Z}{\partial r_4} = \frac{-p_5r_3[p_2((r_3+e_3)e_4-r_3)+p_4+p_5+p_3e_4+p_1((r_2+e_2)e_3e_4-r_2e_4+(r_2+e_2)r_3(e_4-1))]}{[-p_1r_2r_3r_4 - p_3r_3r_4 + p_4r_4(r_3+e_3) + p_5((r_3+e_3)(r_4+e_4)-e_3)]^2}. \text{(xxv)}$$

It is easy to show that since $r_i = 1 + R_i > 1$ and $e_i = 1 + E_i > 1$, all three of these derivatives are negative. That is, any increase in risk aversion R_i will cause Z to diminish and hence the risk premium to increase.

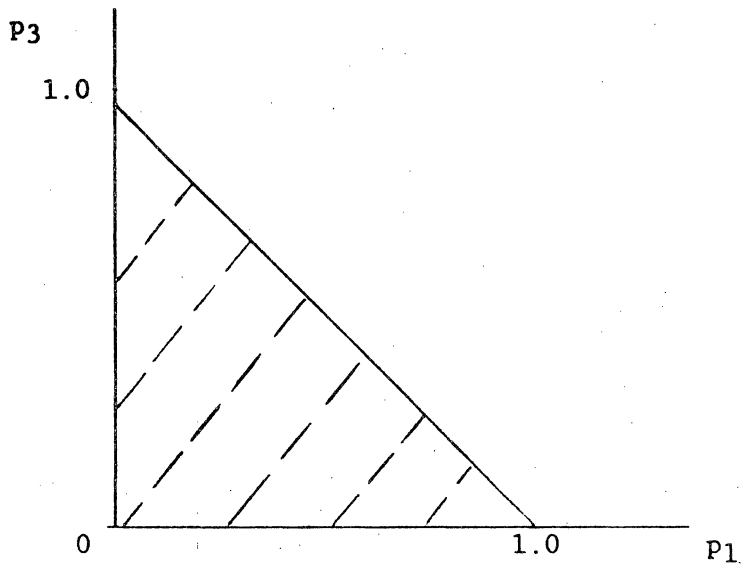

FIGURE 1

Iso-mean Lines

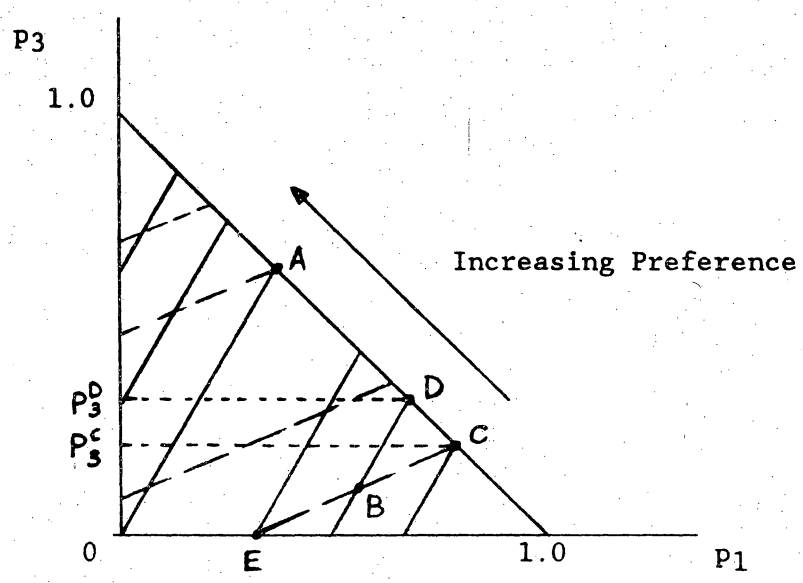

FIGURE 2

Iso-mean Lines (dashed) and Indifference Curves (solid) for an Expected Utility Function

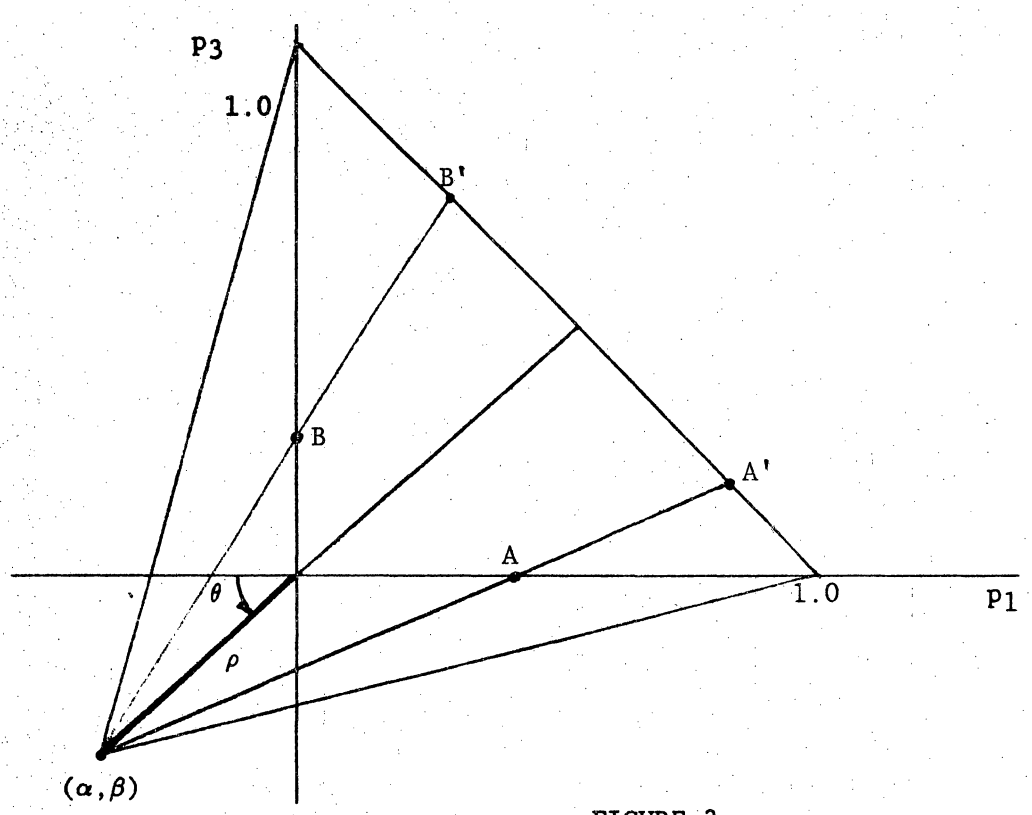

FIGURE 3

Indifference Curves for a Weighted Expected Utility Function

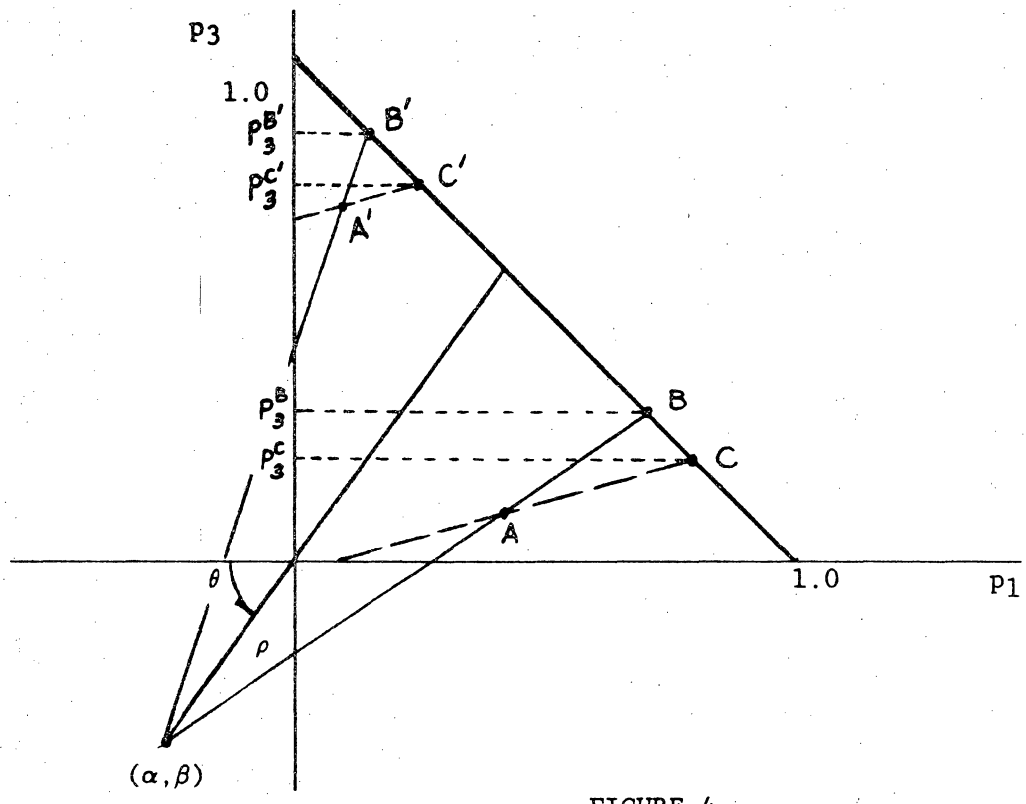

FIGURE 4

The Probability Premium and Its Relationship to the Hub Point

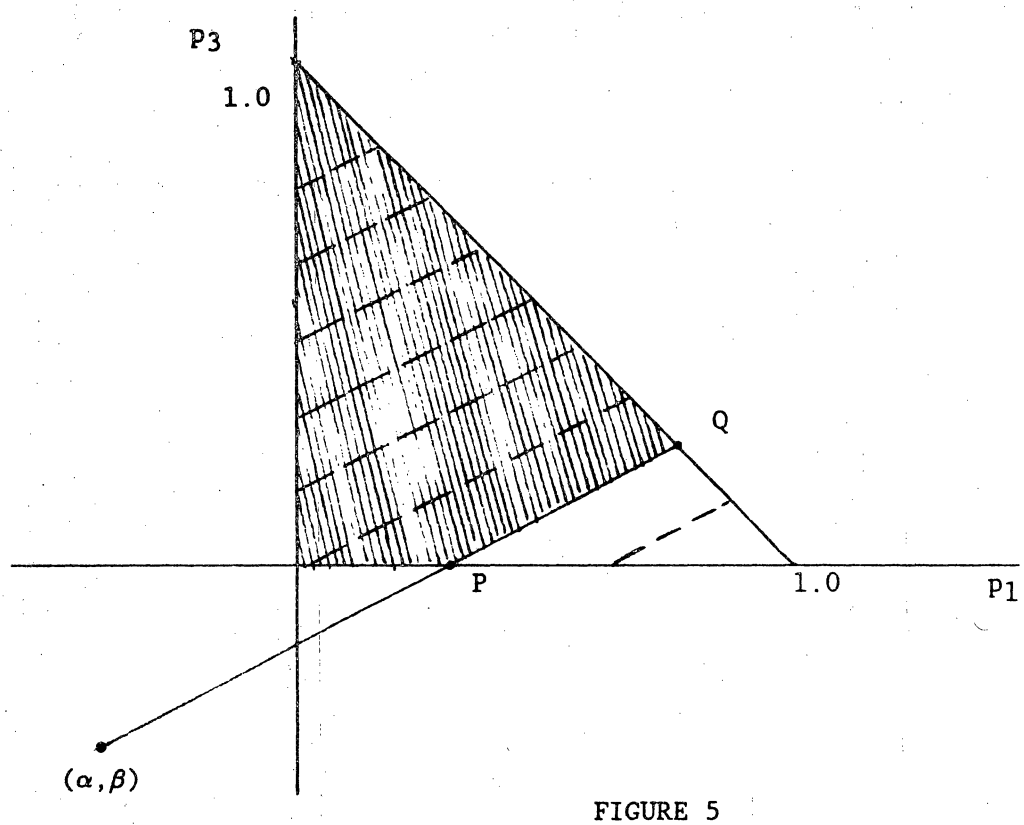

FIGURE 5

Increasing Risk and Weighted
Expected Utility

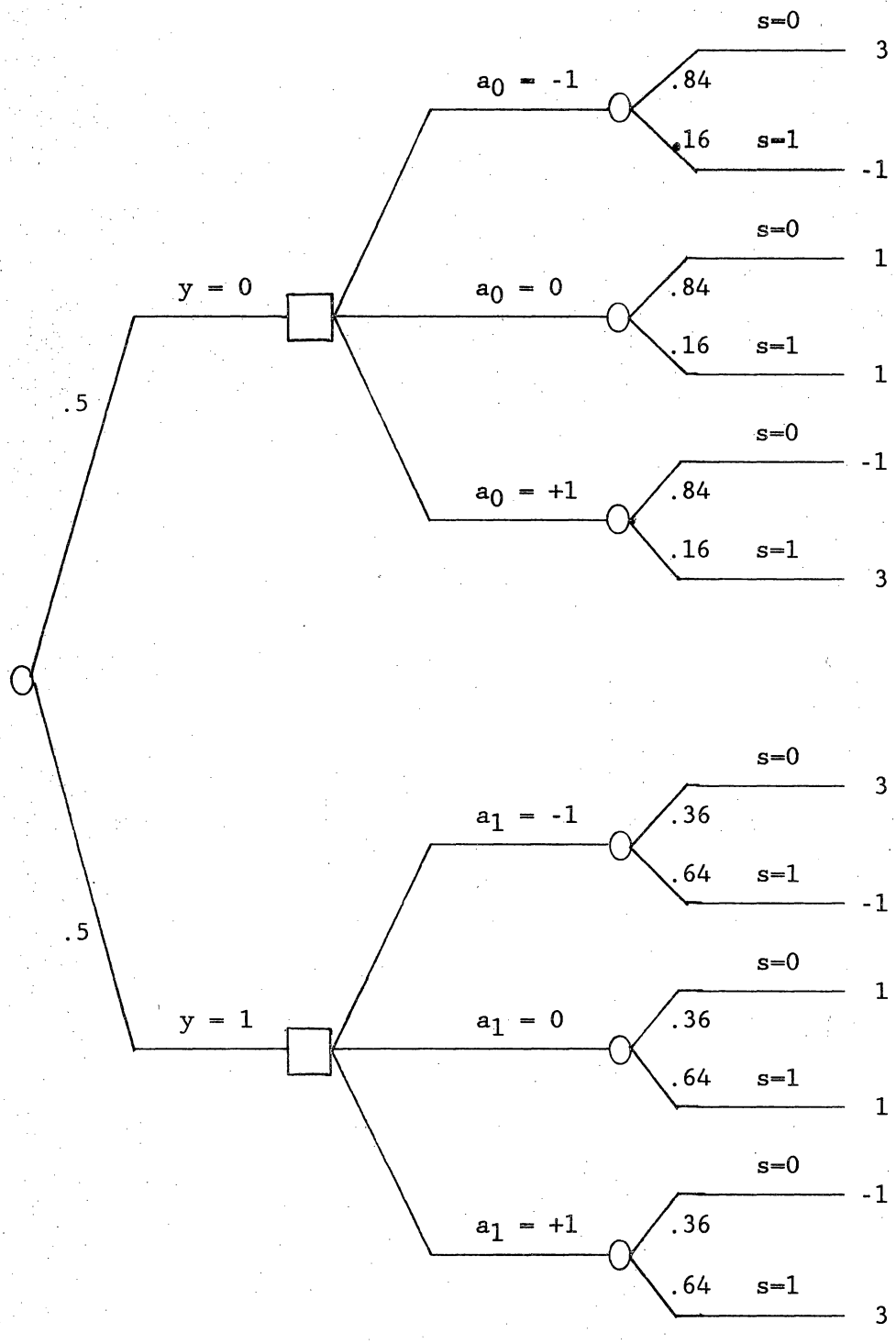

FIGURE 6
Decision Tree

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Footnotes

1. To see this, recall that the slopes of the indifference curves increase with an increase in risk aversion. In particular, the slope of the indifference curve through lottery B will increase. This in turn increases p_3^D, which increases the probability premium, p.

2. "Increase is risk" is defined as a mean preserving spread in the probability distribution over possible outcomes. For continuous distributions over the interval [a,b], a cumulative probability distribution G(x) is a mean preserving spread of a cumulative probability distribution F(x) if

$$\int_a^t (G(x)-F(x))\, dx \geq 0, \text{ for all } t \in [a,b], \tag{1n}$$

$$\int_a^t (G(x)-F(x))\, dx = 0, \text{ for } t = b. \tag{2n}$$

The inequality (1n) has the consequence that more weight is put on extreme outcomes with distribution G than with distribution F, whereas equality (2n) implies that the means of the two distributions are identical.

3. A Ricatti equation is of the form $x'(t) + x(t)^2 = f(t)$.