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THE ENVELOPE THEOREM IN DYNAMIC OPTIMIZATION

by

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Staff Paper 87-2

September, 1987

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September 24, 1987

ABSTRACT

This paper is concerned with the effects of a change in an exogenous parameter on a control problem's optimal performance function. A dynamic envelope theorem is obtained which generalizes and relates results from earlier studies. The theorem derived is applied to a problem of dynamic lifetime consumption theory, giving rise to an intertemporal analogue of the Slutsky equation.

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I. Introduction

The envelope theorem is a well-known and powerful tool in static economic analysis (Samuelson [1947, 1960a, 1960b] and Silberberg [1971, 1974, 1978]). However, the questions of the existence and application of a dynamic analogue to the envelope theorem have not been completely addressed in the literature, nor does the relatively simple nature of the problem appear to be well-understood.

Oniki [1973] alluded to the existence of a dynamic version of the envelope theorem when he suggested that it is possible to obtain a simple expression by substituting the optimum values of the control and state variables back into a control problem's objective function, and then differentiate with respect to the paremeter. Hochman, LaFrance and Zilberman [1984] correctly applied (but did not attempt to demonstrate the validity of) a special form of the envelope theorem in dynamic optimization problems. Utilizing a special form of the Hamilton-Jacobi equation associated with autonomous systems, intertemporal duality in a dynamic model of the firm has been examined by Epstein [1981] and McLaren and Cooper [1980], and in consumer theory by Cooper and McLaren [1980]. Another important, but not generally well-known, result is a dynamic analogue to the envelope theorem obtained by Epstein [1978] for the special case of no nondifferential constraints, and an autonomous system in current value form. Finally, Epstein and Denny [1983] apply the duality results of Epstein [1981] and McLaren and Cooper [1980] to the multivariate flexible accelerator model.

The purpose of this paper is to derive a general statement of the envelope theorem in dynamic optimization problems, and to relate the theorem to the previous results discussed above. The paper focuses on the effects of a change in an exogenous parameter on the optimal performance function arising from a general dynamic optimization problem. The objective functional may depend upon a variety of parameters including output prices, input prices, the discount rate, and the initial value of the state variables (e.g., resource stocks). The optimal performance function can depend on the parameters in two ways: first, the parameter may enter the problem explicitly through the integrand of the functional; second, the parameter may enter the problem's constraints and therefore impact on the optimized objective function only indirectly by changing the optimal controls and state variables.

The paper proceeds as follows. In Section 2 a general dynamic optimization problem containing an exogenous parameter is set forth. Section 3 presents the main result on the impact of a change in the parameter on the problem's optimal performance function to obtain a dynamic version of the envelope theorem. The basic theorem is more general than can be found elsewehere in the literature; the objective functional, the state equations, and any or all nondifferential constraints may depend explicitly on the parameter, and the system need not be autonomous. An application of the paper's principal result to a problem of consumer choice in a dynamic setting is given in Section 4. Section 5 contains a summary and conclusion.

II. A Dynamic Optimization Problem

The problem of interest is to

(1) $$\text{maximize } J = \int_0^T f(x(p,t),u(t),p,t)dt$$

subject to

(2) $\partial x_i(p,t)/\partial t = g^i(x(p,t),u(t),p,t)$, $x_i(p,0) = x_i^o$,

fixed $\forall\ p \in (a,b)$, $i = 1, \ldots, n$,

(3) $h^j(x(p,t),u(t),p,t) = 0$, $j = 1, \ldots, k$,

(4) $h^j(x(p,t),u(t),p,t) \leqq 0$, $j = k+1, \ldots, \ell$,

where x is an n-vector of state variables, u is an m-vector of control variables, p is a scalar parameter with observable values on an open, convex real interval (a,b), t indexes time, and $\ell < m$. By virtue of (2) - (4), the state variables x depend upon the parameter p $\forall\ t \in (0,T]$. However, even if g and h are independent of p, the optimal choice functions for the controls u will depend upon p through (1), and hence so too will the optimal path for the state variables.

We make the following assumptions on the structure of the problem:

(A1) $f,g,h \in C^2$;

(A2) $\hat{L}(x,u,p,t,\lambda,\hat{\mu}) = f(x,u,p,t) + \lambda' g(x,u,p,t) + \hat{\mu}'\hat{h}(x,u,p,t)$ is strictly concave in u, where λ is the n-vector of costate variables, $\hat{\mu} = [\mu_1,\ldots,\mu_k]'$ is the k-vector of Lagrange multipliers for the nondifferential equality constraints $\hat{h} = [h^1,\ldots,h^k]'$, and ' denotes vector transpositon;

(A3) $L(x,u,\lambda,\mu,p,t) = f(x,u,p,t) + \lambda' g(x,u,p,t) + \mu' h(x,u,p,t)$ is strictly concave in x, where $\mu = [\mu_1,\ldots,\mu_\ell]'$ is the ℓ-vector of Lagrange multipliers for all of the nondifferential constraints $h = [h_1,\ldots,h_\ell]'$;

(A4) The set $\Omega = \{u \in R^m\colon h^j(x,u,p,t) \leqq 0,\ j=k+1,\ldots,\ell\}$ is closed, strictly convex, and has a nonempty interior; where (A1) - (A4) are assumed to hold $\forall\ p \in (a,b)$, $\forall\ t \in [0,T]$, and $\forall\ (x,u) \in R^{n+m}$ such that conditions (3) and (4) are satisfied.

Clearly conditions (A1) - (A4) are not the most general (weakest) assumptions that could be applied to this problem. However, they are sufficient for the Arrow, Hurwicz and Uzawa [1961] constraint qualification to be satisfied, and as is demonstrated below, for the optimal controls and Lagrange multipliers to be unique and C^1 in (x,λ,p,t), and hence (from the development by Oniki [1973]) for the optimal paths of the state and costate variables to be C^1 in (p,t), along with the existence and continuity of $\partial^2 x(p,t)/\partial p \partial t \;\; \forall\, p \in (a,b)$, and $\forall\, t \in [0,T]$. These properties are essential to the straightforward development of a simple statement and proof of the dynamic envelope theorem. Although a more general statement can be obtained by allowing such considerations as corners along the optimal path or bounded state variables, most economic applications satisfy regularity conditions similar to (A1) - (A4).

The Hamiltonian for the problem (1) subject to (2) - (4) is defined by

(5) $H = f(x(p,t),u(t),p,t) + \lambda(t)'g(x(p,t),u(t),p,t)$.

The optimal control solution maximizes (5) with respect to u for each $t \in [0,T]$ subject to (3) and (4). The Lagrangian function for this static optimization problem is given by

(6) $L = f(x(p,t),u(t),p,t) + \lambda(t)'g(x(p,t),u(t),p,t) + \mu(t)'h(x(p,t),u(t),p,t)$

Since $L \in C^2$ and is strictly concave in u, the necessary and sufficient conditions for a constrained maximum with respect to u are for each $t \in [0,T]$,

(7) $\partial L/\partial u = \partial f/\partial u + (\partial g/\partial u)'\lambda + (\partial h/\partial u)'\mu = 0$,

(8) $\partial L/\partial \mu = h(x,u,p,t) = 0$.

By (A1) - (A4) and the Theorem of the Maximum, the solution functions $\hat{u}(x,\lambda,p,t)$ and $\hat{\mu}(x,\lambda,p,t)$ are C^1 in all of their arguments (see, e.g., Varian [1978], Appendix A).

In addition to (7) and (8), along the optimal path, x and λ must satisfy the conjugate differential equations on $[0,T] \;\forall\, p \in (a,b)$ (Hestenes [1965, 1966]),

(9) $$\frac{\partial x}{\partial t} = \frac{\partial L}{\partial \lambda} = g(x,u,p,t), \ x(p,0) = x^o,$$

(10) $$\frac{\partial \lambda}{\partial t} = - \frac{\partial L}{\partial x} = - \left(\frac{\partial f}{\partial x} + \frac{\partial g'}{\partial x} \lambda + \frac{\partial h'}{\partial x} \mu \right), \ \lambda(p,T) = 0.$$

Furthermore, since f, g, h, ε C^2 and there are no corners (see (8)), it follows from Oniki's [1973] Lemma 1 that the solution functions to (9) and (10), $x^*(p,t)$ and $\lambda^*(p,t)$, are C^1 in (p,t) $\forall$ $t \in [0,T]$ and $\forall$ $p \in (a,b)$.

Now, substituting $x^*(p,t)$ and $\lambda^*(p,t)$ for x and λ in $\hat{u}(x,\lambda,p,t)$ and $\hat{\mu}(x,\lambda,p,t)$ gives us the optimal controls and Lagrange multipliers as C^1 functions of (p,t) for every $t \in [0,T]$ and $p \in (a,b)$,

(11) $$u^*(p,t) = \hat{u}(x^*(p,t),\lambda^*(p,t),p,t).$$

(12) $$\mu^*(p,t) = \hat{\mu}(x^*(p,t),\lambda^*(p,t),p,t).$$

Finally, along the optimal path the state equations (2) can be written as

(13) $$\partial x^*(p,t)/\partial t = g(x^*(p,t),u^*(p,t),p,t).$$

Since g, x^* and u^* are all at least C^1 in p, it follows that

(14) $$\frac{\partial^2 x^*}{\partial t \partial p} = \frac{\partial g}{\partial x}\frac{\partial x^*}{\partial p} + \frac{\partial g}{\partial u}\frac{\partial u^*}{\partial p} + \frac{\partial g}{\partial p},$$

exists and is continuous for $t \in [0,T]$ and $p \in (a,b)$. With these preliminary developments, we are now prepared to state and prove the main result in the next section.

III. The Dynamic Envelope Theorem

Consider the optimal performance function

(15) $$J^*(x^o,p,T) = \int_0^T f(x^*(p,t),u^*(p,t),p,t)dt.$$

In this section we state and prove the main result which provides two equivalent ways of expressing the marginal effect of a change in the parameter p on J^*. This result is the following:

Theorem 1. If the dynamic optimization problem (1) subject to (2) - (4) satisfies conditions (A1) - (A4), then $J^* \varepsilon C^1$ in p, and

$$(16) \qquad \frac{\partial J^*(x^o,p,T)}{\partial p} = \int_0^T \left(\frac{\partial f}{\partial x}\frac{\partial x^*}{\partial p} + \frac{\partial f}{\partial u}\frac{\partial u^*}{\partial p} + \frac{\partial f}{\partial p}\right)dt$$

$$= \int_0^T \left(\frac{\partial f}{\partial p} + \lambda^{*\prime}\frac{\partial g}{\partial p} + \mu^{*\prime}\frac{\partial h}{\partial p}\right)dt$$

$$= \int_0^T \frac{\partial L(x^*(p,t),u^*(p,t),\lambda^*(p,t),\mu^*(p,t),p,t)}{\partial p}\,dt.$$

Proof: $J^* \varepsilon C^1$ in p follows from (A1) and the fact that $x^*, u^* \varepsilon C^1$. Therefore, the first equality in (16) follows from partially differentiating (15) with respect to p and applying Liebnitz' rule to the right hand side. Along the optimal path for each $t \varepsilon [0,T]$, the nondifferential constraints are identities

$$(17) \qquad h(x^*(p,t),u^*(p,t),p,t) \equiv 0.$$

Partially differentiating (17) with respect to p and taking the vector product with $\mu^*(p,t)$ gives

$$(18) \qquad \mu^*(p,t)'\left(\frac{\partial h}{\partial x}\frac{\partial x^*}{\partial p} + \frac{\partial h}{\partial u}\frac{\partial u^*}{\partial p} + \frac{\partial h}{\partial p}\right) = 0$$

for $p \varepsilon (a,b)$ and $t \varepsilon [0,T]$. Therefore, adding zero to the integrand in (16) does not change the value of the definite integral, i.e.

$$(19) \qquad \frac{\partial J^*(x^o,p,T)}{\partial p} = \int_0^T \left(\left(\frac{\partial f}{\partial x}' + \mu^{*\prime}\frac{\partial h}{\partial x}\right)\frac{\partial x^*}{\partial p} + \left(\frac{\partial f}{\partial u}' + \mu^{*\prime}\frac{\partial h}{\partial u}\right)\frac{\partial u^*}{\partial p}\right.$$

$$\left. + \frac{\partial f}{\partial p} + \mu^{*\prime}\frac{\partial h}{\partial p}\right)dt.$$

Now, the first order condition (7) for a constrained maximum in u implies

$$(20) \qquad \frac{\partial f}{\partial u}' = -\left(\lambda^{*\prime}\frac{\partial g}{\partial u} + \mu^{*\prime}\frac{\partial h}{\partial u}\right) \quad \forall\, t \varepsilon [0,T].$$

Substituting this expression for $\partial f/\partial u'$ into (19) and cancelling common terms,

$$(21)\qquad \frac{\partial J^*(x^o,p,T)}{\partial p} = \int_0^T \Big(\Big(\frac{\partial f}{\partial x}' + \mu^{*\prime}\frac{\partial h}{\partial x}\Big)\frac{\partial x^*}{\partial p} - \lambda^{*\prime}\frac{\partial g}{\partial u}\frac{\partial u^*}{\partial p} + \frac{\partial f}{\partial p} + \mu^{*\prime}\frac{\partial h}{\partial p}\Big)dt.$$

If we solve (14) for $(\partial g/\partial u)\partial u^*/\partial p$, substitute the result into (21), and group terms, we obtain

$$(22)\qquad \frac{\partial J^*(x^o,p,t)}{\partial P} = \int_0^T \Big(\frac{\partial f}{\partial x}' + \lambda^{*\prime}\frac{\partial g}{\partial x} + \mu^{*\prime}\frac{\partial h}{\partial x}\Big)\frac{\partial x^*}{\partial p}dt - \int_0^T \lambda^{*\prime}\frac{\partial^2 x^*}{\partial p\partial t}dt + \int_0^T \Big(\frac{\partial f}{\partial p} + \lambda^{*\prime}\frac{\partial g}{\partial p} + \mu^{*\prime}\frac{\partial h}{\partial p}\Big)dt$$

where $\partial^2 x^*/\partial t\partial p = \partial^2 x^*/\partial p\partial t$ by Young's Theorem.

The proof is completed by demonstrating that the first two integrals on the right hand side of (22) sum to zero. To show this, we integrate the second term by parts

$$(23)\qquad \int_0^T \lambda^*(p,t)'\frac{\partial^2 x^*(p,t)}{\partial p\partial t}dt = \lambda^*(p,t)'\frac{\partial x^*(p,t)}{\partial p}\Big|_0^T - \int_0^T \frac{\partial \lambda^*(p,t)'}{\partial t}\frac{\partial x^*(p,t)}{\partial p}dt = -\int_0^T \frac{\partial \lambda^*(p,t)'}{\partial t}\frac{\partial x^*(p,t)}{\partial p}dt,$$

where the second equality follows from $x(p,0) = x^o$, fixed $\forall\ p \in (a,b)$ and the variable endpoint transversality condition $\lambda^*(p,T) = 0$. But the differential equations for the costate variables (10) imply that this last expression in (23) is just the negative of the first expression in (21). Thus,

$$(24)\qquad \frac{\partial J^*(x^o,p,T)}{\partial P} = \int_0^T \Big(\frac{\partial f}{\partial p} + \lambda^{*\prime}\frac{\partial g}{\partial p} + \mu^{*\prime}\frac{\partial h}{\partial p}\Big)dt.$$

<u>Q.E.D.</u>

Remarks:

1. Theorem 1 is a dynamic analogue to the familiar envelope theorem associated with static constrained optimization problems. The static envelope theorem asserts that the rate of change in an indirect objective function with respect to a parameter, allowing all choice variables to adjust to the change, is equal to the partial derivative of the Lagrangian with respect to the parameter, holding the choice variables constant at their optimal level. The intertemporal version states that the rate of change in a control problem's optimal performance function with respect to a parameter, allowing the state variables, controls, auxiliary variables and multipliers to adjust to the change, is equal to the integral of the partial derivative of the Lagrangian with respect to the parameter, where all the state variables, controls, auxiliary variables and multipliers are held constant with respect to p at their optimal levels for each point in time $t \in [0,T]$.

2. Theorem 1 is a generalization of the following commonly analyzed special case, first derived for finite horizon problems by Epstein [1978], and subsequently for infinite planning periods by Cooper and McLaren [1980], McLaren and Cooper [1980], and Epstein [1981]. Consider the infinite horizon problem where $t \to \infty$, $f(x,u,p,t) = v(x,u,p)e^{-rt}$, $\partial g(x,u,p,t)/\partial t \equiv 0$, and $h(x,u,p,t) \equiv 0 \; \forall \; t \in [o,\infty)$. We assume that the transversality condition $\lim_{t\to\infty} \lambda^*(p,t)'x^*(p,t) = 0$ is satisfied at the optimal solution for any possible p (see Arrow and Kurz [1970, p.46] and Halkin [1974]).

Define the current-value Hamiltonaian by

$$\tilde{H}(x,u,\tilde{\lambda},p) = v(x,u,p) + \tilde{\lambda}'g(x,u,p) \tag{25}$$

$$= e^{rt}[f(x,u,p,t) + \lambda'g(x,u,p)] = e^{rt}H(x,u,\lambda,p,t),$$

where $\tilde{\lambda} = e^{rt}\lambda$. The current-value optimal performance function for the time interval $[t,\infty)$,

$$\tilde{J}(x_t,p,t) = e^{rt} \int_t^\infty v(x^*,u^*,p)e^{-r(\tau-t)}\,d\tau \tag{26}$$

$$= \int_o^\infty v(x^*,u^*,p)e^{-r\tau}\,d\tau$$

does not depend upon t explicitly. Furthermore, the maximized current-value Hamiltonian is constant throughout $t \in [0,\infty)$; see, e.g. Arrow and Kurz [1970, pp. 47-51].

Since $\tilde{J}(x_t,p,t) = e^{rt}J^*(x_t,p,t)$, it follows that at initial time t=0, $\tilde{J}(x^o,p,0) = J^*(x^o,p,0)$, where J^* is the present value optimal performance function for the time interval $[t,\infty)$,

$$J^*(x_t,p,t) = \int_t^\infty v(x^*,u^*,p)e^{-r(\tau-t)}d\tau. \tag{27}$$

The proof of Theorem 1 remains unchanged for this infinite horizon problem. Therefore, equation (24) takes the form

$$\frac{\partial J^*}{\partial p} = \int_o^\infty \left(\frac{\partial v}{\partial p} + \tilde{\lambda}'\frac{\partial g}{\partial p}\right)e^{-rt}\,dt \tag{28}$$

$$= \frac{\partial \tilde{H}}{\partial p} \int_o^\infty e^{-rt}\,dt = \frac{1}{r}\frac{\partial \tilde{H}}{\partial p},$$

so that $r\partial J^*/\partial p = \partial\tilde{H}/\partial p$. This is the link between the intertemporal duality results of Cooper and McLaren [1980], McLaren and Cooper [1980], Epstein [1981], and Epstein and Denny [1983] and the general envelope theorem in dynamic optimization.

IV. Applications

Like the familiar static envelope theorem, the dynamic result obtained above can be applied to a variety of problems. In this section the dynamic envelope theorem is applied to a problem of a consumer maximizing discounted utility from consumption subject to a lifetime budget constraint. A dynamic version of the Slutsky equation is obtained, first for the case of constant prices or static expectations, and then for the case of time dependent prices and perfect foresight.

4.1 A Lifetime Consumption Problem.

Consider a consumer choosing the time path of an n - vector of consumption c(t) to

(29) maximize $U = \int_0^T u(c(t))e^{-\rho t}dt$

subject to the budget constraint

(30) $\partial m(p,t)/\partial t = p'c(t)e^{-rt}$, $m(p,0) = 0$, $m(p,T) = M \; \forall \; p \in \Phi$,

where ρ is the consumer's personal discount rate, u is the instantaneous utility function; p is an n-vector of constant prices with values in the set Φ a subset of R_+^n, r is the market rate of discount, and M is the present value of the consumer's lifetime earnings. The optimal consumption path can be written as $c^M(p,M,t)$, and the optimal level of discounted utility flows is given by

(31) $U^*(p,M,T) = \int_0^T u(c^M(p,M,t))e^{-\rho t}dt.$

Now consider the dual problem of choosing a consumption path c(t) to minimize the present value of lifetime expenditures on consumption, subject to a constraint which holds discounted lifetime utility constant at some level, say U^o,

(32) minimize $J = \int_0^T p'c(t)e^{-rt}dt$

subject to

(33) $\partial U(t)/\partial t = u(c(t))e^{-\rho t}$, $U(0) = 0$, $u(T) = U^o$.

Denote this problem's solution by $c^U(p,U^o,t)$, and the minimal net present value of consumption expenditures by

(34) $$J^*(p,U^o,T) = \int_0^T p'c^U(p,U^o,t)e^{-rt}dt.$$

Both of these problems can be represented as isoperimetric problems by rewriting (30) and (33) in integral form

(30') $$\int_0^T p'c(t)e^{-rt}dt = M$$

(33') $$\int_0^T u(c(t))e^{-\rho t}dt = U^o.$$

A well-known result from the calculus of variations states that, for isoperimetric problems of this type, the solution (31) is equivalent to (34) if the variable M appearing in (30) is equal to J* (for a proof see Clegg [1968, pp. 117-121]). Therefore, we have the following identity:

(35) $$c^U(p,U^o t) \equiv c^M(p,J^*(p,U^o,T),t) \;\forall\; t \in [0,T].$$

Then, the effect on the consumption flow of good j at each point in time t from a change in the price of good k is given by

(36) $$\partial c_j^U(p,U^o,t)/\partial p_k = \partial c_j^M(p,J^*(p,U^o,T),t)/\partial p_k$$
$$+ \;(\partial c_j^M(p,J^*(p,U^o,T),t)/\partial M)\partial J^*(p,U^o,T)/\partial p_k.$$

Applying the dynamic envelope theorem to $J^*(p,U^o,t)$ obtains

(37) $$\frac{\partial J^*(p,U^o,t)}{\partial P_k} = \int_0^T c_k^U(p,U^o,t)e^{-rt}dt.$$

Substituting (37) into (36) gives

(38) $$\partial c_j^M/\partial p_k = \partial c_j^U/\partial p_k - (\partial c_j^M/\partial M) \int_0^T C_k^M \, e^{-rt}dt.$$

The above expression is a dynamic version of the Slutsky equation. The impact of a change in p_k on consumption of commodity j can be decomposed into two effects: a "utility-held-constant" substitution effect and a wealth effect. The derivation of (38) parallels the derivation of an "instant" Slutsky equation for the standard single period model of a consumer found in Silberberg [1978, p. 248 - 250].

4.2 The Lifetime Consumption Problem with Time Dependent Prices

In the previous example, the assumption that prices are constant or that consumers have static expectations plays an important role in the structure of the optimal consumption path. This assumption also appears in the intertemporal duality studies discussed above. However, dynamic economic problems frequently contain uncontrolled state variables which are not necessarily constant over the planning horizon. Examples include time dependent prices of inputs and outputs in a model of a competitive firm, a time dependent discount rate, and the size of the labor force in a model of economic growth.

In those cases where some of the state variables vary over time but are not subject to control, we might wish to assume that decision makers form their expectations about the movement of these variables such that the initial values are not (necessarily) expected to remain constant throughout the planning period. The optimal solution to such a problem will contrast significantly from the solution which arises from assuming, for example, that decision makers have static expectations but are able to update these expectations instantaneously (i.e., the dynamic decision problem is characterized by open loop control with feedback and the decision maker only carries out the $t = 0$ part of his planned sequence of actions - see Epstein [1981]). In particular, without the static expectations assumption, the special form of the Hamilton-Jacobi equation, $rJ^* = \tilde{H}$, exploited by Cooper and McLaren [1980], Epstein [1981], Epstein and Denny [1983], and McLaren and Cooper [1980] does not arise because the current value Hamiltonian is no longer constant.

To introduce uncontrolled time dependent state variables into the analysis, consider the following problem:

(39) $$\text{maximize } J = \int_0^T f(x,u,p,t)dt$$

subject to

(40) $$\partial x/\partial t = g(x,u,p,t),\ x^o \text{ fixed},$$

(41) $$dp/dt = \eta(p,t),\ p^o \text{ fixed},$$

(42) $$h^j(x,u,p,t) = 0,\ j = 1,\ldots,k,$$
$$h^j(x,u,p,t) \leqq 0,\ j = k+1,\ldots,\ell.$$

We assume that the solution to (41), $p(t) = \phi(p^o,t)$, exists uniquely and is twice continuously differentiable. Note that even if f,g,η,h do not depend upon t explicitly, the solution to (41) will in general be an explicit function of t. For example, if

(43) $$dp/dt = \alpha + \beta p,\ p(0) = p^o,$$

then

(44) $$p(t) = \phi(p^o,t) = p^o e^{\beta t} + \alpha(e^{\beta t}-1)/\beta\ \forall t.$$

Clearly, even if $f = ve^{-rt}$, the inclusion of p among the arguments of v,g or h with any sort of time dependent structure for p makes it impossible to generate an autonomous system in current value form if decision makers do not have static expectations about future values of p(t).

Substituting $\phi(p^o,t)$ for p in (39) - (42), under conditions (A1) - (A4) unique, continuously differentiable solution functions $x^*(p^o,t)$, $u^*(p^o,t)$, $\lambda^*(p^o,t)$ and $\mu^*(p^o,t)$ exist. Indeed, the arguments of sections 2 and 3 carry over to this situation unchanged. From Theorem 1, the impact of a change in p^o on the optimal performance function is given by

(45) $$\partial J^*/\partial p^o = \int_0^T (\partial f/\partial p + \lambda^{*\prime}\partial g/\partial p + \mu^{*\prime}\partial h/\partial p)\partial\phi/\partial p^o dt.$$

In general, (45) will not reduce to an expression like (28) except when $\phi(p^o,t) \equiv p^o$ and $f = ve^{-rt}$, where v,g,h do not depend upon t explicity.

Now we wish to examine the lifetime consumption problem for the case of time dependent prices and perfect foresight. Let p(t) be an n-vector of prices which depends upon the initial set of prices and time,

$$p(t) = \phi(p^o, t). \tag{46}$$

The consumer is fully aware of (46), or equivalently, the differential equation system (43), and wishes to maximize (29) subject to the budget constraint (30), where p is replaced by $\phi(p^o,t)$. The optimal consumption path can be written as $c^M(p^o,M,t)$.

Similarly, the optimal consumption path for the dual problem of minimizing (32) subject to (33) using (46) to describe the motion of p can be written as $c^U(p^o,U^o,t)$, and the minimal present value of consumption expenditures is

$$J^*(p^o,U^o,T) = \int_0^T \phi(p^o,t)'c^U(p^o,U^o,t)e^{-rt}dt. \tag{47}$$

Applying Theorem 1 to (47) implies

$$\partial J^*/\partial p_k^o = \int_0^T (\partial\phi/\partial p_k^o)'c^U\, e^{-rt}\, dt. \tag{48}$$

Setting $M = J^*(p^o,U^o,T)$ and partially differentiating

$$c^M(p^o,J^*(p^o,U^o,T),t) \equiv c^U(p^o,U^o,t)$$

with respect to p_k^o for a given $t \in [0,T)$ therefore implies

$$\partial c_j^M/\partial p_k^o = \partial c_j^U/\partial p_k^o - \partial c_j^M/\partial M \int_0^T (\partial\phi/\partial p_k^o)'c^M e^{-rt}dt. \tag{49}$$

Note that the "wealth effect" of the dynamic Slutsky euqation (49) includes the marginal impact on <u>all</u> values of <u>all</u> future prices due to a change in the initial value of the k^{th} price, which is considerably more complicated than the static expectations case (38).

V. Conclusion

The envelope theorem is a powerful tool in economics. In static models, it provides a link between indirect objective functions and demand and supply euqations, greatly simplifying estimation problems. It is also quite useful in conducting comparative statics, in welfare analysis, and in interpreting Lagrange multipliers.

In this paper, we have shown that the solutions to dynamic optimization problems are characterized by an exact intertemporal analogue to the static envelope theorem. The result obtained is general enough to encompass all of the special cases that have appeared in recent literature, as well as dynamic processes on the parameters affecting the intertemporal choice problem. As such, the results presented here should be useful for the interpretation of the results contained in previous studies, and in future applications of dynamic optimization.

References

Arrow, K. J., L. Hurwicz, and H. Uzawa, "Constraint Qualification in Nonlinear Programming," Naval Research Logistics Quarterly 8(1961).

Arrow, K. J. and M. Kurz, "Public Investment, the Rate of Return, and Optimal Fiscal Policy, Baltimore: John-Hopkins Press, 1970.

Clegg, J. C., Calculus of Variations, New York: Interscience Publishers Inc., 1968.

Cooper, R. J. and K. R. McLaren, "Atemporal, Temporal and Intertemporal Duality in Consumer Theory," International Economic Review 21(1980): 599 - 609.

Epstein, L. G., "The Le Chatelier Principle in Optimal Control Problems," Journal of Economic Theory 19(1978): 103 - 22.

Epstein. L. G., "Duality Theory and Functional Forms for Dynamic Factor Demands," Review of Economic Studies 18(1981): 81 - 95.

Epstein, L. G. and M. G. S. Denny, "The Multivariate Flexible Accelerator Model: Its Empirical Restrictions and an Application to U.S. Manufacturing," Econometrica 51(1983): 647 - 674.

Halkin, H., "Necessary Conditions for Optimal Control Problems with Infinite Horizons," Econometrica 42(1974): 267 - 272.

Hestenes, M. R., "On Variational Theory and Optimal Control Theory," J. SIAM Control 3(1965): 23 - 48.

Hestenes, M. R., Calculus of Variations and Optimal Control Theory, New York: John Wiley and Sons, 1966.

Hochman, E., J. LaFrance, and D. Zilberman, "Solar Energy and Hydroelectric Power Generation in the Dead Sea: A Dynamic Analysis," Water Resources Research 20(1984): 1469 - 1476.

McLaren, K. R. and R. J. Cooper, "Intertemporal Duality: Application to the Theory of the Firm," Econometrica 48(1980): 1755 - 62.

Oniki, H., "Comparative Dyanmics (Sensitivity Analysis) in Optimal Control Theory," Journal of Economic Theory 6(1973): 265 - 83.

Pontryagin, L. S., Ordinary Differential Equations, Reading, Mass.: Addison Wesley, 1962.

Pontryagin, L. S., V. G. Boltyanski, R. V. Gamkrelidge, and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, New York: Interscience, 1962.

Samuelson, P. A., Foundations of Economic Analysis, Cambridge, Mass.: Harvard University Press, 1947.

Samuelson, P. A., "An Extension of the Le Chatelier Principle," Econometrica 28(1960a): 368 - 379.

Samuelson, P. A., "Structure of a Minimum Equilibrium System," in R. W. Pfouts (ed.), Essays in Economics and Econometrics: A Volume in Honor of Harold Hotelling, Chapel Hill: The University of North Carolina Press, 1960b.

Silberberg, E., "The Le Chatelier Principle as a Corollary to a Generalized Envelope Theorem, Journal of Economic Theory 3(1971): 146 - 155.

Silberberg, E., "A Revision of Comparative Statics Methodology, or, How to Do Economics on the Back of an Envelope, "Journal of Economic Theory 7(1974): 159 - 172.

Silberberg, E., The Structure of Economics: A Mathematical Analysis, New York: McGraw-Hill Book Company, 1978.

Varian, H. R., Microeconomic Analysis, New York: W. W. Norton and Company, 1978.