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Toward a Resolution of the Allocatable Input Dilemma

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TOWARD A RESOLUTION OF THE ALLOCATABLE INPUT DILEMMA

C. Richard Shumway

Abstract

Technology specification for the multiple-product firm with allocatable inputs is explored and extended. Technical interdependence does not hinder reduction of a system of constrained individual production functions to a single-equation multiple-product generic-input specification. When costs are associated with input allocations, however, the generic-input equation cannot be derived. Reversible duality relationships apply to the generic-input equation even when the behavioral objective is partially maintained in its specification. However, two-way duality relationships apply to a system of production functions only if they are technically independent with no binding constraints or non-constant marginal costs associated with the allocation of fixed inputs.

C. Richard Shumway is a professor of Agricultural Economics, Texas A&M University.

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Notational Glossary

Because of the large amount of notation required, all notation defined in the text are repeated here in alphabetical order for convenient reference.

Roman

F is general functional notation,

f^i is general functional notation,

G is general functional notation,

h is an output index number,

i is an output index number,

j is a variable-input index number,

K is general functional notation,

k is a fixed-input index number,

m is number of outputs,

n is number of generic variable inputs,

$P = (p_1,\ldots,p_m)$ is the vector of output prices,

$P' = (p_2/p_1,\ldots,p_m/p_1)$ is the vector of normalized output prices,

p_i is price of output i,

q is number of generic fixed inputs,

$R = (r_1,\ldots,r_n)$ is the vector of variable-input prices,

$R' = (r_1/p_1,\ldots,r_n/p_1)$ is the vector of normalized variable-input prices,

r_j is the price of variable input j.

$X = (x_{11},\ldots,x_{1n},\ldots,x_{m1},\ldots,x_{mn})$ is the vector of variable-input allocations,

$\bar{X} = (\bar{x}_1,\ldots,\bar{x}_n)$ is the vector of generic variable-input quantities,

$X_i = (x_{i1},\ldots,x_{in})$ is the vector of variable inputs allocated to y_i,

$X_{\bar{1}} = (x_{22},\ldots,x_{2n},\ldots,x_{m2},\ldots,x_{mn})$ is the vector of variable-input allocations except all allocations of the first variable input and all allocations to the first output,

$\bar{x}_j$ is the quantity of the jth generic variable input,

$\bar{x}_j^*(\cdot)$ is general functional notation,

x_{ij} is the quantity of the jth variable input allocated to the ith output,

$x_{ij}(\cdot)$ is general functional notation,

$Y = (y_1,\ldots,y_m)$ is the vector of output quantities,

$\underline{Y}_{-i} = (y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_m)$ is the vector of output quantities excluding output i,

y_i is the quantity of output i,

$y_i^*(\cdot)$ is general functional notation,

$Z = (z_{11},\ldots,z_{iq},\ldots,z_{m1},\ldots,z_{mq})$ is the vector of fixed-input allocations,

$\bar{Z} = (\bar{z}_1,\ldots,\bar{z}_q)$ is the vector of generic fixed-input quantities,

$Z_1^- = (z_{21},\ldots,z_{2q},\ldots,z_{m1},\ldots,z_{mq})$ is the vector of variable input allocations except all allocations to the first output,

$\bar{z}_k$ is the quantity of the kth generic fixed input,

z_{ik} is the quantity of the kth fixed input allocated to the ith output,

$z_{ik}(\cdot)$ is general functional notation.

Greek

Γ is general functional notation,

∂ is partial derivative notation,

θ is general functional notation,

π is short-run profit (net returns over variable costs),

π^* is short-run maximum profit for given prices of outputs and variable inputs and quantities of fixed inputs,

$\pi^{*\prime} = \pi^*/p_1$ is normalized maximum profit,

ϕ is general functional notation,

ψ is general functional notation.

TOWARD A RESOLUTION OF THE ALLOCATABLE INPUT DILEMMA

Agricultural production is characterized by many price-taking firms, most of which produce multiple products and use many inputs, some fixed and most clearly allocated to individual products. Economic analysis of issues of concern to such producers typically requires explicit or implicit consideration of this multiple-product, multiple-allocatable-input technology. Appropriate specification of the technology (production function(s)), identification of equilibrium conditions, and recognition of duality limitations are necessary to correctly examine the economic behavior of such firms.

More than two decades ago, Pfouts focused attention on a dilemma associated with the allocatable fixed input. He documented that when costs are associated with the allocation of a constraining fixed input, only one of Samuelson's four conditions of price-taking cost-minimizing single-product firms in equilibrium apply to the multiple-product firm. Even when each production function is technically independent in each input, only the first of these properties hold for the allocations: (a) variable-input shadow price equals input cost, (b) cross-derivatives of input demands with respect to other input prices are symmetrical, (c) input demands are downward-sloping in own price, and (d) marginal cost is equal to the Lagrangian multiplier (Samuelson, pp. 57-69). Pfouts (p. 657) also noted that only when excess capacity exists in all fixed inputs do all four conditions hold, and that is the only situation in which the multiple-product firm can meaningfully be considered a simple collection of single-product firms.

Since Pfouts, economists have generally ignored the allocation issue, apparently by implicitly assuming no allocation costs. This has led to the common practice of writing technology for the multiple-product firm as a single equation, i.e., with one (or more) product quantities expressed as an explicit or implicit function of all other product and generic-input quantities (e.g., Lau, 1972; Hall). All of Samuelson's four conditions as well as the duality relationships that apply to a single-product firm can be shown to apply to the generic inputs of such a multiple-product firm. However, there appears to be nothing in the duality literature about relationships for the multiple-product firm with a technology more appropriately expressed as a system of production functions subject to common constraint(s).

Two recent papers in the agricultural economics literature have refocused attention on dilemmas associated with allocatable inputs. Shumway, Pope, and Nash extended Pfouts' work by demonstrating that even without allocation costs, the presence of a constraining allocatable input is a cause of joint production in inputs (Lau, 1972) even for firms with technically-independent production functions. Consequently, Pfouts' assertion about the limited condition under which multiple-product firms can be treated as a collection of single-product firms is actually even more limited. Separate production functions can be written, but profits can be maximized (or costs minimized) only subject to the constraints on allocations. Multiple-product models must be formulated and the production of each commodity examined as part of a system when a constraint is binding even if there are no allocation costs.

Just, Zilberman, and Hochman addressed technology for the multiple-product firm from a Pfouts' perspective. Although they ignored allocation costs, they wrote the firm's technology as a system of production functions subject to input allocation equations. They assert that a system of m independent output production functions with n+q input allocation equations can be reduced to a single-equation specification of technology (ala Lau, 1972; Hall) with one (or more) product quantities as a function of all other product and generic-input quantities only if (a) m = 1, (b) n+q = 1, (c)

$m(n+q)-m-(n+q)+1 = (m-1)(n+q-1)$ of the allocations do not appear in either the production functions or the constraints, or (d) the firm's behavioral objective is considered. The first condition implies a single-product firm. The second is a multiple-product firm with only one input (e.g., land) clearly allocated among commodities; no other inputs are distinguished by the commodity to which they are applied (an unlikely case in agriculture). The third allows cases where inputs may be allocated to groups of commodities so long as the number of allocations does not exceed the number of distinct commodities and generic inputs less one. Since most realistic cases in agriculture involve multiple-product firms with more than one input clearly allocated among all products, the fourth condition, i.e., maintaining hypotheses about a behavioral objective, is most often required to reduce technology to a single-equation specification in product and generic-input quantities. The resulting equation is a mixture of technology and economic theory. Thus, it cannot be value-free. It is a correct specification only if the underlying theory is also correct.

Objectives

Several questions remain unanswered or inadequately answered by these works. For example, is there a "correct" way to express the technology of a multiple-product firm? Can a system of technically-interdependent production functions with binding allocation constraints be reduced to a single-equation generic-input specification? What effect do allocation costs have on our ability to reduce the technology to a single equation? When the firm's behavioral objective must be considered in deriving a single-equation generic-input technology specification, what, if any, duality relationships apply to it? What duality relationships apply to the multiple-product technology without embodying economic theory in the specification? Answers to each of these questions will be sought in this paper.

Expressing Technology for the Multiple-Product Firm

Technology for the multiple-product firm has most often been expressed explicitly or implicitly either as

$$(1) \quad \phi(Y) = \psi(\bar{X},\bar{Z})$$

or

$$(2) \quad y_i = \Gamma(\underline{Y}_i,\bar{X},\bar{Z})$$

where $Y = (y_1,\ldots,y_m)$ is the vector of product quantities, $\bar{X} = (\bar{x}_1,\ldots,\bar{x}_n)$ is the vector of generic variable-input quantities (i.e., the total quantity of each input used in all products), $\bar{Z} = (\bar{z}_1,\ldots,\bar{z}_q)$ is the vector of generic fixed-input quantities, $\underline{Y}_i = (y_1,\ldots, y_{i-1}, y_{i+1},\ldots,y_m)$ is the vector of product quantities excluding product i. Writing the technology as (1) maintains the hypothesis of separability between outputs and inputs. Both (1) and (2) maintain either (a) restrictions on the dimensions of $Y,\bar{X}$ and $\bar{Z}$, (b) restrictions on the allocations of $\bar{X}$ and $\bar{Z}$, or (c) a behavioral objective as noted by Just, et al. Thus, neither (1) nor (2) is suitable as a general specification of multiple-product technology, a conclusion also noted by Mittelhammer, et al.

When inputs are allocated, a single-equation extension of (2) is to replace $\bar{X}$ and $\bar{Z}$ by their allocations (e.g., Naylor; Nash):

$$(3) \quad y_i = \theta(\underline{Y}_i,X,Z),$$

where $X = (x_{11},\ldots,x_{1n},\ldots,x_{m1},\ldots,x_{mn})$ is the vector of variable input allocations, and $Z = (z_{11},\ldots,z_{1q},\ldots,z_{m1},\ldots,z_{mq})$ is the vector of fixed-input allocations. A logical problem with (3) becomes apparent when we consider partial derivatives of y_i with respect to x_{hj} or z_{hk}, $h\neq i$

(Mittelhammer, et al.). Since x_{hj} is the quantity of input j used in another product h, x_{hj} cannot be altered without also altering y_h unless its marginal product is zero (which violates standard regularity conditions) or another input allocated to y_h is simultaneously altered. Because a change in x_{hj} induces a simultaneous change either in y_h or another input allocated to y_h, it is not possible even in principle for the vector $(\underline{Y}_i, X, Z)$ to be a set of independent variables. There are simply too many variables on the right-hand-side of (3) to be mathematically consistent as a reduced-form equation. Further, the above problem is not solved when the constraints on allocations of the fixed inputs,

$$(4) \quad \sum_{i=1}^{m} z_{ik} \leq \bar{z}_k, \; k = 1,\ldots,q,$$

are binding. The vector $(\bar{z}_1,\ldots,\bar{z}_q)$ can be substituted, for example, for $(z_{11},\ldots,z_{1q})$ in Z. However, a change in x_{hj} must still be accompanied by a change either in y_h or some other input allocated to y_h, both of which would remain as right-hand-side variables in the modified equation (3).

It appears that the only way to appropriately initiate a specification of the multiple-product firm is to write individual production functions for all products. Although Pfouts and Just, et al. wrote their systems of production functions to be technically independent, there is no reason to expect that such a restrictive assumption is generally valid for all multiple-product firms. Even when many inputs are clearly allocated among products, others may not be allocated. In such a case we can still write individual production functions with other output quantities appearing on the right-hand-side of some or all equations:

$$(5)\quad y_i = f^i(\underline{Y}_i, X_i, Z_i), \quad i = 1,\ldots,m,$$

where $X_i = (x_{i1},\ldots,x_{in})$ is the vector of variable inputs allocated to y_i, and $Z_i = (z_{i1},\ldots,z_{iq})$ is the vector of fixed inputs allocated to y_i. For simplicity, nonallocated inputs are not identified in (5), but the total quantity of any generic input could be substituted as appropriate for its allocations in all equations of (5). The system of equations (5) constitute a general specification of multiple-product technology that does not impose *a priori* assumptions of either nonjointness or separability. Consequently, it is preferred as an initial specification to any of the single-equation specifications, (1), (2), or (3), or to the system of independent production functions of Pfouts or Just, et al.

Reducing the Technology System to a Single Equation

Just, et al.'s statements governing the reduction of a system of individual production functions to a single-equation specification of multiple-product technology are now elaborated. The reduction to an equation in total quantities of outputs and generic inputs is illustrated in two parts. First, assuming that sufficient conditions are satisfied for taking inverses of the individual production functions, Just, et al.'s assertions are confirmed governing restrictions required to derive a single-equation generic-input specification without considering the producer's behavioral objective. Secondly, it is shown that profit maximization can be accomplished in two steps, the first of which yields a single-equation generic-input specification in the general case of multiple products with multiple allocated inputs even with technically-interdependent production functions, as long as there are no allocation costs.

Take the system of technically-interdependent production functions (5) subject to constraints on allocations of the fixed inputs (4) and the following set of identities on allocations of the variable inputs:

$$(6) \quad \sum_{i=1}^{m} x_{ij} \equiv \bar{x}_j, \; j = 1,\ldots,n.$$

Assuming that the constraints in (4) are binding (i.e., fixed inputs are fully allocated) and inverses of (5) exist, these m+n+q equations in m+m·n+m·q+n+q variables can be reduced to a single-equation production function in m-1+n+(m-1)(n-1)+q+(m-1)q = m(n+q) other variables. For y_1 this implies:

total quant
allocation except for 1

$$(7) \quad y_1 = F(\underline{Y}_1, \bar{X}, X_{\bar{1}}, \bar{Z}, Z_{\bar{1}}),$$

where $X_{\bar{1}} = (x_{22},\ldots,x_{2n},\ldots,x_{m2},\ldots,x_{mn})$, and $Z_{\bar{1}} = (z_{21},\ldots,z_{2q},\ldots,z_{m1},\ldots,z_{mq})$.[1]

math!

The logical inconsistency that appeared in (3) does not trouble the single-equation derived specification of technology in (7). Since the total quantity of each generic input is included in (7) and a necessary number of allocations are excluded, it is possible to take a partial derivative of y_1 with respect to any right-hand-side variable without altering any other right-hand-side variable. For example, a change of x_{22} would be offset (a) by an equal and opposite change in x_{12}, (b) by a change in x_{21}, and (c) by an equal and opposite change in x_{11} (none of which are arguments in (7)), sufficient to keep $\bar{x}_2$, y_2, and other right-hand-side variables unaltered.

Thus, (7) is a correct general reduced-form specification of the production system (4) - (6). It is clearly derivable from (4) - (6) when the constraints in (4) are binding and inverses of the production functions (5) exist. Yet, unless n = 1, m+q = 1, or a sufficient number of allocations do not appear in (5), (7) includes allocations of variable and/or fixed inputs as well as the generic input quantities. To remove all of the $(m-1)\cdot(n-1)+(m-1)\cdot q$ allocations requires that $(m-1)\cdot(n-1)+(m-1)\cdot q$ more equations be included in the system.

Without imposing arbitrary assumptions on the technology, we turn to the behavioral objective. Assuming price-taking profit-maximizing behavior, the objective function may be written with (7) as:

$$(8)\quad \max_{\underline{Y}_1,\bar{X},X_1^-,Z_1^-} \pi = p_1F(\underline{Y}_1,\bar{X},X_1^-,\bar{Z},Z_1^-) + \sum_{i=2}^{m} p_iy_i - \sum_{j=1}^{n} r_j\bar{x}_j,$$

where π is short-run profit (net returns over variable costs), p_i is price of output i, and r_j is price of variable input j. The first-order equations are

$$(9a)\quad \partial\pi/\partial y_i = p_1\partial F/\partial y_i + p_i = 0, \qquad i = 2,\ldots,m,$$

$$(9b)\quad \partial\pi/\partial\bar{x}_j = p_1\partial F/\partial\bar{x}_j - r_j = 0, \qquad j = 1,\ldots,n,$$

$$(9c)\quad \partial\pi/\partial x_{ij} = p_1\partial F/\partial x_{ij} = 0, \qquad i = 2,\ldots,m;\ j = 2,\ldots,n,$$

$$(9d)\quad \partial\pi/\partial z_{ik} = p_i\partial F/\partial z_{ik} = 0, \qquad i = 2,\ldots,m;\ k = 1,\ldots,q.$$

Assuming $p_1 > 0$, equations (9c) and (9d) imply $\partial F/\partial x_{ij} = 0$ and $\partial F/\partial z_{ik} = 0$.[2] This gives $(m-1)\cdot(n-1)+(m-1)\cdot q$ additional equations which, like (7), are functions only of $(\underline{Y}_1,\bar{X},X_1^-,\bar{Z},Z_1^-)$. Assuming the relevant Jacobians are non-zero, these equations can be solved using the implicit function theorem to yield the $(m-1)(n-1)+(m-1)q$ allocations (X_1^-,Z_1^-) as functions of $(\underline{Y}_1,\bar{X},\bar{Z})$:

$$(10)\quad x_{ij} = x_{ij}(\underline{Y}_1,\bar{X},\bar{Z}), \qquad i = 2,\ldots,m;\ j = 2,\ldots,n,$$

$$z_{ik} = z_{ik}(\underline{Y}_1,\bar{X},\bar{Z}), \qquad i = 2,\ldots,m;\ k = 1,\ldots,q.$$

By substitution into (7) we obtain

$$y_1 = F[\underline{Y}_1, \overline{X}, X_1^-(\underline{Y}_1,\overline{X},\overline{Z}), \overline{Z}, Z_1^-(\underline{Y}_1,\overline{X},\overline{Z})],$$

which by collecting terms gives

$$(11) \quad y_1 = G(\underline{Y}_1,\overline{X},\overline{Z}).$$ Same as (2) ?

We have thus obtained the reduced-form specification of one product in other product and generic-input quantities from a system of technically-interdependent product functions with binding constraints. However, for a multiple-product firm, (11) is value-free (i.e., not dependent on the behavioral objective) only if no more than one input, variable or fixed, is effectively allocated among all products. Otherwise it is dependent on those first-order conditions that do not depend on prices, e.g., (9c) and (9d) for a behavioral model such as (8).

Allocation Costs

If costs are associated with allocating the fixed inputs to individual products, (8) can be rewritten following Pfouts as

$$(12) \quad \underset{\underline{Y}_1,\overline{X},X_1^-,Z_1^-}{\text{Max}} \; \pi = p_1F(\underline{Y}_1,\overline{X},X_1^-,\overline{Z},Z_1^-) + \sum_{i=2}^{m} p_iy_i - \sum_{j=1}^{n} r_j\overline{x}_j - K(\overline{Z},Z_1^-),$$

where $K(\overline{Z},Z_1^-)$ is the allocation cost function. First-order equations for (12) consist of (9a) - (9c) and

$$(13) \quad \partial\pi/\partial z_{ik} = p_1\partial F/\partial z_{ik} - \partial K/\partial z_{ik} = 0, \quad i = 2,\ldots,m; \; k = 1,\ldots,q.$$

Because (13) includes a second term, $\partial K/\partial z_{ik}$, it is not possible to solve the $(m-1)(n-1)+(m-1)q$ equations in (9c) and (13) for (X_1^-, Z_1^-) as functions only of $(\underline{Y}_1, \bar{X}, \bar{Z})$ unless $\partial K/\partial z_{ik}$ is constant, $i = 2,\ldots,m$; $k = 1,\ldots,q$. Otherwise the vector of marginal allocation costs and p_1 remain as arguments in the $(m-1)q$ equations for Z_1^-. Consequently, if there are non-constant marginal costs associated with allocating any fixed input, we are unable to obtain enough equations that are functions only of the technology variables to derive a reduced-form generic-input production function such as (11), even using the behavioral objective.

This finding is particularly troublesome. It is apparent that there are indeed costs associated with allocating a number of quasi-fixed inputs in agriculture. For example, some modifications are required to convert equipment used to harvest corn in order to harvest soybeans, grain sorghum, or wheat. These modifications constitute costs of allocating a fixed input to more than one product. The full allocation cost is associated with the first unit of the second crop harvested, so marginal allocation costs are not constant across all units.

It will be shown in the next section that reversible duality relations do not exist for the system of production functions (5) with binding constraints. They do exist for (11), but since non-constant marginal allocation costs prevent derivation of (11), reversible duality relations cannot be demonstrated for any technology specification when there are non-constant marginal costs of allocating binding fixed inputs.

Duality Relationships

Since we did not use first-order conditions in $\underline{Y}_1$ and $\bar{X}$, i.e., (9a) and (9b), in deriving (11), those conditions can be restated and used in the second step of profit maximization when there are no costs of allocation. The second-step objective function is

$$(14)\quad \underset{\underline{Y}_1,\bar{X}}{\text{Max}}\ \pi_2 = p_1 G(\underline{Y}_1,\bar{X},\bar{Z}) + \sum_{i=2}^{m} p_i y_i - \sum_{j=1}^{n} r_j \bar{x}_j.$$

Its first-order equations are

$$(15a)\quad \partial\pi_2/\partial y_i = p_1 \partial G/\partial y_i + p_i = 0, \qquad i = 2,\ldots,m,$$

$$(15b)\quad \partial\pi_2/\partial\bar{x}_j = p_1 \partial G/\partial\bar{x}_j - r_j = 0, \qquad j = 1,\ldots,n.$$

Assuming appropriate curvature and smoothness properties are satisfied in $\underline{Y}_1$ and $\bar{X}$, these m+n-1 equations can be solved by the implicit function theorem for $(\underline{Y}_1,\bar{X})$ as functions of $(P,R,\bar{Z})$:

$$(16a)\quad y_i = y_i^*(P,R,\bar{Z}), \qquad i = 2,\ldots,m,$$

$$(16b)\quad \bar{x}_j = \bar{x}_j^*(P,R,\bar{Z}), \qquad j = 1,\ldots,n,$$

where $P = (p_1,\ldots,p_m)$ and $R = (r_1,\ldots,r_n)$. By substituting (16a) and (16b) into (11), we obtain

$$(16c)\quad y_1 = G[\underline{Y}_1^*(P,R,\bar{Z}),\ \bar{X}(P,R,\bar{Z}),\ \bar{Z}] = y_1^*(P,R,\bar{Z}).$$

We now have product supply equations (16a) and (16c) and generic-input demand equations (16b) as functions of the exogenous product and variable-input prices and fixed-input quantities. It is apparent that the indirect restricted profit function can be specified using (16a) - (16c):

$$(17)\quad \pi^* = p_1 y_1^*(P,R,\bar{Z}) + \sum_{i=2}^{m} p_i y_i^*(P,R,\bar{Z}) - \sum_{j=1}^{n} r_j \bar{x}_j^*(P,R,\bar{Z}) = \pi^*(P,R,\bar{Z}).$$

The envelope theorem is valid for its partial derivatives in product and variable input prices and permit direct recovery of (16a) and (16c) and the negative of (16b).

The Hessian of (11) is square with dimension m+n+q-1. It has the same dimensions as the Hessian of the normalized indirect profit function, $\pi^{*'} \equiv \pi^*/p_1$, in normalized prices, $(P',R') \equiv (p_2/p_1,\ldots,p_m/p_1, r_1/p_1,\ldots,r_n/p_1)$.[3] Lau has shown that the Hessian of $\pi^{*'}$ is fully identified by the Hessian of G and vice-versa. Thus, a two-way duality between the multiple-product generic-input production function (11) and its restricted profit function applies to both first and second derivatives just as a two-way duality exists for the single-product firm's production function. The fact that (11) may embody behavioral properties does not alter this two-way duality.

Unless there is only one allocatable input, it is not possible to demonstrate this same reversible duality between the Hessians of the restricted profit function and the derived production function (7). The problem is that a lack of information imbedded in the dual problem makes the matrices noncomformable. The dimension of the square Hessian of (7) is m-1+n+(m-1)(n-1)+q+(m-1)q = m(n+q) as compared to the above-noted dimensions of $\pi^{*'}$. The restricted profit function (17) can indeed be derived from (7), either by solution of the entire set of first-order equations in (9a) - (9d) or by the above two-step procedure, but the parameters of (7) cannot be fully identified from (17). This problem prevents dual identification of the parameters of any production function with binding constraints on the allocation of a fixed input if costs are associated with its allocation since (7) does not reduce to (11) in that case.

Further, it is not possible to write separate profit functions for each product unless (a) the system of production functions (5) is technically independent among products (as assumed by Pfouts and by Just, et al.), (b) the constraints on allocations of the fixed inputs (4) are all nonbinding, and (c) complete data are available on all allocations. Therefore, unless all these unlikely conditions exist for the multiple-product firm, the same problem of nonconformable matrices prevents complete identification of the system of production functions (5) from parameter estimates of the profit function. The multiple-product restricted profit function (17) can be correctly derived from (5) even if there exists technical interdependence in production of multiple products and/or binding constraints on allocations of fixed inputs and/or incomplete data on input allocations, but (5) cannot be derived from (17). There is simply not enough information available from the dual estimation.

Conclusions

Unless production of multiple products is separable in outputs and inputs or only one input is effectively allocated among all outputs, one cannot write the value-free technology specification as a single equation. When either of these conditions is not satisfied, the technology should be expressed as a system of individual, possibly interdependent, production functions subject to any constraints that exist on allocations. If no non-constant marginal costs are associated with specific allocations, a single-equation reduced-form specification of technology in product and generic-input quantities can be derived in this general case by incorporating those first-order equations from the behavioral objectives that are not functions of prices. Technical interdependence does not limit this derivation capability. However, with or without independence, the reduced-form equation is not value-free. Allocation costs prevent derivation of a reduced-form generic-input specification of the technology, even using information based on the behavioral objective.

Complete reversible duality relations exist between individual profit functions and individual technically-independent production functions without binding allocation constraints on fixed inputs or costs of allocation. Complete reversible duality also exists between the reduced-form generic-input specification, with or without partial incorporation of the behavioral objective, and the multiple-product restricted profit function. Duality relations for the system of individual production functions subject to binding allocation constraints on fixed inputs, however, can be demonstrated in only one direction, from the technology system to the restricted profit function. This is true also for both the system of individual production functions and the single-equation specification of technology when allocation costs are nonzero. Limitations of dual models for extracting information about multiple-product technologies are thus further documented.

Footnotes

1. The derivation of (7) from (4) - (6) can be shown by rewriting (4) and (6) as

$$\text{(4a)}\quad z_{1j} = \bar{z}_j - \sum_{i=2}^{m} z_{ij}, \qquad j = 1,\ldots,q,$$

$$\text{(6a)}\quad x_{1j} \equiv \bar{x}_j - \sum_{i=2}^{m} x_{ij}, \qquad j = 1,\ldots,n,$$

and taking inverses of the production functions, $y_2,\ldots,y_m$, in $x_{21},\ldots,x_{m1}$:

$$\text{(5a)}\quad x_{i1} = f^{i-}(Y, x_{i2},\ldots,x_{in}, z_{i1}, \ldots,z_{iq}), \qquad i = 2,\ldots,m.$$

Substituting (4a), (5a) and (6a) into the first equation of (5),

$$\text{(5b)}\quad y_1 = f^1(Y_1, x_{11},\ldots,x_{in}, z_{11},\ldots,z_{1q}),$$

we obtain

$$\text{(5c)}\quad y_1 = f^1[Y_1, \bar{x}_1 - \sum_{i=2}^{m} f^{i-}(Y, x_{i2},\ldots,x_{in}, z_{i1},\ldots,z_{iq}),$$
$$\bar{x}_2 - \sum_{i=2}^{m} x_{i2}, \ \ldots, \ \bar{x}_n - \sum_{i=2}^{m} x_{in}, \ \bar{z}_1 - \sum_{i=2}^{m} z_{i1}, \ \ldots, \ \bar{z}_q - \sum_{i=2}^{m} z_{iq}]$$

which, by collecting terms, may be specified as

$$\text{(7)}\quad y_1 = F(Y_1, \bar{X}, x_{22},\ldots,x_{2n},\ldots,x_{m2},\ldots,x_{mn}, \bar{Z}, z_{21},\ldots,z_{2q},\ldots,z_{m1},\ldots,z_{mq}).$$

2. These conditions mean that the marginal rate of technical substitution for each pair of inputs allocated to one product is the same as for allocations of the same pair to every other product. This can be proven by taking the partial derivative of y_1 with respect to any allocation in the vector $(\bar{X_1},\bar{Z_1})$ from equation (5c) in Footnote 1 and setting it equal to zero.

3. Maximizing normalized profit gives the same solution in the choice variables as maximizing profit since we divide by a constant. Further, the envelope theorem also applies to normalized profit such that the partial derivatives of $\pi^{*'}$ with respect to (P',R') give $(Y_1,-\bar{X})$.

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