GRAFTED POLYNOMIALS AS APPROXIMATING FUNCTIONS

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The use of segments of polynomials to approximate production surfaces and time-series trends is described and illustrated. These segmented curves are restricted to be continuous and have a continuous derivative(s) at the join points.

The goal in many studies is to represent a response variable, y, as a relatively simple analytic function of an input variable(s). Most functions furnish only an approximation in a limited range. Thus, in a practical situation, the choice of functional form will rest upon theoretical considerations, ease of estimation and acceptance by the data.

Discussions of the properties of some common forms used in agricultural response studies are given in Heady and Dillon [4] and Mason [9]. Two articles by Anderson [1 and 2] contain a discussion of estimation problems associated with form.

Generally, a function is desired that

1. is continuous,
2. possesses continuous first derivatives,
3. is easy to estimate (i.e., linear in the parameters), and
4. permits easy computation of optima.

Obviously this is not an exhaustive listing of desirable properties, nor are the listed properties of equal importance. One function that satisfies these criteria to an admirable degree is the quadratic. However, the quadratic does not always furnish an adequate approximation over the entire experimental space.

We shall show how it is possible to 'graft' quadratic (and other polynomial) functions to increase the domain of approximation in such a way that Properties 1, 2, and 3, and to a large extent Property 4, are retained.

Consider first the one-dimensional case. We assume that we can approximate the response with

$$y = a_0 + a_1X + a_2X^2 \qquad X \leqslant C$$
$$y = b_0 + b_1X + b_2X^2 \qquad X \geqslant C$$

where C is a specified number and the parameters (the a's and b's) are restricted so that the curve and the derivative are continuous at the point C. These restrictions can be written:

$$a_0 + a_1C + a_2C^2 = b_0 + b_1C + b_2C^2$$
$$a_1 + 2a_2C = b_1 + 2b_2C.$$

These restrictions are linear in the parameters and reduce the number

* Journal Paper No. *J-6226* of the Iowa Agriculture and Home Economics Experiment Station, Ames, Iowa. Project No. 1005. A portion of this work was conducted during the 1964-65 academic year while the author was a National Science Foundation Fellow at Stanford University.

of independent parameters in the model from six to four. We choose, for reasons which become evident below, to estimate directly the parameters

$$a_0, a_1, a_2, b_2 - a_2.$$

The b_i are then defined in terms of these four parameters by

$$b_0 = a_0 + C^2 (b_2 - a_2)$$
$$b_1 = a_1 - 2C (b_2 - a_2)$$
$$b_2 = a_2 + (b_2 - a_2).$$

Estimates of a_0, a_1, a_2, and $(b_2 - a_2)$ are obtained from the regression equation

$$y = a_0 + a_1X + a_2X^2 + (b_2 - a_2) Z$$

where

$$\begin{aligned} Z &= 0 & X &\leqslant C \\ &= (X - C)^2 & X &> C. \end{aligned}$$

The format permits estimating a single quadratic as well as the modified quadratic by simply deleting the independent variable, Z, from the regression. Also, the 't' for the regression coefficient estimating $(b_2 - a_2)$ indicates immediately the 'gain' obtained by adding a parameter to the model.

Obviously one can join several quadratics in this manner and could, for example, represent a curve with increasing and then decreasing returns as in Figure 1.

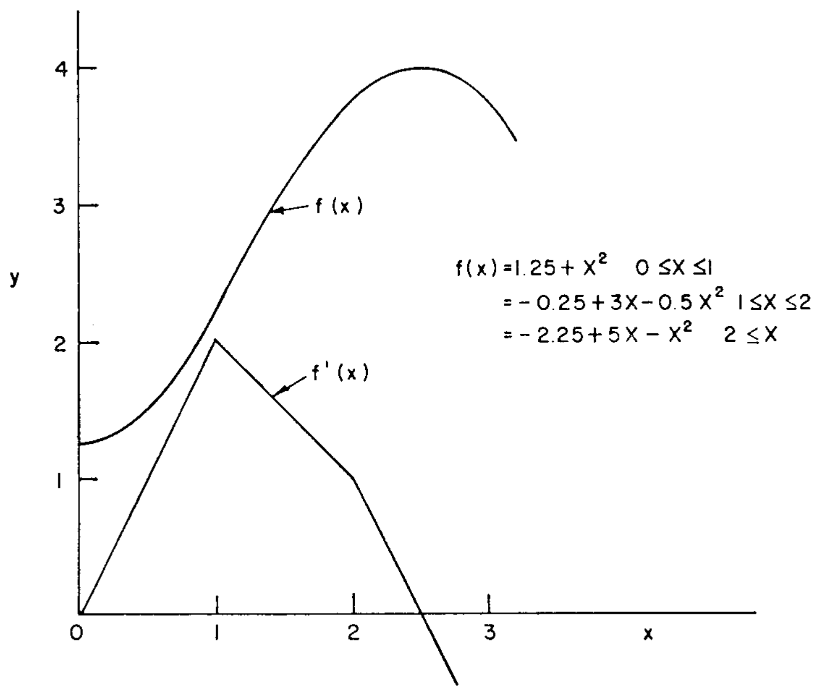

Fig. 1—Example of a Grafted Quadratic.

The procedure is easily extended to higher-dimensional surfaces by cutting the domain of the function with planes. We illustrate the procedure in two dimensions. Assume, as illustrated in Figure 2, that the two-variable quadratic

$$y = a_0 + a_1X_1 + a_2X_2 + a_3X^2{}_1 + a_4X^2{}_2 + a_5X_1X_2$$

holds in the region

$$X_1 \leqslant K_0 + K_1X_2 \qquad K_1 \neq 0$$

and that

$$y = b_0 + b_1X_1 + b_2X_2 + b_3X^2{}_1 + b_4X^2{}_2 + b_5X_1X_2$$

holds in the region

$$X_1 \geqslant K_0 + K_1X_2$$

subject to the restrictions that, on the line $K_0 + K_1X_2 = X_1$:

(i) the functions are equal,

(ii) the partial derivatives of the two functions w.r.t. X_1 are equal, and

(iii) the partial derivatives of the two functions w.r.t. X_2 are equal.

These restrictions specify five independent linear restrictions upon the 12 parameters. If we specify the parameters to be estimated directly by

$$a_0, a_1, a_2, a_3, a_4, a_5, (b_5 - a_5),$$

then the remaining parameters are defined by

$$\begin{aligned}
b_0 &= a_0 - K^2{}_0 \,(2K_1)^{-1}\,(b_5 - a_5)\\
b_1 &= a_1 + K_0K^{-1}{}_1\,(b_5 - a_5)\\
b_2 &= a_2 - K_0\,(b_5 - a_5)\\
b_3 &= a_3 - (2K_1)^{-1}\,(b_5 - a_5)\\
b_4 &= a_4 - 2^{-1}K_1\,(b_5 - a_5)\\
b_5 &= a_5 + (b_5 - a_5).
\end{aligned}$$

Thus, the regression equation becomes

$$y = a_0 + a_1X_1 + a_2X_2 + a_3X^2{}_1 + a_4X^2{}_2 + a_5X_1X_2 + (b_5 - a_5)\,Z$$

where

$$\begin{aligned}
Z &= 0 && X_1 \leqslant K_0 + K_1X_2\\
&= -(2K_1)^{-1}\,(X_1 - K_0 - K_1X_2)^2 && X_1 > K_0 + K_1X_2.
\end{aligned}$$

We note that the functions will differ in the two subdomains only if the interaction differs in the two domains (i.e., $a_5 \neq b_5$). If the response displays no interaction, one gains little by dividing the domain along a line perpendicular to neither axis. Thus, one procedure is to divide the domain by planes perpendicular to axes obtained by the rotation such that the interaction term is eliminated. That is, if one fits the quadratic

$$y = a_0 + a_1X_1 + a_2X_2 + a_3X^2{}_1 + a_4X^2{}_2 + a_5X_1X_2$$

to the data of the shaded region of Figure 2, the suggested cutting planes would be of the form

$$\begin{aligned}
C_1 &= X_1 \cos\theta - X_2 \sin\theta\\
C_2 &= X_1 \sin\theta + X_2 \cos\theta
\end{aligned}$$

where θ is defined by

$$\begin{aligned}
\tan 2\theta &= a_5/(a_3 - a_4) && a_3 \neq a_4.\\
\theta &= \pi/4 && a_3 = a_4.
\end{aligned}$$

We now outline a procedure for dividing the domain into more than two subdivisions. Such a subdivision is illustrated in Figure 3. If the sub-

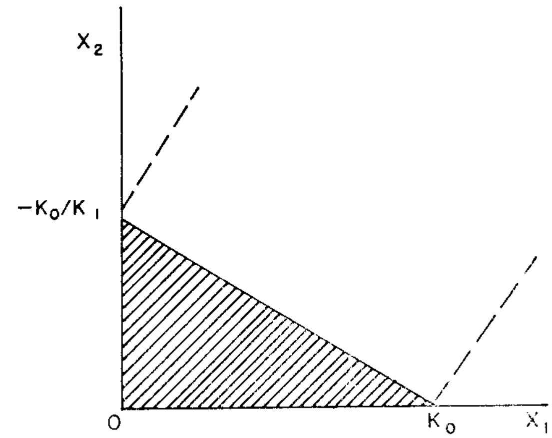

FIG. 2—A possible subdivision of the plane for estimation of a grafted quadratic.

divisions are identified as in the figure, we associate the four quadratic functions with the four subdivisions

1 $y = a_0 + a_1X_1 + a_2X_2 + a_3X^2{}_1 + a_4X^2{}_2 + a_5X_1X_2 = f_1$
2 $y = b_0 + b_1X_1 + b_2X_2 + b_3X^2{}_1 + b_4X^2{}_2 + b_5X_1X_2 = f_2$
3 $y = c_0 + c_1X_1 + c_2X_2 + c_3X^2{}_1 + c_4X^2{}_2 + c_5X_1X_2 = f_3$
4 $y = d_0 + d_1X_1 + d_2X_2 + d_3X^2{}_1 + d_4X_2 + d_5X_1X_2 = f_4$

The restrictions are:

A. On the line $X_1 = C_1$

(1a)	$f_1 = f_2$	(2a)	$f_3 = f_4$
(1b)	$\delta f_1/\delta X_1 = \delta f_2/\delta X_1$	(2b)	$\delta f_3/\delta X_1 = \delta f_4/\delta X_1$
(1c)	$\delta f_1/\delta X_2 = \delta f_2/\delta X_2$	(2c)	$\delta f_3/\delta X_2 = \delta f_4/\delta X_2$

B. On the line $X_2 = C_2$

(1a)	$f_1 = f_3$	(2a)	$f_2 = f_4$
(1b)	$\delta f_1/\delta X_1 = \delta f_3/\delta X_1$	(2b)	$\delta f_2/\delta X_1 = \delta f_4/\delta X_1$
(1c)	$\delta f_1/\delta X_2 = \delta f_3/\delta X_2$	(2c)	$\delta f_2/\delta X_2 = \delta f_4/\delta X_2$.

This system permits the independent estimation of eight parameters. If we identify the parameters to be estimated directly as

$$a_0, a_1, a_2, a_3, a_4, a_5, b_3 - a_3, c_4 - a_4,$$

the remaining parameters are given by

$b_0 = a_0 + C^2{}_1\ (b_3 - a_3)$	$c_3 = a_3$
$b_1 = a_1 - C_1\ (b_3 - a_3)$	$c_5 = a_5$
$b_2 = a_2$	$d_0 = a_0 + C^2{}_1\ (b_3 - a_3) + C^2{}_2\ (c_4 - a_4)$
$b_4 = a_4$	$d_1 = b_1 = a_1 - 2C_1\ (b_3 - a_3)$
$b_5 = a_5$	$d_2 = c_2 = a_2 - 2C_2\ (c_4 - a_4)$
$c_0 = a_0 + C^2{}_2\ (c_4 - a_4)$	$d_3 = b_3 = a_3 + (b_3 - a_3)$
$c_1 = a_1$	$d_4 = c_4 = a_4 + (c_4 - a_4)$
$c_2 = a_2 - 2C_2\ (c_4 - a_4)$	$d_5 = a_5$.

We see immediately that the estimates are obtained from the regression equation:

$$y = a_0 + a_1X_1 + a_2X_2 + a_3X^2{}_1 + a_4X^2{}_2 + a_5X_1X_2 + (b_3 - a_3)\, Z_1 + (c_4 - a_4)\, Z_2$$

where

$$\begin{aligned} Z_1 &= 0 && X_1 \leqslant C_1 \\ &= (X_1 - C_1)^2 && X_1 > C_1 \\ Z_2 &= 0 && X_2 \leqslant C_2 \\ &= (X_2 - C_2)^2 && X_2 > C_2. \end{aligned}$$

Notice that the coefficient on the interaction term is identical in the four subdomains. Thus, it is suggested that one use this subdivision after rotating the axis to remove most of the interaction. Another possibility would be to further subdivide the domain by a dashed line as in Figure 3.

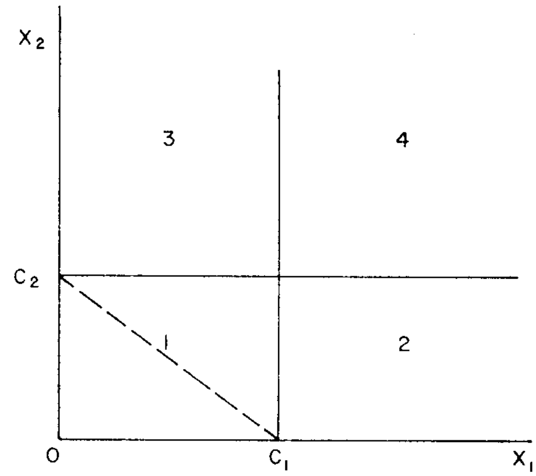

FIG. 3—Division of the domain of a production function into several subdomains.

An alternative way of approaching the problem is to approximate the response surface by sums of functions, each of which is continuous and has continuous first derivatives. We note that the functions

$$\begin{aligned} Z_1 &= (X_1 - C)^2 && X_1 > C \\ &= 0 && X_1 \leqslant C \\ Z_3 &= (X_2 - \alpha_0 - \alpha_1X_1)^2 && X_2 > \alpha_0 + \alpha_1X_1 \\ &= 0 && \text{otherwise} \end{aligned}$$

satisfy these requirements. Obviously, the functions obtained by reversing the inequalities satisfy the requirements equally well. Note that the function

$$\begin{aligned} Z_4 &= (X_1 - C_1)\, (X_2 - C_2) && X_1 < C_1, X_2 < C_2 \\ &= 0 && \text{otherwise} \end{aligned}$$

does not satisfy the restriction.

If we desire that higher-order derivatives of our approximating surface be continuous, we use higher-order polynomials. In general, for example, the following functions where γ_i and C are constants, are continuous with $m - 1$ continuous derivatives:

$$T_1 = (X - C)^m \qquad X > C$$
$$= 0 \qquad \text{otherwise}$$
$$T_2 = (X_g - \sum_{i=0}^{g-1} \gamma_i X_i)^m \qquad X_g > \sum_{i=0}^{g-1} \gamma_i X_i$$
$$= 0 \qquad \text{otherwise.}$$

The function

$$T_3 = \prod_{i=1}^{g} (X_i - C_i)^{m_i} \qquad X_i < C_i, \quad i = 1, 2, \ldots, g$$
$$= 0 \qquad \text{otherwise}$$

where m_i ($m_i > 0$) and m are integers with $\Sigma m_i = m$, has continuous derivatives of order $m_1 - 1$ where m_1 is the minimum of the m_i.

Thus, if we desire an approximating curve in one dimension with continuous first and second derivatives, we would fit the regression equation, e.g.,

$$y = a_0 + a_1 X + a_2 X^2 + a_3 X^3 + a_4 T$$

where

$$T = (X - C)^3 \qquad X > C$$
$$= 0 \qquad \text{otherwise.}$$

Of course, any number of functions of the T-type might be included in the equation.

The grafted polynomials we have been discussing can be used to approximate the trend in a time series. In the use of moving averages to remove trend, one often makes the assumption: 'the trend over a distance of A observations is adequately approximated by a polynomial of degree m'. On the basis of this assumption, one then constructs K weights [8, p. 366] to estimate the trend value at the midpoint of the interval. Consider now a similar assumption: 'The series may be divided into segments each containing A observations and the trend in each is adequately approximated by a polynomial of degree m. The trend is continuous from period to period and the first r ($r \leqslant m - 1$) derivatives of the trend are continuous.' Using the latter of the two quotations, we would fit a grafted polynomial to the data and obtain the estimated trend for the entire series. The latter assumption might heuristically be judged somewhat more restrictive, but making this assumption permits one to obtain a 'smooth' trend estimate for the entire series.

Let us consider in detail the case when we are willing to approximate the trend by a quadratic with continuous first derivative. Assume the data are indexed by ($t = 1, 2, \ldots, n$). Dividing the data in segments of A observations, we consider the regression variables:

$$Z_{it} = (t - (i-1)A)^2 \qquad t > (i-1)A$$
$$= 0 \qquad \text{otherwise}$$

where

$$i = 1, 2, \ldots, M$$
$$M = \text{an integer such that } |AM - n| < A.$$

The series is divided into M segments and $M - 1$ is the number of grafts or joins in our function. In this formulation we assume that, if $n \neq AM$, the last segment, containing observations indexed by

$i = A(M-1)+1, A(M-1)+2, \ldots, n$, is the only one to contain more or less than A observations. While obviously any of the segments can be varied in length, this formulation will simplify the presentation and construction of regression variables. At the formal level all we need do to estimate the trend is to regress our series upon $t, Z_1, Z_2, \ldots Z_M$ the estimated trend being given by

$$\hat{Y}_t = \hat{C}_0 + \hat{C}_1 t + \sum_{i=1}^{M} \hat{d}_i Z_{it}$$

where $\hat{C}_0$, $\hat{C}_1$ and $\hat{d}_i$ are the estimated regression coefficients. However, if M is at all large, we can expect to encounter numerical problems in obtaining the inverse and regression coefficients. To reduce the correlation, we suggest that a simple linear combination of the Z's be taken to form the variables

$$X_{it} = Z_{it} - 3Z_{i+1,\,t} + 3Z_{i+2,\,t} - Z_{i+3,\,t} \qquad i = 1, 2, \ldots, M$$

where for convenience we define

$$Z_{M+1} = Z_{M+2} = Z_{M+3} = 0.$$

Note that

$$\begin{aligned} X_{it} &= (t-(i-1)A)^2 & (i-1)A < t < iA \\ &= (t-(i-1)A)^2 - 3(t-iA)^2 & iA \leqslant t < (i+1)A \\ &= (t-(i+2)A)^2 & (i+1)A \leqslant t < (i+2)A \\ &= 0 \quad \text{otherwise.} & \end{aligned}$$

Since the function, X_{it}, for $i < M-3$ is symmetric about $(i+\frac{1}{2})A$, the X variables can be written down immediately. The X_i remain correlated, but there should be little trouble in obtaining the inverse.

This procedure is thought to have merit when one is called upon to extrapolate a series. Since polynomials tend to plus or minus infinity at a rate equal to the highest power of t as t increases, practitioners typically hesitate to use high-order polynomials in extrapolation. Thus, though a series may display a nonlinear trend, we might wish to extrapolate on the basis of a linear trend. To accomplish this we approximate the trend of the last K observations of a series of n observations by a straight line. By imposing restrictions this linear trend can be made continuous to and tangent at the point of join to a non-linear trend for the earlier portion of the time series. Using grafted quadratics for the earlier portion of the series, one could construct the regression variables

$$\begin{aligned} X_{1t} &= t & \\ Z_{2t} &= (t-K)^2 & t < n-K \\ &= 0 & \text{otherwise} \\ Z_{2+i,\,t} &= (t-K-iA)^2 & t < n-K-iA,\ i = 1, 2, \ldots \\ &= 0 & \text{otherwise} \end{aligned}$$

where it is assumed that time is coded from 1 to n, n being the last observation. If there is a large number of Z's they should be transformed into the lesser correlated X's.

With either the Z's or X's, the forecast equation is

$$y = \hat{b}_0 + \hat{b}_1 t$$

where the $\hat{b}$'s are the least squares coefficients.

One may desire to approximate a curve with 'locally linear segments' but would like the region of the graft of the two curves to be smooth to the extent that the first derivative is continuous. The function

$$
\begin{aligned}
Z &= 0 && t < C_1 \\
&= (t - C_1)^2 && C_1 \leqslant t \leqslant C_2 \\
&= C^2{}_1 - C^2{}_2 + 2(C_2 - C_1)\, t && t \geqslant C_2
\end{aligned}
$$

is such a function.

Using these ideas we could fit a trend line to a time series which would be linear for the first K_1 observations, linear for the last K_2 observations and a grafted quadratic in the remainder. The required variables would be, e.g.,

$$
\begin{aligned}
X_{1t} &= t \\
X_{2t} &= (t - K_1)^2 && t > K_1 \\
&= 0 && \text{otherwise} \\
X_{2+i,\,t} &= (t - K_1 - i\,A)^2 && t > K_1 + iA,\ i = 1, 2, \ldots, r-3 \\
&= 0 && \text{otherwise} \\
X_{r,\,t} &= (X - n + A + K_2)^2 && n - A - K_2 < t < n - K_2 \\
&= (n - A - K_2)^2 - (n - A)^2 + 2, At && t > n - K_2 \\
&= 0 && \text{otherwise}
\end{aligned}
$$

Example One

We consider first a fertilizer experiment with the objective of estimating the response surface for corn yield as a function of nitrogen, N, and phosphorus, P. These data are taken from Heady, Pesek and Brown [6]. The data have also been analysed in [3] and [5]. The experiment was conducted on Ida Silt Loam in Western Iowa in 1952. The experiment contains 57 treatments (combinations of nitrogen and phosphate) in two replicates. Nitrogen and phosphorus are coded in units of 40 pounds. The treatment means for a portion of the experiment that forms a 5 x 5 factorial are given in Table 1. As a preliminary analysis, the data were analysed in four separate sections with a range of 160 pounds of nutrient in each section. The data for the highest levels of both nutrients displayed little or no treatment effects. (The F for treatment was less than one.) That for the high levels of one and low levels of the other generally showed treatment effects only for the nutrient at the low level and little interaction. The data for low P and high N suggested that response to P was not well approximated by a quadratic. The data for the low levels of both nutrients displayed significant effects for both variables and significant high order interaction. On this basis, the domain was divided into four subdomains by the lines $N = 4$ and $P = 4$. The functions

$$
\begin{aligned}
Z_1 &= (N - 4)^2 && N < 4 \\
&= 0 && \text{otherwise} \\
Z_2 &= (P - 4)^2 && P < 4 \\
&= 0 && \text{otherwise} \\
Z_3 &= (N + P - 4)^2 && N + P < 4 \\
&= 0 && \text{otherwise}
\end{aligned}
$$

$$
\begin{aligned}
Z_4 &= (N-4)^2\,(P-4)^2 = Z_1Z_2 && N<4 \text{ and } P<4 \\
&= 0 && \text{otherwise} \\
Z_5 &= (P-2)^2 && P<2 \\
&= 0 && \text{otherwise} \\
Z_6 &= (P-1)^2 && P<1 \\
&= 0 && \text{otherwise} \\
Z_7 &= (P-4)^3 && P<4 \\
&= 0 && \text{otherwise}
\end{aligned}
$$

were constructed. The first four of these are given for the twenty-five treatments in Table 1.

Below are three equations estimated and the square root function as reported in [6]. We give the residual mean square (R.M.S.) on a per observation basis for each.

$$
\text{(i)}\quad y = \underset{(1\cdot7)}{126\cdot6} - \underset{(0\cdot23)}{6\cdot22Z_1} - \underset{(0\cdot50)}{4\cdot60Z_2} - \underset{(6\cdot7)}{14\cdot7Z_3} + \underset{(0\cdot42)}{1\cdot28Z_4} - \underset{(7\cdot8)}{35\cdot3Z_6} \qquad \text{R.M.S.} = 155
$$

$$
\text{(ii)}\quad y = \underset{(1\cdot8)}{126\cdot4} - \underset{(0\cdot24)}{6\cdot21Z_1} - \underset{(0\cdot80)}{3\cdot70Z_2} - \underset{(7\cdot0)}{15\cdot1Z_3} + \underset{(0\cdot44)}{1\cdot31Z_4} - \underset{(3\cdot2)}{12\cdot2Z_5} \qquad \text{R.M.S.} = 168
$$

$$
\text{(iii)}\quad y = \underset{(1\cdot8)}{126\cdot4} - \underset{(0\cdot25)}{6\cdot21Z_1} - \underset{(1\cdot70)}{0\cdot94Z_2} - \underset{(7\cdot0)}{16\cdot3Z_3} + \underset{(0\cdot45)}{1\cdot38Z_4} - \underset{(0\cdot43)}{1\cdot45Z_7} \qquad \text{R.M.S.} = 177
$$

$$
\text{(iv)}\quad y = -5\cdot682 - 0\cdot316N - 0\cdot417P + 6\cdot351\sqrt{N} + 8\cdot516\sqrt{P} + 0\cdot341\sqrt{NP} \qquad \text{R.M.S.} = 215
$$

The square root function is given in the original units as reported in [6] and the other equations are in the coded 40 pound units. The predicted values computed with equations (i) and (iv) are given in Table 1.

The analysis of variance presented in [6] gives the error mean square as 156. The F test comparing the residual mean square for any of the four equations to the error mean square would be nonsignificant. A more interesting comparison of the equations is obtained by considering the change in residual sum of squares associated with the addition of the variables unique to one equation to the other equation and vice versa.

To compare, for example, equation (i) and (ii), we add the variables occurring in (i) but not in (ii) and test the sum of squares associated with this addition against error. The only variable in equation (i) but not (ii) is Z_6. The resulting F with 1 and 57 degrees of freedom is 812/156. If we add the variable Z_5 occurring in (ii) but not in (i) to equation (i), the resulting F is 146/156. On this basis, we would conclude that equation (i) is significantly superior to equation (ii). If we add the variables in the square-root function to equation (i), the F with 5 and 57 degrees of freedom is 194/156, which is nonsignificant. If we add the variables in equation (i) to the square-root function, the F with 5 and 57 degrees of freedom is 799/156 which is highly significant, and we conclude that equation (i) is a superior representation of the data. One might well point out that the data were used to estimate the join points of the grafted quadratic. Thus, perhaps we should consider the model to contain nine parameters rather than five (the five regression coefficients, plus the four join lines). This would certainly reduce the level of significance of the F test; but in our case, if we charged ourselves for four additional parameters, we would obtain a ratio of 444/156 which is highly significant if compared with the tabular value of F for 9 and 57 degrees of freedom. This is obviously being overly harsh on equation (i) since we did not choose the join lines to obtain the absolute minimum sum of squares. Additional testing indicates that equation (i) is significantly superior to all other equations. It is interesting that the grafted quadratic is considerably superior to the cubic in approximating the response to P.

TABLE 1

Fertilizer Experiment Data

Nitrogen*	Phosphorus*	z_1	z_2	z_3	z_4	Corn Yield**	Predicted yield grafted quadratic	Predicted yield square root**
0	0	16	16	16	256	15·4	11·0	−5·7
0	2	16	4	4	64	26·4	31·9	37·1
0	4	16	0	0	0	23·0	27·1	35·3
0	6	16	0	0	0	36·5	27·1	26·1
0	8	16	0	0	0	11·6	27·1	13·1
2	0	4	16	4	64	17·6	16·1	25·8
2	2	4	4	0	16	107·4	103·8	95·9
2	4	4	0	0	0	105·4	101·8	105·4
2	6	4	0	0	0	102·5	101·8	104·8
2	8	4	0	0	0	100·2	101·8	99·2
4	0	0	16	0	0	12·2	17·8	24·0
4	2	0	4	0	0	94·2	108·2	105·4
4	4	0	0	0	0	123·0	126·6	119·6
4	6	0	0	0	0	126·7	126·6	122·6
4	8	0	0	0	0	122·8	126·6	120·0
6	0	0	16	0	0	11·5	17·8	16·8
6	2	0	4	0	0	119·0	108·2	106·9
6	4	0	0	0	0	127·4	126·6	124·6
6	6	0	0	0	0	117·6	126·6	130·4
6	8	0	0	0	0	133·4	126·6	130·1
8	0	0	16	0	0	22·0	17·8	6·8
8	2	0	4	0	0	105·5	108·2	104·1
8	4	0	0	0	0	129·2	126·6	124·9
8	6	0	0	0	0	137·9	126·6	133·0
8	8	0	0	0	0	123·4	126·6	134·7

* In 40 pound units of N and P_2O_5
** Taken from [6, p. 305]

Example Two

We shall fit a trend line to the adjusted corn yield data of Shaw and Durost [11, p. 97]. We somewhat arbitrarily assume that, for a segment of 8 years, the trend is adequately approximated by a quadratic. We approximate the trend for the last 2 years by a straight line used for extrapolation. The trend is required to have continuous first derivatives. Accordingly, the variables

$$
\begin{aligned}
Z_{1t} &= (t - 1961)^2 && t < 1961 \\
&= 0 && \text{otherwise} \\
Z_{2t} &= (t - 1952)^2 && t < 1952 \\
&= 0 && \text{otherwise} \\
Z_{3t} &= (t - 1944)^2 && t < 1944 \\
&= 0 && \text{otherwise} \\
Z_{4t} &= (t - 1936)^2 && t < 1936 \\
&= 0 && \text{otherwise}
\end{aligned}
$$

are constructed.

These were transformed as suggested and

$$
\begin{aligned}
X_{1t} &= Z_{1t} - 3Z_{2t} + 3Z_{3t} - Z_{4t} \\
X_{2t} &= Z_{2t} - 3Z_{3t} + 3Z_{4t} \\
X_{3t} &= Z_{3t} - 3Z_{4t} \\
X_{4t} &= Z_{4t}
\end{aligned}
$$

together with the variable $(t - 1962)$ were used in the regression. The resulting estimated equation is

$$
\begin{aligned}
y_t = \underset{(1{\cdot}7)}{74{\cdot}2} + \underset{(0{\cdot}45)}{3{\cdot}18}\,(t - 1962) + \underset{(0{\cdot}04)}{0{\cdot}20}\,X_{1t} + \underset{(0{\cdot}06)}{0{\cdot}33}\,X_{2t} + \\
\underset{(0{\cdot}10)}{0{\cdot}42}\,X_{3t} + \underset{(0{\cdot}12)}{0{\cdot}57}\,X_{4t}.
\end{aligned}
$$

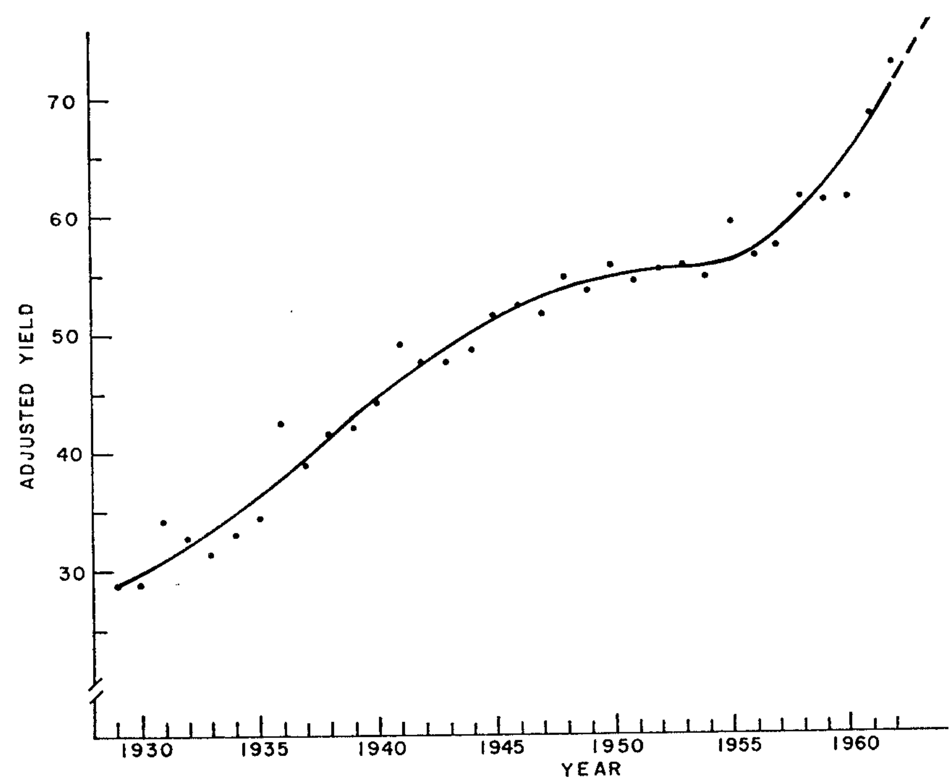

FIG. 4—Estimated Trend in Adjusted Corn Yields 1929-62 (Source [11; p. 97]).

When the variables are coded in this manner, we see that the constant term is the forecasted yield for the next observation, in our case 1962. The forecast for additional observations is given by

$$y_t = 74{\cdot}2 + 3{\cdot}18\ (t - 1962).$$

The data and the fitted trend line are plotted in Figure 4.

Determining the Join Points

In our discussion of estimation procedure, we have considered the domains of our curves to be established from sources other than the data. Yet, in our fertilizer example, the data are used in some degree to subdivide the domain. In our examples, we did not attempt to find those lines that would have minimized the residual sum of squares. It is our belief that, in a great many situations, one may quite readily establish the approximate area of the join line. One may then use an approximate solution as we did, or one may estimate the point (or line) by constructing several Z-functions and choosing the one (or combination) that gives the smallest residual sum of squares. The estimation of the join point is clearly a nonlinear problem. For discussion of the estimation of join points, see [7], [10] and [12].

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