THE MAXIMISATION OF REVENUE FROM NEW ZEALAND SALES OF BUTTER ON THE UNITED KINGDOM MARKET—A DYNAMIC PROGRAMMING PROBLEM

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Introduction

The objective of this paper is to consider and illustrate the potential usefulness of the method of analysis known as Dynamic Programming[1] in one field of market analysis. The illustrative problem solved is that of maximising revenue from New Zealand sales of butter on the United Kingdom market.

Since April 1962 the United Kingdom has restricted imports of butter by placing quotas for twelve month periods on importing countries. It is reasonable to assume that under normal supply conditions, these quotas will be fulfilled. Under abnormal conditions, such as those due to the winter experienced in the Northern Hemisphere in 1962-63, a good market information service should make it possible to predict the expected level of supply of butter from the countries concerned. United Kingdom producers are not subject to any restriction as to the quantity of butter sold on the home market, but the expected level of supply should be predictable in any year.

Statement of Problem

In this example we consider that the aim of the New Zealand butter marketing authority is to maximise revenue from butter sales in the United Kingdom over a twelve month quota period. The problem to be solved will be formulated as the selection of the monthly levels of New Zealand butter sales that maximise revenue for New Zealand from the butter quota allocated.

The problem will first be solved using Dynamic Programming. The disadvantages of this method of solution will then be discussed and a simpler solution to the problem, which holds for linear demand functions, will then be given.

Dynamic Programming Approach

Dynamic programming refers to sequential decisions in time or in space, or in general, to any n-dimensional problem that can be split into n one-dimensional problems. Selection of the monthly levels of butter sales that maximise New Zealand revenue over a twelve month period is a sequential or multi-stage allocation process. In this example the term 'stage' refers to time, i.e. one month. In the absence of dynamic

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[1] Richard Bellman, *Dynamic Programming*, Princeton University Press, Princeton, N.J., 1957.

programming we would be faced with one twelve-dimensional problem: deriving the level of butter sales in each month that maximises revenue from New Zealand sales of butter over a twelve month period.

The rationale behind dynamic programming is given by Bellman's principle of optimality[2]: "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." The state of a process at any particular stage describes the condition of the process at that stage. Decision making at a given stage controls[3] the state of the process in the following stage. For a problem to be non-trivial, in addition to the sequential nature of the process, the decision variables must be interdependent.

The decision variables in this example are the level of sales of New Zealand butter in each month. We will denote sales of New Zealand butter in the i-th month by v_i. Now we may write:

$$v_i = u_{i-1} + a_i - u_i \tag{1}$$

where u_i is the level of New Zealand butter in store at the end of the i-th month, and a_i is the quantity of New Zealand butter that has arrived in the United Kingdom during the i-th month. Sales of New Zealand butter in the i-th month therefore affect the possible level of sales in the $i + 1$-th month because, on rearranging (1), we have:

$$u_i = u_{i-1} + a_i - v_i$$

and,

$$v_{i+1} = u_i + a_{i+1} - u_{i+1}$$

therefore,

$$v_{i+1} = u_{i-1} + a_i - v_i + a_{i+1} - u_{i+1}$$

The New Zealand butter marketing authority is therefore clearly faced with the problem of making sequential decisions where the decision made in any one month affects the decisions that can be made in the following months. In this example it is convenient to consider that the level of arrivals of New Zealand butter in the United Kingdom are known[4]. We note then that the variable describing the state of any particular stage in this example, is the level of inventory at the beginning of that stage, while the decision variable is the level of inventory at the end of that stage.

Having established that marketing New Zealand butter in the United Kingdom may be considered as a sequential process where the decisions made in subsequent stages are interdependent, we now apply Bellman's principle of optimality to our example.

Bellman's principle of optimality leads to a recurrence relation connecting the members of the sequence in which we are interested. This recurrence relation may be conveniently summarised by use of the following notation:

Let $F_n(u_{i-1})$ = the n-stage return obtained starting from an initial state u_{i-1} and using an optimal policy.

[2] Richard Bellman *op. cit.*, p. 83.
[3] In this example only deterministic control will be considered.
[4] Any set of New Zealand butter arrivals could be used in the Dynamic Programming procedure to calculate optimum monthly *sales* of New Zealand butter.

and $f_i(v_i) = f_i(u_{i-1} + a_i - u_i)$
= the return from sales, v_i, in the i-th stage (month) of an n-stage process.

The fundamental recurrence relationship for this example may then be written:

$$F_n(u_{i-1}) = \underset{v_i}{\text{Max}} \left[f_i(v_i) + F_{n-1}(u_i) \right]$$

$$= \underset{u_i}{\text{Max}} \left[f_i(u_{i-1} + a_i - u_i) + F_{n-1}(u_i) \right] \quad (2)$$

for $n \geqslant 2$. The mathematical expression (2) provides a convenient summary of Bellman's principle of optimality. Thus we interpret:

$$F_3(u_9) = \underset{u_{10}}{\text{Max}} \left[f_{10}(u_9 + a_{10} - u_{10}) + F_2(u_{10}) \right]$$

in the context of our example, as the maximum return from the three stage process (of selling butter in the 10th, 11th and 12th months) when the level of inventory at the beginning of the 10th month is given by u_9.

To evaluate $F_3(u_9)$ we first note that as no profit can be made from inventory held at the end of the 12th month we may set $u_{12} = 0$. We could then calculate, (parametrically if necessary), the optimum level of inventory u_{11} for each inventory level u_{10} to obtain $F_2(u_{10})$. This information allows us to put a value on each level of u_{10} under an optimal policy in successive stages. We now consider the effect of the level of u_{10} on returns from the 10th month, given a fixed level of inventory u_9. By combining this information on the way the inventory level u_{10} affects returns in the 10th and successive months, (decisions as to inventory levels in these latter months are optimal), we may calculate the optimum level of u_{10} for each level of u_9, i.e. $F_3(u_9)$.

The information given by $F_3(u_9)$ then allows us, if we so desire, to evaluate each level of u_9, where decisions in successive stages are optimal. We can then select the level(s) of inventory, u_9^*, that allows maximum returns from the three month process. Given this *maximum maximorum* value of u_9 for these three months, we may calculate the optimising quantities u_{10}^* and u_{11}^* ($u_{12}^* = 0$). We may also, of course, use the information given by $F_3(u_9)$ together with the effect of the level of u_9 on returns from butter sales in the 9th month, to calculate the optimum inventory level u_9 for any given level of inventory u_8, i.e. $F_4(u_8)$.

Before applying Bellman's principle of optimality to the example of maximising returns from New Zealand sales of butter on the United Kingdom market, it is necessary to derive a revenue equation for the period of interest.

Market Relationships

In this example we will consider the period: June 1958→May 1959. The first requirement is a demand relationship at wholesale prices. A recent study by Candler and Townsley[5] was unable to show any entirely satisfactory wholesale demand relationship for New Zealand butter. However, a reasonably satisfactory regression estimate of United

[5] "A Study of the Demand for Butter in the United Kingdom", Wilfred Candler and Robert Townsley, *The Australian Journal of Agricultural Economics*. Vol. 6, No. 2, p. 36, December, 1962.

Kingdom retail demand for butter[6] for the period 8 June 1958 to 1 November 1959 was given by:

$$\hat{Y} = 169 - 1{\cdot}38X, \quad r^2 = {\cdot}88, \quad d = 0{\cdot}28 \tag{3}$$

where Y is the C.E.C. "index of retail sales" of butter per week[7] and X is the retail price of New Zealand butter in pence per lb. For the years 1958 and 1959, an index of 100 for each of these years corresponds approximately to 6,900 and 7,600 tons per week respectively. In this example we will assume that an index of 100 corresponds to 7,000 tons per week over the period of interest, so that multiplication of the coefficients of (3) by 70 gives a prediction of butter consumed in tons per week thus:

$$\hat{Y}' = 11830 - 96{\cdot}60X \tag{4}$$

where Y' is total retail demand for butter per week in tons. The influence of the wholesale price of butter on the retail price has been shown by Candler and Townsley to be direct and immediate as illustrated by the regression equation[8]:

$$\hat{X} = 1{\cdot}19 + {\cdot}12Z; \quad r^2 = {\cdot}99 \tag{5}$$

where X is the retail price in pence per lb. for New Zealand butter in "Multiple" stores, and Z is the wholesale price of butter in the *previous week* in shillings per hundredweight (or pounds per ton). Now if we assume that this week's retail sales were last week's wholesale sales[9] we may substitute for X in equation (4) as given by equation (5), to obtain:

$$\hat{Y}' = 11830 - 96{\cdot}60(1{\cdot}19 + {\cdot}12Z)$$

$$\therefore \hat{Y}' = 11715{\cdot}046 - 11{\cdot}592Z \tag{6}$$

where Y' is now total weekly butter sales at wholesale and Z is wholesale price.

Now rearranging (6) to obtain price as a function of wholesale sales per week we have:

$$Z = 1010{\cdot}6150 - {\cdot}0862664Y' \tag{7}$$

It will be convenient to consider periods of equal length. In our example then, we have 12 months, each $\frac{52}{12}$ weeks in length. Multiplying the $-{\cdot}0862664$ in (7) by $\frac{52}{12}$ we get:

$$Z_i' = 1010{\cdot}6150 - {\cdot}0199076q_i \tag{8}$$

where q_i is total sales of butter at wholesale in the i-th month, and Z'_i is the average wholesale price of butter in the i-th month.

[6] Candler and Townsley, *op. cit.,* found that retail demand for butter in the United Kingdom exhibited a hysteresis effect estimated by the three demand functions given in Table 3, p. 44. For exposition, only one of these demand functions has been used in the present study.

[7] This index has been published regularly from April 1959 in the C.E.C. *Intelligence Bulletin* and does not profess to show total butter sales in the United Kingdom as it refers to sales of a limited number of firms. However, this index provides the only available indicator of total retail sales of butter.

[8] Candler and Townsley, *op. cit.,* p. 43.

[9] This assumption may be very weak, but is necessary because of the paucity of information on actual retail and wholesale sales of all butter. This assumption also ignores speculative purchases which, together with cross-elasticities of demand for other types of butter, may be important in the solution of the problem.

Now let New Zealand sales in the i-th month $= v_i$, and let other sales of butter in the i-th month $= s_i$.

Then $q_i = v_i + s_i$.

Now revenue from New Zealand sales of butter in the i-th month, $f_i(v_i)$ is given by:

$$f_i(v_i) = \text{price } (Z_i) \times \text{quantity } (v_i)$$

Thus:

$$f_i(v_i) = (1010{\cdot}6150 - {\cdot}0199076(v_i + s_i))v_i$$
$$= 1010{\cdot}6150v_i - {\cdot}0199076v_i^2 - {\cdot}0199076v_is_i$$

If all quantties are measured in thousands of tons, and revenue is measured in hundreds of thousand pounds, we have:

$$f_i(v_i) = 10{\cdot}106150v_i - {\cdot}199076v_i^2 - {\cdot}199076v_is_i \qquad (9)$$

As before, let u_{i-1} equal the level of New Zealand butter stocks in cool stores at the end of the $i - 1$-th month (beginning of the i-th month). Similarly for u_i, u_{i+1}, etc. Also, let a_i equal the arrivals of New Zealand butter into the United Kingdom in the i-th month. Then sales of New Zealand butter in the i-th month, v_i, are given by:

$$v_i = u_{i-1} + a_i - u_i \qquad (10)$$

Substituting for v_i as given by (10) in the revenue equation (9), and expanding, we have:

$$f_i(v_i) = (10{\cdot}106150 - {\cdot}199076s_i - {\cdot}398152a_i)u_{i-1}$$
$$- (10{\cdot}106150 - {\cdot}199076s_i - {\cdot}398152a_i)u_i$$
$$+ {\cdot}398152u_{i-1}u_i - {\cdot}199076u_{i-1}^2 - {\cdot}199076u_i^2$$
$$- {\cdot}199076a_i^2 - {\cdot}199076s_ia_i + 10{\cdot}106150a_i \qquad (11)$$

It will be noted that no allowance for storage costs has been made in (11). In practice there is no difficulty in incorporating costs of this sort, given the relevant information. For the moment, it is assumed that storage costs are negligible.

Table 1 sets out information on sales of butter and arrivals of New Zealand butter in the United Kingdom over the period June 1958 to May 1959. Total sales of New Zealand butter recorded for the 12 months were 193·54 thousand tons, while total sales of other butter were estimated at 254·59 thousand tons. For this illustrative example we will consider these quantities to be New Zealand's quota and the total of expected sales by other countries, respectively.

Dynamic Programming Solution

For the 12 month period June 1958 to May 1959 total sales of New Zealand butter were 193·54 thousand tons, while total arrivals of New Zealand butter over this period were 161·80 thousand tons. Thus stocks of New Zealand butter in store at the beginning of this 12 month period must have totalled at least 193·54 — 161·80 = 31·74 thousand tons.

The problem to be solved, then, is the selection of the level of New Zealand butter sales in each month, $v_i (i = 1, \ldots, 12)$, that maximises total revenue, $\sum_{i=1}^{12} f_i(v_i)$, subject to certain restrictions:

$$v_i \geqslant 0 \quad (i = 1, 2, \ldots, 12)$$
$$u_i \geqslant 0 \quad (i = 0, 1, \ldots, 12)$$
$$u_0 - u_{12} = 31{\cdot}74 \text{ thousand tons,} \qquad (12)$$

TABLE 1

Butter Sales and Arrivals in the United Kingdom (1958-59)

Month	q_i Total Sales (a)	v_i N.Z. Sales (b)	s_i Other Sales (c)	a_i N.Z. Arrivals (d)
	('000 tons)			
1958				
June	40·95	15·80	25·15	9·00
July	41·88	18·40	23·48	9·60
August	40·79	15·67	25·12	13·10
September	39·52	14·49	25·03	14·20
October	40·87	19·00	21·87	11·60
November	38·34	22·14	16·21	15·30
December	36·45	16·47	19·99	16·40
1959				
January	33·38	12·07	21·31	12·30
February	32·38	11·17	21·21	14·50
March	35·73	16·41	19·32	18·70
April	34·24	16·60	17·64	17·50
May	33·59	15·32	18·27	9·60
Totals	448·13	193·54	254·59	161·80

Sources: (a) C.E.C. Monthly *Intelligence Bulletins.* Index of retail sales, 100 = 7000 tons/week.
(b) New Zealand Dairy Products Marketing Commission, *Annual Report* 1959.
(c) By Subtraction: $s_i = q_i - v_i$.
(d) C.E.C. Monthly *Intelligence Bulletins.*

and given the schedules of New Zealand butter arrivals and sales of other butter, as given in Table 1.[10] From the above restrictions we may conveniently set $u_{12} = 0$, and $u_0 = 31{\cdot}74$.

Because we have assumed that the New Zealand marketing authorities wish to fulfil their "quota" of 193·54 thousand tons of butter, and hence (12) must hold, we have been able to choose values for both u_{12} and u_0. In this situation the Dynamic Programming solution to be illustrated, may be started at either end of the time period in question. We will follow the usual practice of first considering the last stage and working sequential towards the first stage (month).

Method of Application

12th Month: Consider, now, income from New Zealand sales of butter in the 12th month. From Table 1 we have:

$$a_{12} = 9{\cdot}60$$
$$s_{12} = 18{\cdot}27$$

In this case we also have: $u_{12} = 0$.

[10] We could solve this particular problem by the classical calculus approach using Lagrangian multipliers. However, in more complicated examples the method of dynamic programming has considerable advantage. For a short discussion on this point see: *Operations Research—Methods and Problems,* by M. Sasieni, A. Yaspan and L. Friedman; John Wiley and Sons, Inc., N.Y., 1959, p. 274.

Now, from equation (11) we have:

$$\begin{aligned} f_{12}(v_{12}) = &(10{\cdot}106150 - {\cdot}199076s_{12} - {\cdot}398152a_{12})u_{11} \\ &- (10{\cdot}106150 - {\cdot}199076s_{12} - {\cdot}398152a_{12})u_{12} \\ &+ {\cdot}398152u_{11}u_{12} - {\cdot}199076u_{11}^2 - {\cdot}199076u_{12}^2 \\ &- {\cdot}199076a_{12}^2 - {\cdot}199076s_{12}a_{12} + 10{\cdot}106150a_{12} \end{aligned}$$

Substituting the values for a_{12}, s_{12} and u_{12} given above, and simplifying, we have:

$$f_{12}(v_{12}) = 43{\cdot}75586 + 2{\cdot}646770u_{11} - {\cdot}199076u_{11}^2$$

This expresses the revenue in the last month, as a simple quadratic function of stocks, u_{11}, at the end of the penultimate month.

11th Month: From Table 1 we have: $a_{11} = 17{\cdot}50$
$s_{11} = 17{\cdot}64$

Substituting these values in equation (11) and simplifying we have for revenue from sales of New Zealand butter in the 11th month:

$$\begin{aligned} f_{11}(v_{11}) = &54{\cdot}43584 - {\cdot}373211u_{10} + {\cdot}373211u_{11} \\ &+ {\cdot}398152u_{10}u_{11} - {\cdot}199076u_{10}^2 - {\cdot}199076u_{11}^2 \end{aligned}$$

Now total revenue from the 11th and 12 months is:

$$\begin{aligned} \sum_{i=11}^{12} f_i(v_i) &= f_{11}(v_{11}) + f_{12}(v_{12}) \\ &= 98{\cdot}19170 - {\cdot}373211u_{10} + 3{\cdot}019981u_{11} \\ &\quad - {\cdot}199076u_{10}^2 - {\cdot}398152u_{11}^2 \\ &\quad + {\cdot}398152u_{10}u_{11} \end{aligned} \tag{13}$$

We now apply Bellman's principle of optimality to obtain $F_2(u_{10})$, the revenue from two stages (the 11th and 12th months) starting from an initial state u_{10}, and using an optimal policy. The level of u_{11} which maximises the expression for revenue from the 11th and 12th month, $\sum_{i=11}^{12} f_i(v_i)$, for any fixed level of u_{10} is obtained from:

$$\frac{d\left[\sum_{i=11}^{12} f_i(v_i)\right]}{du_{11}} = 3{\cdot}019981 - {\cdot}796304u_{11} + {\cdot}398152u_{10} = 0$$

i.e. $u_{11}^* = 3{\cdot}79250 + {\cdot}5u_{10}$, (14)

as the second derivative is negative.

Substituting this optimising value for $u_{11}(= u_{11}^*)$ in (13) we therefore obtain $F_2(u_{10})$.

$$\begin{aligned} \text{i.e. } F_2(u_{10}) &= \max_{u_{11}} \sum_{i=11}^{12} f_i(v_i) \\ &= 98{\cdot}19170 - {\cdot}373211u_{10} + 3{\cdot}019981(3{\cdot}79250 + 5u_{10}) \\ &\quad - {\cdot}199076u_{10}^2 - {\cdot}398152(3{\cdot}79250 + {\cdot}5u_{10})^2 \\ &\quad + {\cdot}398152u_{10}(3{\cdot}79250 + {\cdot}5u_{10}) \\ &= 103{\cdot}91834 + 1{\cdot}136779u_{10} - {\cdot}099538u_{10}^2 \end{aligned}$$

We have now obtained the maximum feasible revenue in the last two months, as a function of stocks at the end of the tenth month.

10th Month: From Table 1 we have: $a_{10} = 18{\cdot}70$
$s_{10} = 19{\cdot}32$

Substituting these values in equation (11) and simplifying we have for revenue from sales of New Zealand butter in the 10th month:

$$\begin{aligned} f_{10}(v_{10}) &= f_{10}(u_9 + a_{10} - u_{10}) \\ &= 47{\cdot}44714 - 1{\cdot}185441u_9 + 1{\cdot}185441u_{10} \\ &\quad + {\cdot}398152u_9u_{10} - {\cdot}199076u_9^2 - {\cdot}199076u_{10}^2 \end{aligned}$$

Now from (2) we may write:

$$F_3(u_9) = \underset{u_{10}}{\text{Max}}\left[f_{10}(u_9 + a_{10} - u_{10}) + F_2(u_{10})\right]$$

for the revenue from the three stages (10th, 11th and 12th months) starting from an initial state u_9 and using an optimal policy. Substituting for f_{10} $(u_9 + a_{10} - u_{10})$ and $F_2(u_{10})$ in this expression we obtain:

$$\begin{aligned} F_3(u_9) &= \underset{u_{10}}{\text{Max}}\Big[47{\cdot}44714 - 1{\cdot}185441u_9 + 1{\cdot}185441u_{10} \\ &\quad + {\cdot}398152u_9u_{10} - {\cdot}199076u_9^2 \\ &\quad - {\cdot}199076u_{10}^2 + 103{\cdot}91834 + 1{\cdot}136779u_{10} \\ &\quad - {\cdot}099538u_{10}^2\Big] \\ &= \underset{u_{10}}{\text{Max}}\Big[151{\cdot}36548 - 1{\cdot}185441u_9 + 2{\cdot}322220u_{10} \\ &\quad - {\cdot}199076u_9^2 - {\cdot}298614u_{10}^2 + {\cdot}398152u_9u_{10}\Big] \end{aligned} \tag{15}$$

Maximising (15) with respect to the level of New Zealand stocks of butter at the end of the 10th month, u_{10}, for any fixed level of u_9, we differentiate the expression within the square brackets with respect to u_{10}, set the differential equal to zero, and solve for u_{10}, (where the second derivative is negative). Doing this for (15) we obtain:

$$\frac{dF_3(u_9)}{du_{10}} = 2{\cdot}322220 - {\cdot}597228u_{10} + {\cdot}398152u_9 = 0$$

which gives:

$$u_{10}^* = 3{\cdot}88833 + {\cdot}666667u_9$$

This value for u_{10} $(= u_{10}^*)$ allows us to maximise revenue from sales of New Zealand butter in the 10th, 11th and 12th months for any given level of inventory at the end of the 9th month, u_9. Substituting u_{10}^* in (15) gives:

$$F_3(u_9) = 155{\cdot}88026 + {\cdot}362706u_9 - {\cdot}066358u_9^2$$

In this way the fundamental recurrence relationship (2) is repeated backwards month by month to obtain successively:

$$\begin{gathered} F_4(u_8) \\ F_5(u_7) \\ \vdots \\ F_{12}(u_0) \end{gathered}$$

To preserve some level of continuity, the calculations for the expressions $F_4(u_8)$ to $F_{11}(u_1)$ have been omitted. The analysis for the first month (stage) of the process follows.

1st Month: From Table 1 we have: $a_1 = 9{\cdot}00$
$s_1 = 25{\cdot}15$
$u_0 = 31{\cdot}74$

By repeating the above process we eventually obtain:

$$F_{11}(u_1) = 491{\cdot}96387 + {\cdot}422910u_1 - {\cdot}018098u_1^2 \qquad (16)$$

Now as before we may write:

$$F_{12}(u_0) = \underset{u_1}{\text{Max}}\Big[f_1(u_0 + a_1 - u_1) + F_{11}(u_1)\Big] \qquad (17)$$

for the revenue from twelve stages (1st to 12th months inclusive) starting from an initial state u_0 and using an optimal policy. Now with total New Zealand sales in the 12 month period fixed at 193·54 thousand tons, and with $u_{12} = 0$, we know from (12) that the initial state variable $u_0 = 31{\cdot}74$ thousand tons. Substituting the above values for a_1, s_1, and u_0 in equation (11) we obtain:

$$\begin{aligned} f_1(v_1) &= f_1(u_0 + a_1 - u_1) \\ &= -122{\cdot}66681 + 11{\cdot}121323u_1 - {\cdot}199076u_1^2 \end{aligned}$$

Substituting for $f_1(v_1)$ and $F_{11}(u_1)$ in (17) we obtain:

$$\begin{aligned} F_{12}(u_0 = 31{\cdot}74) = \underset{u_1}{\text{Max}}\Big[&- 122{\cdot}66681 + 11{\cdot}121323u_1 \\ &- {\cdot}199076u_1^2 + 491{\cdot}96387 \\ &+ {\cdot}422910u_1 - {\cdot}018098u_1^2\Big] \end{aligned}$$

Maximising this expression for u_1 we have:

$$\frac{dF_{12}(u_0 = 31{\cdot}74)}{du_1} = 11{\cdot}544233 - {\cdot}434348u_1 = 0$$

therefore,

$$u_1^* = 26{\cdot}57830$$

Substituting for u_1^* in the equation for $F_{12}(u_0 = 31{\cdot}74)$ we obtain:

$$\begin{aligned} F_{12}(u_0 = 31{\cdot}74) &= 522{\cdot}71 \\ &= £52{\cdot}271 \text{ million} \end{aligned}$$

The optimum inventory level at the end of the first month, u_1^*, allows us to maximise revenue from sales of New Zealand butter over the 12 month period where the level of inventory at the beginning of this period, u_0, equals 31·74 thousand tons.

The values u_2^*, u_3^*, . . ., u_{11}^*, that maximise revenue are calculated successively as illustrated for u_{11}^*, equation (14), in the body of the text. From the relationship $v_i = u_{i-1} + a_i - u_i$ we may then calculate the quantities of butter that should be sold each month. This information is summarised in Table 2.

Dynamic Programming Solution—Discussion

Utilising the monthly revenue function (9), and the information on actual New Zealand and other sales of butter recorded in Table 1, total revenue from New Zealand butter can be estimated as £50·485 million.

The difference between estimated maximum revenue and estimated actual revenue from New Zealand butter sales over this 12 month period is then:

$$£52 \cdot 271 - £50 \cdot 485 = £1 \cdot 786 \text{ million.}$$

Summary of Dynamic Programming Solution

TABLE 2

Arrivals of N.Z. Butter (,000's tons)	Quantity in Store at End of Month (,000's tons)	Optimum Monthly Sales (given $u_0 = 31 \cdot 74$) (,000's tons)
	$u_0 = 31 \cdot 74$	
$a_1 = 9 \cdot 00$	$u_1^* = 26 \cdot 58$	$v_1 = 14 \cdot 16$
$a_2 = 9 \cdot 60$	$u_2^* = 21 \cdot 18$	$v_2 = 15 \cdot 00$
$a_3 = 13 \cdot 10$	$u_3^* = 20 \cdot 10$	$v_3 = 14 \cdot 18$
$a_4 = 14 \cdot 20$	$u_4^* = 20 \cdot 08$	$v_4 = 14 \cdot 22$
$a_5 = 11 \cdot 60$	$u_5^* = 15 \cdot 88$	$v_5 = 15 \cdot 80$
$a_6 = 15 \cdot 30$	$u^* = 12 \cdot 55$	$v_6 = 18 \cdot 63$
$a_7 = 16 \cdot 40$	$u_7^* = 12 \cdot 21$	$v_7 = 16 \cdot 74$
$a_8 = 12 \cdot 30$	$u_8^* = 8 \cdot 43$	$v_8 = 16 \cdot 08$
$a_9 = 14 \cdot 50$	$u_9^* = 6 \cdot 79$	$v_9 = 16 \cdot 13$
$a_{10} = 18 \cdot 70$	$u_{10}^* = 8 \cdot 42$	$v_{10} = 17 \cdot 08$
$a_{11} = 17 \cdot 50$	$u_{11}^* = 8 \cdot 00$	$v_{11} = 17 \cdot 92$
$a_{12} = 9 \cdot 60$	$u_{12}^* = 0 \cdot 00$	$v_{12} = 17 \cdot 60$

One aspect of this Dynamic Programming analysis needs further discussion. If arrivals of New Zealand butter were very unevenly distributed, or sales of other butter were very unevenly distributed, over the year, it is possible that the optimum values of one or more u_i will be less than zero. This means in effect that actual arrivals together with the quantity of butter initially in store, u_0, have been insufficient to maintain sales at the optimum level. The practical approach to this problem would be to ensure that arrivals did not become out of phase with the desired level of sales, after due consideration of expected levels of other sales. Where this solution is not possible the initial 12 month period of interest must be subdivided into smaller time periods. Thus, if u_4^* and u_7^* are both less than zero, the procedure is to set $u_7 = 0$ and maximise revenue over months 8 to 12 inclusive. If $u_i \geqslant 0 (i = 8, 9, \ldots, 12)$ we may then attempt to maximise revenue over months 1 to 7 inclusive, with $u_7 = 0$. In this way maximum *feasible* revenue over the 12 month period may be calculated.

The real practical difficulty that can be seen in the Dynamic Programming procedure for selecting the optimum level of New Zealand butter sales in each month, is the necessity to have a reasonable estimate of sales of other butter for some months in advance (twelve months in this example). If this information were available, Dynamic Programming could be utilised to organise optimum sales of New Zealand butter each month.

Under the quota system of butter marketing that exists at present, however, it is probably reasonable to assume that estimates of total sales of other butter for a twelve month quota period may be obtained. A simplified solution to the problem of maximising revenue from the New Zealand quota of butter sales over a 12 month period is now presented.

Simplified Solution

The principles involved in this simplified solution are the same as those involved in Dynamic Programming and most other maximisation techniques. Revenue from a given quantity of New Zealand butter will be maximised when the marginal revenue from New Zealand butter sales is equated betwen months.

That this condition does in fact hold may be checked using the Dynamic Programming solution already obtained. The marginal revenue in any month may be obtained by evaluating the first differential of equation (9):

$$\frac{df_i(v_i)}{dv_i} = 10{\cdot}106150 - {\cdot}398152v_i - {\cdot}199076s_i.$$

Using the optimum values for v_i obtained from the Dynamic Programming solution, and the values for s_i given in Table 1, the marginal revenue for New Zealand butter is found to be *minus* £54 per ton in each month.[11]

In this simplified solution we need make only the assumption that the total demand for butter at wholesale is of the linear form:

$$q_i = a - bp_i \tag{18}$$

where q_i are the total butter sales in the i-th month and p_i is the average price in the i-th month.[12]

Rearranging (18) we may write:

$$\begin{aligned} p_i &= m - nq_i \\ &= m - n(v_i + s_i) \end{aligned}$$

where v_i are sales of New Zealand butter in the i-th month, and s_i are sales of other butter in the i-th month. Then revenue from New Zealand sales in the i-th month, $f_i(v_i)$, is given by:

$$f_i(v_i) = v_i\,[m - n(v_i + s_i)]$$

$$\text{i.e. } f_i(v_i) = mv_i - nv_i^2 - nv_is_i \tag{19}$$

Marginal revenue of New Zealand butter sales in any month may be obtained from (19) by evaluating:

$$\frac{df_i(v_i)}{dv_i} = m - 2nv_i - ns_i$$

In the case where New Zealand revenue from a given total quantity of butter *sold,* is to be maximised over a 12 month period, marginal revenue is equated between months.

Thus we have:

$$\frac{df_i(v_i)}{dv_i} = K, (i = 1, 2, \ldots, 12),$$

and therefore:

$$m - 2nv_i - ns_i = K, (i = 1, 2, \ldots, 12). \tag{20}$$

[11] Where revenue may be expressed as a quadratic function, with stationary point a maximum, total revenue is maximised where marginal revenue is zero. In this example then New Zealand's revenue could be increased by selling less than the allocated quota. The *maximum maximorum* solution can be found using Dynamic Programming, with the alteration that no assumption is made as to the total quantity of New Zealand butter sold. Thus, the value of u_0 that maximises the expression $F_{12}(u_0)$ is calculated as an additional step to the procedure outlined.

[12] This analysis could just as easily be made on a weekly, fortnightly, etc., basis.

Evaluating (20) for each month and adding we obtain:

$$\sum_{i=1}^{12} \frac{df_i(v_i)}{dv_i} = 12m - 2n \sum_{i=1}^{12} v_i - n \sum_{i=1}^{12} s_i = 12K,$$

and dividing by 12 we have:

$$m - 2n\bar{v}_i - n\bar{s}_i = K \qquad (21)$$

where $\bar{v}_i$ = average New Zealand monthly butter sales, and $\bar{s}_i$ = average other monthly butter sales.

Now equating (20) and (21) we have:

$$m - 2nv_i - ns_i = m - 2n\bar{v}_i - n\bar{s}_i$$
$$\therefore 2nv_i = 2n\bar{v}_i + n(\bar{s}_i - s_i)$$
$$\therefore v_i = \bar{v}_i + \cdot 5(\bar{s}_i - s_i) \qquad (22)$$

Application of equation (22) to give optimum sales of New Zealand butter in the i-th month, is considerably simpler than the Dynamic Programming analysis, and only requires estimates of total sales of New Zealand and other butter during the overall period in which we wish to maximise New Zealand revenue, and an estimate of other sales in the ith month.

Summary

This paper illustrates the mechanism of Dynamic Programming in one context of marketing analysis. The method is extremely flexible but its usefulness could suffer because of the amount of information required about future levels of competitor's sales and the wholesale demand function. In addition, a simplified solution to the problem of maximising revenue from New Zealand's quota of butter sales on the United Kingdom market, that requires considerably less information, is given.