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ESTIMATING AN UPPER BOUND ON THE PRATT RISK AVERSION COEFFICIENT WHEN THE UTILITY FUNCTION IS UNKNOWN*

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The use of stochastic efficiency techniques frequently requires knowledge of a risk aversion coefficient (RAC). For example, use of mean-variance programming models (for example, Freund 1956) or stochastic dominance with respect to a function (Meyer 1977) usually requires knowledge of either specific RACs or at least a range of values. When investigators possess reliable estimates of decision makers' utility functions, as well as data on wealth, they can use the Pratt RAC definition $[-u''(X)/u'(X)]$ where $u(X)$ is the utility function at wealth level X. However, utility function estimates are not always available, can be expensive to obtain, are personalistic (thus, not necessarily applicable to groups of decision makers) and are sometimes of questionable reliability (for example, Whittaker and Winter 1980).

In a situation where the resolution of risky choices is desirable and appropriate utility functions are unknown, incomplete rankings are often all that is possible without assumptions on RACs. Analysts facing this quandary have utilised RACs from other studies [for example, Holt and Brandt (1985) used the RACs from Kramer and Pope (1981)]. Such a procedure is questionable since individual characteristics influencing utility functions, the dispersion of the risky prospect, and wealth levels would change between studies. Furthermore, RACs have been used which have been too large (Grube 1986), and this can cause numerical difficulties [for example, in the programme implementing Meyer's (1977) procedure, numerical overflows often occur because of large RACs, while expected value-variance models do not utilise all of their resources] or yield an excessively large non-dominated set. Thus, needs arise for inferring the magnitude of the RAC when the appropriate utility function is not available. This paper provides formulae which satisfy these needs.

RACs are needed either to aid in developing the 'best' risky prospects or to resolve choices among risky prospects. RACs aid in developing 'best' risky prospects when using techniques such as expected value-variance analysis where the RAC value is specified *a priori*. On the other hand, the RAC may be used *ex post* after the risky prospects have been fully described to identify the dominant set. In the *ex post* case, when dealing with known prospects, the RACs needed to rank these prospects can be developed utilising procedures such as those described by Grube (1986), McCarl (1988) or Hammond (1974). Thus, in this paper attention is restricted to identifying bounds for *a priori* specification of the RAC.

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A Review: The Pratt Risk Aversion Coefficient and the Risk Premium

Pratt (1964) defined a local RAC which measures risk aversion [$r(X)$] which is given by:

$$r(X) = -u''(X)/u'(X)$$

where $u(X)$ is the decision maker's utility function of wealth, X is the decision maker's level of wealth, $u'(X)$ is the first derivative of $u(X)$ with respect to X, and $u''(X)$ is the second derivative of $u(X)$ with respect to X.

Pratt (1964, p. 125) also developed a relationship between the risk premium, the variance of the risky prospect and $r(X)$ as:

$$\Pi(X, Y) = 0.5\sigma_Y^2 r(X) + o(\sigma_Y^2) \tag{1}$$

where $\Pi(X, Y)$ is the risk premium given a level of wealth X and a risky prospect Y, σ_Y^2 is the variance of the risky prospect, $r(X)$ is the RAC at level of wealth X and $o(\sigma_Y^2)$ are the higher order terms in the Taylor series expansion of the expected utility function around the mean of X. The risk premium is the amount that expected income differs from a certain income level with equal utility and is defined as:

$$u[X + E(Y) - \Pi(X, Y)] = E[u(X + Y)] \tag{2}$$

where $E(\cdot)$ is the expectation operator.

Inferring a Bound on the Risk Aversion Coefficient

Given $u(X)$, X and σ_Y^2, the above risk premium is a logical result of the equations. However, when $u(X)$ is unknown, there may be cases when the above equation may be manipulated to yield information on $r(X)$. This can be done by solving for $r(X)$ from equation (1), which yields

$$r(X) = 2[\Pi(X, Y) - o(\sigma_Y^2)]/\sigma_Y^2 \tag{3}$$

If, following Tsiang (1972) the dispersion of the risk prospect is assumed small relative to wealth, then the term $o(\sigma_Y^2)/\sigma_Y^2$ may be neglected.[1] Thus, $r(X)$ is approximately given by:

$$r(X) \cong 2\Pi(X, Y)/\sigma_Y^2 \tag{4}$$

Now suppose a bound on $r(X)$ is desired. An $r(X)$ bound can be developed by bounding $\Pi(X, Y)$ and manipulating. Three approaches may be taken to developing this bound:

(a) The bound may be established such that the certainty equivalent ignoring wealth [$E(Y) - \Pi(X, Y)$] is non-negative. (Equivalently, one restricts the risk premium to be no greater than the mean.)
(b) The bound may be established such that the risk premium is bounded above by a confidence interval.
(c) The bound may be established such that the risk premium does not exceed those found in applied MOTAD studies.

Each of these approaches is discussed below.

[1] The Tsiang (1972) assumption of small risk relative to wealth will be used herein so that all of the risk aversion coefficient discussion properly relates to Pratt's 'local' risk aversion coefficient. Major wealth changes will not be considered.

Non-negative Certainty Equivalent

Pursuing the first approach, the certainty equivalent ignoring wealth is given by:

$$CE = E(Y) - \Pi(X, Y) \tag{5}$$

In turn, if the certainty equivalent is non-negative, then

$$E(Y) \geq \Pi(X, Y) \text{ or} \tag{6}$$

$$E(Y) \geq 0.5\sigma_Y^2 r(X) + o(\sigma_Y^2) \tag{7}$$

and a bound on $r(X)$ is

$$r(X) \leq [2E(Y)/\sigma_Y^2] - [2o(\sigma_Y^2)/\sigma_Y^2] \tag{8}$$

Assuming the $o(\sigma_Y^2)/\sigma_Y^2$ is close to zero yields the bound:

$$r(X) \leq 2E(Y)/\sigma_Y^2 \tag{9}$$

This bound is equivalent to twice the inverse of the coefficient of variation divided by the standard deviation.

Confidence Interval

Yet a second bound may be derived based on results found using classical decision theory. Many such applications have found decision makers willing to take small risks in order to attain highly probable gains (Keeney and Raiffa 1976). In other words, decision makers have been found to be willing to risk unlikely values of the distribution. The likelihood of such items has been expressed in terms of a confidence interval utilising the standard deviation. An $r(X)$ bound may be derived by assuming that the number of standard deviations (D) in the confidence interval is related to the risk premium. Suppose a $D\sigma_Y$ confidence interval is established. In turn, the risk premium may be bounded as:

$$\Pi(X, Y) \leq D\sigma_Y \tag{10}$$

Substituting equation (1) into inequality (10) yields

$$\Pi(X, Y) = 0.5\sigma_Y^2 r(X) + o(\sigma_Y^2) \leq D\sigma_Y \tag{11}$$

Manipulating inequality (11) and again neglecting $o(\sigma_Y^2)$ yields a second bound:

$$r(X) \leq 2D/\sigma_Y \tag{12}$$

Note that this bound is tighter than the first when the inverse of the coefficient of variation exceeds D.[2]

MOTAD

The third approach to developing a bound comes from the numerous applied studies with MOTAD (Hazell 1971). In such studies, as discussed in Hazell (1971) or later in Brink and McCarl (1978), a transformation is applied such that the objective function can be interpreted as:

$$\max\ U_Y - \psi\sigma_Y \tag{13}$$

[2] Two related bounds have been derived. Tsiang (1972, p. 358) pursues an approach in the case of constant absolute risk aversion which results in the bound $r(X) < 1/\sigma_Y$ after manipulation. Paris (1979, p. 273) finds a bound of D/σ_Y.

where U_Y is the mean of the risky prospect, ψ is the risk aversion parameter and σ_Y is the standard deviation of the risky prospect. In this case, the term $\psi\sigma_Y$ is the risk premium. In applied studies as reviewed in Apland, McCarl and Miller (1980) or Hazell, Norton, Parthasarthy and Pomereda (1983), there have been estimates of ψ based on the best correspondence with observed data. The range of ψ values found has fallen between zero and 2.5. Using the maximum ψ usually reported (2.5) and bounding the risk premium yields

$$\Pi(X, Y) \leq 2.5\sigma_Y \tag{14}$$

and, as above, a bound on $r(X)$ arises where

$$r(X) \leq 5/\sigma_Y \tag{15}$$

Note that this bound is tighter than the first when the inverse of the coefficient of variation exceeds 2.5 and is tighter than the second when D exceeds 2.5.

Inferring a Magnitude for D in the Second Bound

The second approach described above yields bounds on $r(X)$ given a number of standard deviations to use for the confidence interval (D). Estimates of D are needed to use the bound. This term can be estimated based upon the interrelationship of the RAC with the risk premium under assumptions of normality or no distributional knowledge.

Inference assuming normality

Assuming the prospects are normally distributed, then the D coefficient is equivalent to a Z value in the standardised normal distribution. Suppose that one is willing to bound the risk premium to no more than two standard deviations.

$$\Pi(X, Y) \leq 2\sigma_Y \tag{16}$$

This corresponds to a situation where in making the decision one would compare the values of $E(Y) - 2\sigma_Y$ and choose the prospect with the greater one of these. In turn, the bound on $r(X)$ would be $4/\sigma_Y$. Generalising, suppose under the assumption of normality and a confidence interval with the probability of observations falling outside of it equal to $0 - \alpha$, where α is between zero and 1, then the D value could be set to the one-tailed value of Z_α arising from the standard normal table and the $r(X)$ coefficient would be given by:

$$r(X) = 2Z_\alpha/\sigma_Y \tag{17}$$

For example, for α no greater than 0.005, then $D = 2.57$ and the bound on the value of $r(X)$ would be[3]

$$0 \leq r(X) \leq 5.14/\sigma_Y \tag{18}$$

Developing bounds based on Chebyshev's inequality

A bound on $r(X)$ may be developed using a less restrictive distributional assumption. Chebyshev has shown that the probability of being further than D standard deviations away from the mean is less than or equal to

[3] This equation assumes risk aversion. One could, however, examine risk preference with the same analytical framework.

$1/D^2$. In turn, one can solve for a value of D such that the probability of being more than $D\sigma$ deviations below the mean is greater than or equal to α. This value of D is given by:

$$D = \alpha^{-0.5} \tag{19}$$

and the Pratt RAC would fall within the range

$$0 \leq r(X) \leq 2\alpha^{-0.5}/\sigma_Y \tag{20}$$

This again gives a bound for $r(X)$. Solving the equation for a probability level of 0.005 yields a value for $r(X)$ of 28 over the standard deviation of the risky prospect. Assuming reasonable decision makers would not wish to be more than 99.5 per cent certain in their risk premium level, then $28/\sigma_Y$ is another upper bound on $r(X)$. This is an extreme value for $r(X)$ because of the conservative nature of the Chebyshev inequality.

Use of the Formulae

The above relationships develop upper bounds on the Pratt RAC based on expected utility theory and empirical evidence accumulated using decision theory and MOTAD. The order of magnitude for the bounds, assuming risk aversion, is between zero and 28 over a relevant standard deviation of income ($5/\sigma_Y$ seems to be a more realistic bound). Naturally, the relevant standard deviation of income may not always be known *a priori*, and some *ex post* or iterative procedure may be required to develop the appropriate magnitudes for RACs. However, often one can develop an estimate of the relevant standard deviation of income by calculating the standard deviation of commonly used crop plans or the standard deviation of the risky prospects being compared. We recommend that the standard deviation be constructed so that it is approximately equal to that of a commonly followed plan (for example, the standard deviation of the plan currently being used).

Transforming RACs Between Studies

Analysts may feel uncomfortable with use of the above results, as the values derived are not based on decision makers' opinions or observed behaviour, but rather are based on assumptions. However, the formulae can be used to transform values of $r(X)$ from one study to another for comparative purposes.[4]

Suppose an analyst is willing to assume that decision makers act in such a manner that the risk premium is proportionally the same in two studies. Suppose that in study 1 values of $r_1(X)$ and σ_{Y1}^2 have been observed, and that the risk premium as a proportion of expected value (U_1) for the risky choice Y_1 is

$$\Pi(X, \sigma_{Y1}^2)/U_1 = 0.5 r_1(X) \sigma_{Y1}^2 / U_1 \tag{21}$$

Then, in a second study where the utility function and $r(X)$ are unknown, one could use the same proportion of the expected value for the risk premium. Setting the proportional risk premiums equal to the resultant $r(X)$ relevant to the second study gives

$$r_2(X) = [r_1(X)\sigma_{Y1}^2/\sigma_{Y2}^2]U_2/U_1 \tag{22}$$

[4] Related transformatio[illegible]h correct for the effect of different monetary units are derived in Raskin and C[illegible] (1986).

which would give an $r(X)$ value for use under the assumption that the risk premium is of the same percentage in both studies.

RAC Comparisons

The potential usefulness of the above formulae depends upon the extent to which inappropriate RACs have been and will be used as well as the degree of accuracy to which the maximum RAC is predicted. These issues are explored using data from eight previously published studies (Table 1). In the context of these studies three items are examined: the maximum RAC used in the study, the RAC bounds derived from the formulae and the maximum RAC required to rank any two prospects in the study. These comparisons will be used to address the magnitude and accuracy issues discussed above.

First, consider the degree of accuracy with which maximum RACs are predicted by the formulae defined in this study. This may be examined by comparing the resultant bounds with the largest RAC required to rank the alternatives. This largest RAC is derived following Hammond (1974) using the assumption of constant absolute risk aversion utility functions and the RISKROOT procedure of McCarl (1988). RISKROOT solves for all RACs at which the expected utility for two prospects is equal [Hammond (1974) and McCarl (1988) provide additional discussion]. The RACs found with the greatest absolute values are presented in Table 1. This comparison shows that the bounds from the formulae derived herein are greater than the maximum RACs required to rank (in some cases being as much as 20 times greater). Generally, the value from the formulae based on the confidence interval under normality and MOTAD was a tighter bound, but in the King and Robison (1984) case the certainty equivalent bound was less. It appears safe to use the smaller of these bounds as the upper bound.

TABLE 1

Comparison Between RAC Values

Previous study	Maximum RAC used in previous study	Inequality (9) bound[a]	Inequalities (12) & (15) bound [a]	Maximum $r(X)$ at which rankings occurred
Danok, McCarl and White (1980)	0.1	0.000123	0.000114	0.000037
Holt and Brandt (1985)	2.02	0.770	0.04	–[c]
King and Robison (1984)	0.00008	0.000123	0.0015	0.000181
Klemme (1985)	–[b]	0.150	0.925	0.0077
Kramer and Pope (1981)	0.03	0.000307	0.000264	0.000135
Lee, Brown and Lovejoy (1985)	–[b]	0.00113	0.0009	0.00003
Lemieux, Richardson and Nixon (1982)	0.003501	0.008781	0.000015	–[c]
Rister, Skees and Black (1984)	0.000063	0.000667	0.00008	–[c]

[a] Tabulated as the largest value from the bound formula over all the prospects considered using D equal to 2.5.
[b] The authors of the study being referenced used first- and second-degree stochastic dominance and therefore no maximum RAC was used.
[c] The data needed for this calculation were not available in the study being referenced.

The issue of whether the bounds might be useful or whether inappropriate RACs may have been used is examined by comparing the formula results with the maximum size of RAC used by the authors of the studies. Under this comparison, it can be noted that two of the studies used maximum risk aversion parameters which were more than 100 times larger than the RAC required to reduce the certainty equivalent below zero. Further, in two of the studies the risk aversion parameters were 20 to 200 times too small, suggesting that the authors' 'strongly risk averse' range might not be so.

Concluding Comments

The RAC bounds that have been derived are subject to an important caveat. It is normally expected that the risk aversion coefficient would be a function of wealth and not of the dispersion of the prospect.[5] This relationship has been neglected in this analysis. The appropriate risk aversion coefficient should fall within the bounds derived above, but the specific coefficient value should vary with other characteristics of the individual.

The above relationships can have important uses. First, one may use the results to devise upper bounds on $r(X)$ and, in turn, explore the interval between zero and the bound. This appears useful as, apparently, some previous studies have used too large or too small a range of $r(X)$ values. Second, the results can be used to translate $r(X)$ between studies. Papers such as Raskin and Cochran (1986), Wilson and Eidman (1983), Kramer and Pope (1981), and King and Robison (1984) report values of the risk aversion coefficient. Obviously, these are relevant in the source studies, but under most cases would not be directly relevant in other studies. However, some studies have directly and inappropriately used the risk aversion coefficients obtained in our study in entirely different contexts [see Holt and Brandt (1985) or several others as identified in Raskin and Cochran (1986)]. Clearly, the appropriate value of $r(X)$ changes between studies and should be elicited or formed as shown herein. Third, one could use the formula for D in relation to $r(X)$ [that is, $D = 0.5r(X)\sigma_Y$] and the normality assumption to interpret the $r(X)$ value in terms of a probability.

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[5] As one of the reviewers pointed out, a bound could be developed based upon wealth and Pratt's (1964) coefficient of local proportional risk aversion ignoring the dispersion of the risky prospects. This other Pratt coefficient has commonly been called the coefficient of relative risk aversion (R) where $R = Xr(X)$. An $r(X)$ bound would be derived by assuming a maximum value of R then dividing it by wealth X.

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