RESOURCE ALLOCATION AND A FITTED PRODUCTION FUNCTION

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I. *Introduction*

Over a considerable period of time, econometricians have been attempting to estimate the marginal productivities of resources by the use of regression analysis. The first use of the most commonly used functional form, the Cobb-Douglas or logarithmic function, was in an attempt to answer some questions concerning the distribution of the product of the economy between capital and labour, using time-series data.[1] Currently, regression methods of estimating marginal productivities have been largely confined to cross-sectional studies of agricultural or pastoral regions.[2]

There exists a considerable body of literature dealing with the statistical problems involved in fitting regression functions for this purpose.[3] There has also been discussion of the validity of the results from a function fitted to a sample of farms when applied to an individual farm. This present article has nothing to say on these problems, and will, in fact, assume that the regression coefficients in the functions fitted provide unbiased estimates of the population coefficients.[4]

However, there exists a quite different problem, largely untouched in the literature in relation to estimated marginal value products. This problem may be defined in a question: "Given that the production function has been estimated, how is an optimum allocation of resources defined, and how should resources be moved to obtain it?"

In the next section the most common method of interpretation of the

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[1] C. W. Cobb and P. H. Douglas, "A Theory of Production," *American Economic Review*, Vol. XVIII, Supplement (March, 1928), pp. 139-165.

[2] For a recent Australian example where a function is fitted to time-series, see G. O. Gutman, "Investment and Production in Australian Agriculture," *Review of Marketing and Agricultural Economics*, Vol. 23, No. 4 (December, 1955), pp. 237-310.

[3] See, for example, H. S. Konijn, "Estimation of an Average Production Function from Surveys," *The Economic Record*, Vol. XXXV, No. 70 (April, 1959), pp. 118-125. The literature has been reviewed by R. M. Parish and J. L. Dillon, "Recent Applications of the Production Function in Farm Management Research," *Review of Marketing and Agricultural Economics*, Vol. 23, No. 4 (December, 1955), pp. 215-236.

[4] This assumption, of course, implies the further assumption that such population coefficients in fact do exist; that is, that all farms in the sample are operating on the same production function.

estimated function will be described, and in subsequent sections an attempt will be made to provide a better approach to the interpretation, based on answering the question posed above.

II. *Marginal Productivities and Market Prices*

At some stage in all studies using Cobb-Douglas functions which the author has seen, the marginal value products of the resources entering the function have been computed at the geometric mean level of income of the sample, and at the geometric mean level of the particular resource being considered. That is, the M.V.P. of a particular resource is computed at its geometric mean, all other resources being held constant at their geometric means. Computationally, this is very simple. If we have a function

$$Y = c \prod_{i=1}^{n} X_i^{a_i} \qquad (1)$$

where Y is gross farm income in money terms and the X_i are the individual resources either in value or physical units, then the M.V.P. of the ith resource is simply

$$MVP_{X_i} = a_i \frac{M_Y}{M_{X_i}} \qquad (2)$$

where M_Y and M_{X_i} are the geometric means of income and of the ith resource, as defined above.

The usual practice is then to compare MVP_{X_i} with p_{X_i} the market price of the ith resource. Where $MVP_{X_i} > p_{X_i}$, it is suggested that the use of the ith resource should be expanded, and, conversely, contracted where $MVP_{X_i} < p_{X_i}$. This is done successively for all resources X_1, ..., X_n entering the function.

It is apparent that this procedure involves several special assumptions in addition to the assumptions regarding the function itself made in this paper.

The first of these special assumptions is that all resources are considered to be variable. This assumption underlies the examination of the relationship between the M.V.P. and the market price of each of the resources taken successively. Generally, however, authors go further than this, and suggest that, whereas the use of one resource, for example, should be expanded, the use of another should be contracted. A necessary condition for the validity of this type of recommendation is that the resource adjustments being considered are purely marginal. Generally, to bring resource use to an optimum, fairly large, and certainly non-marginal, adjustments are necessary. Summarising, we can state that if all resources are considered variable, and the changes in resource allocation required or implied are non-marginal, then, from a given function, recommendations can be made for one resource only, using the conventional approach to these recommendations. Where the above circumstances hold, and more than one resource is considered, then an error is introduced, as the marginal

productivity of any one resource is a function, not only of the level of that resource, but of the level also of all other resources in the function. It seems tedious to spell out this common knowledge in such detail, but, as will be seen in Section V, non-observance of these considerations can lead to wrong conclusions.

The second special assumption is that capital is not limited, i.e. that the conclusions usually drawn indicate movement towards maximising income, without capital restraints, by obtaining an equation of M.V.P. and market price of resources. The use of this assumption can be questioned on two grounds.

Firstly, the maximum income position can be defined only when the sum of the elasticities in the function is less than one, or, when the sum is greater than or equal to one, one or more of the resources are held constant, such that the sum of the elasticities of the variable resources is less than one. These considerations limiting the generality of the criterion of equality between M.V.P. and market price have by no means always been observed.

Secondly, capital-rationing, self-imposed to some extent, is generally recognised in agriculture. The extent of re-allocation of resources open to most farmers thus usually involves the re-allocation of existing resources, or at best, the use of a limited quantity of resources previously not used on the farm.[5] If this be the case, then the appropriate standard against which to compare observed marginal value products is the opportunity cost, rather than the market price.[6]

Some definitions need be clarified before proceeding to the more systematic treatment in the next Section. All variables will be considered to be in value terms, that is, each independent variable X_i (which in practice will be an aggregate of input categories) is the sum of the physical quantities of resources multiplied by the appropriate market prices and including any complementary costs associated with the use of resource aggregate X_i.[7] With this definition of X_i, the test generally used as described above of comparing the marginal value products of resources with their market prices now becomes a comparison of the marginal value product of the resources with unity.

In the next section, the movement towards the final optimum of all marginal value products equal to one will be considered in two steps.

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[5] For some empirical evidence on this point, see E. O. Heady and E. R. Swanson, *Resource Productivities in Iowa Farming with Special Reference to Uncertainty and Capital Use in Southern Iowa,* Research Bulletin 388, Agricultural Experiment Station, Iowa State College (June, 1952).

[6] In this paper, the opportunity cost of resource use will be computed from the function, assuming, initially, that all resources are readily transferable. The assumption that all resources can be expressed in terms of capital and thus allocated among various avenues of expenditure according to some profit maximisation criteria raises the question of the stickiness and lumpiness of many resources. These questions are discussed in G. L. Johnson, "Classification and Accounting Problems in Fitting Production Function to Farm Record and Survey Data," Chapter 9 in E. O. Heady *et al.* (eds.), *Resource Productivity, Returns to Scale, and Farm Size,* The Iowa State College Press, Ames, Iowa, 1956. In the main, these problems will be ignored in the present paper (see. however, Section IIIB below).

[7] This definition is adopted to simplify notation in succeeding sections. A slightly more complex notation would enable an examination of the effects of changes in the prices of resources or of capital.

The first condition for the optimum, the *internal condition* is satisfied when the marginal value products of all variable resources are equal to some number, p. Where $p \neq 1$ the internal condition only is satisfied. The *external condition* is satisfied when $p = 1$.

III. *The Internal Conditions for Equilibrium*

A. *Cobb-Douglas Function—All Resources Variable*

Given that there exists a production function

$$Y = c \prod_{i=1}^{n} X_i^{a_i} \qquad \text{(1)}$$

we shall define the conditions for internal equilibrium as

$$MVPx_i = p \qquad i = 1, \ldots, n \qquad \text{(2)}$$

where p is some number and $MVPx_i$ is the marginal value product of the ith resource. X_i^* is defined as the level of the ith resource when (2) holds.

In this section, we are concerned with the situation where capital is limited, and we wish to define the optimum use of the given total quantum of resources existing on an individual farm. At this stage, we shall assume that all resources are variable and in money terms as defined at the end of Section II.

Then T, the total quantum of resources, is simply

$$T = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_i^* \qquad \text{(3)}$$

Note that X_i^*, p, are functions of T, hence, instead of p, we write p_T as the equilibrium value of all $MVPx_i$, for $\sum_{i=1}^{n} X_{iT}^* = T$.

Then, (2) can be re-written as

$$\left.\frac{\delta Y}{\delta X_i}\right|_{X_{iT}^*} = p_T \quad i = 1, \ldots, n \qquad \text{(4)}$$

When this equality holds, the maximum gross income, Y_T^* is forthcoming.

$$\text{Then } p_T = a_i \frac{Y_T^*}{X_{iT}^*} \qquad i = 1, \ldots, n \qquad \text{(5)}$$

$$X_{iT}^* = a_i \frac{Y_T^*}{p_T} \qquad i = 1, \ldots, n \qquad \text{(6)}$$

Substitute (6) into (3),

$$T = \frac{Y_T^*}{p_T} \sum_{i=1}^{n} a_i \qquad \text{(7)}$$

and $$p_T = \frac{Y_T^*}{T} \Sigma a_i \qquad \text{(8)}$$

Substitute (8) into (6),

$$X_{iT}^* = \frac{a_i T}{\Sigma a_i} \quad i = 1, \ldots, n \qquad \text{(9)}$$

By comparing X_{iT}^* with X_i, the observed level of use of the ith resource, it is possible to recommend resource movements towards the optimum position defined for the existing level of resources.

IIIB. *Cobb-Douglas Function—with Some Fixed Resources*

In the short run, not all resources are variable.[8] The method of analysis developed above can readily be extended to include the situation where the task is to determine the optimum combination of a given quantum of resources among several resource categories, given that the level of one or more resources is fixed.

Consider that X_j, $j = 1, \ldots, m$ are resources which are fixed, and that X_i, $i = 1, \ldots, n$ are variable resources, and that there exists a function

$$Y = c \prod_{j=1}^{m} X_j^{a_j} \prod_{i=1}^{n} X_i^{a_i} \qquad \text{(1)}$$

Let T be the total level of all resources and T′ be the total level of variable resources.

Then,

$$\begin{aligned} T' &= T - \sum_{j=1}^{m} X_j, \\ &= \sum_{i=1}^{n} X_i \\ &= \sum_{i=1}^{n} X_{iT'}^* \qquad \text{(2)} \end{aligned}$$

As before

$$\left.\frac{\delta Y}{\delta X_i}\right|_{X_{iT'}^*} = p_{T'} \quad i = 1, \ldots, n \qquad \text{(3)}$$

[8] An example of such a resource is family labour. A rather more interesting example is land which can be considered as fixed, except in the extremely long period, by the way in which many samples are defined. Properties with non-contiguous land are frequently excluded from the sample, and thus, any increase of the farm area by the addition of non-contiguous land means that the farm is now operating on a different production function. Hence, the only increase in the land input generally considered is of contiguous land, which is generally not available except in the very long run.

In this case p_T' is a number equal to the marginal value products of the variable resources, X_i, at an equilibrium position the level of which is determined by equations (1) and (2).

$$p_T' = \frac{a_i \, Y_{T'}^*}{X_{iT'}} \qquad (4)$$

and

$$X_{iT'}^* = a_i \frac{Y_{T'}^*}{p_T'} \qquad (5)$$

Rewriting (2) by substituting (5), we have that,

$$T' = \frac{Y_{T'}^*}{p_T'} \Sigma a_i \qquad (6)$$

and $$p_T' = \frac{Y_{T'}^*}{T'} \Sigma a_i \qquad (7)$$

From (6) and (7)

$$X_{iT'}^* = \frac{a_i T'}{\Sigma a_i} \qquad i = 1, \ldots, n \qquad (8)$$

Note that $\frac{X_{iT'}^*}{T'}$, the proportion of the total expenditure on variable resources expended on the ith resource, is not a function of either the level or the combination of fixed resources. In practice one would not expect that this property of the Cobb-Douglas function would occur on farms where, for example, the proportions of variable costs represented by expenditure on labour and fuel would vary with the level of investment in machinery, etc.

IIIC. *The Transcendental Function*

The results from IIIA (and IIIB) are readily extended to the situation where the production function is of the form[9]

$$Y = c \prod_{i=1}^{n} X_i^{a_i} e^{b_i X_i} \qquad (1)$$

As in IIIA, let T be the given quantum of resources available on a particular farm, so that, as before,

$$T = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_{iT}^* \qquad (2)$$

As before, we define

$$p_T = \left. \frac{\delta Y}{\delta X_i} \right|_{X_{iT}^*} \qquad i = 1, \ldots, n \qquad (3)$$

[9] See A. N. Halter, H. O. Carter, and J. G. Hocking, "A Note on the Transcendental Production Function," *Journal of Farm Economics*, Vol. XXXIX, No. 4 (November, 1957), pp. 966-974.

Then,

$$p_T = Y^*_T \left\{ \frac{a_i}{X^*_{iT}} + b_i \right\} \qquad \ldots\ldots (4)$$

Y^*_T is defined as in the Cobb-Douglas case.
From (4),

$$X^*_{iT} = \frac{a_i Y^*_T}{p_T - Y^*_T b_i} \qquad \ldots\ldots (5)$$

Substituting (5) into (2) and re-arranging terms,

$$T = Y^*_T \frac{\sum_{i=1}^{n} a_i}{p_T - Y^*_T \sum_{i=1}^{n} b_i} \qquad \ldots\ldots (6)$$

From (6)

$$p_T = Y^*_T \left\{ \frac{\Sigma a_i}{T} + \Sigma b_i \right\} \qquad \ldots\ldots (7)$$

Substituting (7) into (5),

$$X^*_{iT} = \frac{a_i T}{\Sigma a_i + T(\Sigma b_i - b_i)} \qquad \ldots\ldots (8)$$

Note that when all the b_i are zero, the production function, (1), becomes a Cobb-Douglas function and the expression for X^*_{iT}, (8), simplifies to the expression for the Cobb-Douglas case.[10]

IV. *The External Condition for Equilibrium*

In this section it is assumed that the estimated function exhibits decreasing returns to scale, that is, in the Cobb-Douglas case, that the sum of the partial elasticities is less than one.

Unlimited Capital

As before, we have a production function of the form

$$Y = c \prod_{i=1}^{n} X_i^{a_i} \qquad \ldots\ldots (1)$$

We wish to maximise profits, that is, to maximise $Y - \Sigma X_i$, with no restraints upon the quantity of capital. It is assumed that the price of

[10] If b_i is positive, $Y = f(X_i)$ increases at an increasing rate if $a_i \geqslant 1$, or, if $0 < a < 1$, increases at a decreasing rate until $Y = \frac{-a_i + \sqrt{a_i}}{b_i}$, and thence at an increasing rate. See Halter, Carter and Hocking, *op. cit.* If these values of a_i and b_i are observed for one or more of the independent variables in the fitted function, then the method of analysis developed here may become indeterminate. This is not surprising, as we are postulating increasing marginal returns for at least one of the factors.

capital is not a function of the quantity of capital demanded by the individual firm.

To maximise profits, we have

$$\frac{\delta(Y - \Sigma X_i)}{\delta X_i} = 0 \quad \ldots\ldots (2)$$

i.e. $$\frac{a_i}{X_i} Y = 1 \quad \ldots\ldots (3)$$

From (3),

$$X_i = a_i Y \quad \ldots\ldots (4)$$

By taking the a_i^{th} power of both sides of (4), taking the products over $i = 1, \ldots\ldots, n$ and multiplying both sides by c, we have

$$c \prod_{i=1}^{n} a_i^{a_i} Y^{\Sigma a_i} = c \Pi X_i^{a_i}$$

$$= Y \quad \ldots\ldots (5)$$

Divide through by $Y^{\Sigma a_i}$ and let $\epsilon = 1 - \Sigma a_i$

$$c \Pi a_i^{a_i} = Y^{\epsilon} \quad \ldots\ldots (6)$$

$$Y = c^{\frac{1}{\epsilon}} \Pi a_i^{\frac{a_i}{\epsilon}} \quad \ldots\ldots (7)$$

Substituting from (4),

$$X_i = a_i c^{\frac{1}{\epsilon}} \Pi a_i^{\frac{a_i}{\epsilon}} \quad \ldots\ldots (8)$$

X_i here is the optimum quantity of the ith resource to use to maximise profits with capital not limited. As before, we will call this level of the ith resource, X_i^*.

Letting K be the optimum quantity of capital to use, then

$$K = \Sigma X_i^*$$

$$= (1 - \epsilon) c^{\frac{1}{\epsilon}} \Pi a_i^{\frac{a_i}{\epsilon}} \quad \ldots\ldots (9)$$

Then, $$X_i^* = \frac{a_i K}{1 - \epsilon}$$

$$= \frac{a_i K}{\Sigma a_i} \quad \ldots\ldots (10)$$

It is a property of the Cobb-Douglas function that the optimum

proportion of total resources devoted to the ith resource, $\frac{X_i^*}{\Sigma X_i^*}$ is a constant for given resource prices and is not a function of the total level of resources used.

Limited Capital

The results from Section III above enable the definition of the best combination of a given bundle of resources. These results can be extended to indicate whether a greater or a smaller total bundle of resources can profitably be employed.

Computationally, by substituting the values for X_{iT}^* obtained from the use of equation (9) into equation (1) (Section IIIA), the value of Y_T^* is obtained. Substituting this into equation (8) (Section IIIA), the value of p_T is obtained.[11] Where $p_T > 1$ the use of additional resources is profitable.

An intermediate step in the above calculations yields quantities which are useful when making *comparisons* of the efficiency of resource allocation. For instance, do farmers on large properties use the resources under their control more efficiently than small farmers?[12] Separate functions could be fitted to the different groups of farmers from the same district and, for each, compute an "efficiency index," consisting of actual income divided by Y_T^* the expected income when the observed level of resources is combined optimally. Similarly, an index measuring the efficiency of utilisation of individual resources could be computed.

V. *An Example*

Recommendations for the re-allocation of resources which are based on the usual method of interpreting results from a fitted function may well suggest shifts in the use of a particular resource in the opposite

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[11] From the expression,

$$P_T = a_i \frac{Y_T^*}{X_{iT}^*}$$

These quantities are easily computed and have proven readily adapted to automatic treatment on an electronic computer.

[12] A similar type of measure was computed for comparing the efficiency of resource use of share-farmers and owner-operators. See W. S. Miller, "Comparative Efficiency of Farm Tenure Classes in the Combination of Resources," *Agricultural Economics Research*, Vol. IX, No. 1 (January, 1959), pp. 6-16. In this work, however, the index of efficiency was the ratio of actual total costs to the minimum total cost of producing a given level of output. This approach does not allow the use of all the resources at the disposal of the farmer, requires special assumptions about the opportunity costs, and does not indicate inefficiencies arising from operating at an inappropriate level of production.

direction to the direction of movement to an optimum position. This may conveniently be shown by an example.[13]

We shall assume a function of the form

$$Y = 5.424\, X_1^{.2068} X_2^{.1847} X_3^{.4339} X_4^{.1436}$$

In this example, $\Sigma a_i = 0.969$, i.e. there exists diminishing returns to scale. The "observed" levels of the resources together with their marginal value products are set out in Table 1.[14]

Table 1
OBSERVED RESOURCES AND PRODUCTIVITIES

Resource	*Observed Level of Resource*	*Observed Marginal Value Product*
X_1	2500	0.698
X_2	9450	0.165
X_3	1187	3.084
X_4	854	1.419

Interpreting these results conventionally, we note that the marginal value products of resources X_1 and X_2 are less than one, and those of X_3 and X_4 are greater than one. We could then suggest that the use of the last two resources be expanded, and that the use of the first two be curtailed.

Such a re-combination of resources may certainly be an improvement on the observed situation. However, in the case of resource X_1, it would involve a shift *away* from the level of use of X_1 appropriate to the optimum position of the firm, whether or not capital is limited.

Firstly, assume that capital is not limited as is implied by the comparison of marginal value products with one. The observed and the optimum quantities of the four resources are given in Table 2.

Table 2
CAPITAL NOT LIMITED

Resource	*Observed Level*	*Optimum Level*
X_1	2500	124,600
X_2	9450	111,300
X_3	1187	261,400
X_4	854	86,500

[13] An hypothetical example has been used because the author did not succeed in finding in the literature an example which included all the information required.

[14] The marginal value products were computed from $MVP_{X_i} = a_i \hat{Y}/X_i$ where X_i is the observed level of the ith resource and $\hat{Y}$ is the expected gross income computed at the observed levels of all resources.

From Table 2, it appears that, if the consequences of the assumption of unlimited capital are traced out, the results obtained go far beyond the bounds of "reasonable" extrapolation.

The optimum combination of resources when capital is limited to the total of the observed resource levels from Table 1 is set out in Table 3, together with the observed levels of each resource and its marginal value product computed at the observed levels of all resources.

Table 3

CAPITAL LIMITED

Resource	*Observed Level*	*Observed Marginal Value Product*	*Optimum Level*
X_1	2500	0.698	2986
X_2	9450	0.165	2667
X_3	1187	3.084	6265
X_4	854	1.419	2073

The results tabulated above (Table 3) suggest that, although the observed marginal value product of X_1 is less than one, the resource X_1 is actually being used at a level *below* the optimum level as the optimum has been defined in this paper. (At the above level of total resource use, $p_T = 1.12$, and hence the use of additional capital would be profitable.)

VI. *Conclusions*

The commonly used method of computing marginal value productivity of resources at some observed level of all resources, and comparing these with the market prices of resources can lead to incorrect conclusions about the direction of resource shifts required to bring about an optimum combination of resources.

The equilibrium position which is implied by the above procedure may not exist (where the sum of the partial elasticities of variable resources is equal to or greater than one) or may (because of the constant elasticity of production of the Cobb-Douglas function) exist at a level of resource use far greater than anything existing in practice.

A preferable approach is one which allows the definition of an optimum position which is attainable, and which, considering all resources simultaneously, specifies the combination of resources satisfying the required optimum.