2015

RISK AND SUSTAINABLE MANAGEMENT GROUP WORKING PAPER SERIES

FINANCE

TITLE:

Analogy Based Valuation of Currency Options

AUTHOR:

Hammad Siddiqi

Working Paper: F15_2

Schools of Economics and Political Science

The University of Queensland

St Lucia

Brisbane

Australia 4072

Web: www.uq.edu.au

Analogy Based Valuation of Currency Options

Hammad Siddiqi

h.siddiqi@uq.edu.au

University of Queensland

This Version: February 2015

Abstract

The two most intriguing anomalies in currency markets are: 1) the implied volatility smile in currency options, and 2) the forward discount bias in currency exchange rates. I show that if currency options are valued in analogy with the underlying currency and beliefs are heterogeneous, then the forward discount bias causes the smile. The analogy based currency option pricing formula is put forward, which converges to Garman-Kohlhagen formula if there is no forward discount bias. In the presence of the forward discount bias, an increase in belief dispersion increases the slope as well as the curvature of the smile.

JEL Classification: G02, G13, F31

Keywords: Forward Discount Bias, Implied Volatility Smile, Analogy Making, Currency Options

Analogy Based Valuation of Currency Options

Arguably, the most popular currency option pricing model among traders is the Garman-Kohlhagen model, which is the Black Scholes model (see Black and Scholes (1973)) suitably modified for currency options (See Garman and Kohlhagen (1983)). The existence of the implied volatility smile in currency options is inconsistent with the model. If the model is correct then implied volatility should not vary with strike, however, at-the-money currency options typically have lower implied volatility than in-the-money and out-of-the-money options.

Another intriguing anomaly is the existence of the forward discount bias in exchange rates, which means that the forward exchange rate is empirically found to be a biased predictor of future spot rate. Forward exchange rate is equal to the spot exchange rate adjusted for the interest rate differential between the two currencies. It predicts that a currency with a higher interest rate should depreciate against a currency with a lower interest rate. In reality, the opposite is typically observed. This finding is well documented and robust across all major currency pairs. See Lewis (1995), and Engel (1996) for a survey of research in this area.

The above mentioned puzzles of implied volatility smile and the forward discount bias seem unrelated; however, there may be a close connection between them. I show that if the risk of buying a currency call option is perceived as similar to the risk of investing in the underlying currency, and beliefs are heterogeneous, then the forward discount bias in exchange rates leads to the smile. Hence, the observed smile may be a manifestation of the forward discount bias in currency options when beliefs are heterogeneous. Furthermore, when belief dispersion increases, the slope as well as the curvature of the smile increases. This is consistent with empirical evidence showing that heterogeneous beliefs are an important determinant of the shape of the implied volatility smile in currency options. See Beber, A., Buraschi, A., and Breedon, F. (2010).

This article puts forward the associated option pricing formula, termed the analogy based currency option pricing formula. The new formula contains the Garman-Kohlhagen formula as a special case. Specifically, analogy formula converges to the Garman-Kohlhagen formula if the marginal investor in options does not display the forward discount bias.

The Black Scholes model assumes that markets are informationally efficient in the sense that risk adjusted return from a currency option is equal to the risk adjusted return from the underlying currency itself. A pre-requisite for informational efficiency is that all risks are correctly perceived. However, risk is a highly subjective notion, and the argument that risks are often misperceived in significant ways even when stakes are high is frequently made. See Rotheli (2010), Bhattacharya, Goldman, & Sood (2009), Akerlof and Shiller (2009), and Barberis & Thaler (2002) among many others.

In laboratory experiments, it has been found that subjects consider the risk of a call option as similar to the risk of the underlying asset, and value the call option in analogy with the underlying asset by equating their expected returns. See Rockenbach (2004), Siddiqi (2012), and Siddiqi (2011). Furthermore, it is common to find practitioners with decades of experience arguing that a call option is a surrogate for the underlying asset suggesting strong field relevance of the laboratory findings.[1]

The relevance of analogy making for human thinking process has long been acknowledged. Hume wrote in 1748, "*From causes which appear similar, we expect similar effects. This is the sum of all our experimental conclusions*". (Hume 1748, Section IV). Cognitive scientists and psychologists argue that analogy making is the core of cognition and the fuel and fire of thinking (see Hofstadter and Sander (2013)). Hofstadter and Sander (2013) write, *"[…] at every moment of our lives, our concepts are selectively triggered by analogies that our brain makes without letup, in an effort to make sense of the new and unknown in terms of the old and known." (Hofstadter and Sander (2013), Prologue page1).*

Analogy making is the act of perceiving two non-identical situations or objects as the same at some abstract level (see Mitchell. M. (2001)). The analogy between a call option and its underlying asset is very tempting as their payoffs are strongly related. It should not come as a surprise that call options are considered surrogates of the underlying assets and their risks are deemed similar both by experienced market professionals as well as participants in laboratory experiments.

[1] A few examples of investment professionals arguing that a call option is a surrogate for the underlying asset are:
http://finance.yahoo.com/news/stock-replacement-strategy-reduce-risk-142949569.html
http://ezinearticles.com/?Call-Options-As-an-Alternative-to-Buying-the-Underlying-Security&id=4274772,
http://www.investingblog.org/archives/194/deep-in-the-money-options/
http://www.triplescreenmethod.com/TradersCorner/TC052705.asp,
http://daytrading.about.com/od/stocks/a/OptionsInvest.htm

Shouldn't rational arbitrageurs make money at the expense of such analogy makers? Such arbitraging is difficult if not impossible in the presence of transaction costs. An important argument in the behavioral finance literature (see Barberis and Thaler (2002)) is that the absence of "free lunch" does not imply that prices are right due to limits to arbitrage including transaction costs. In our context, the presence of transaction costs is likely to eliminate any "free lunch" at the expense of analogy makers. It is worth mentioning that bid-ask spreads in at-the-money currency options are typically of the order of 5 to 8% of the option price, and spreads are typically 10 to 15% of the option price in out-of-the-money currency options. In continuous time, no matter how small the transaction costs are, the total cost of replicating an option grows without bound. See Soner, Shreve, and Cvitanic (1995). In discrete time, transaction costs are bounded, however, a no-arbitrage interval is created. If analogy price lies within the interval, analogy makers cannot be arbitraged away. The interval gets larger as the required frequency of replicating portfolio adjustments increases. If the underlying currency follows jump diffusion and/or stochastic volatility, then the replicating portfolio argument fails regardless of transaction costs, and analogy makers cannot be arbitraged away.

Siddiqi (2014) studies the implications of analogy making for equity index options and shows that it provides a unified explanation for a number of puzzles such as the implied volatility skew, superior performance of covered call writing, and worse than expected performance of zero beta straddles. Siddiqi (2015) extends the notion of analogy making to commodity options and shows that it provides an explanation for the various types of skews observed in commodity options.

This article extends the analogy argument to currency options. It puts forward an analogy based currency option pricing formula that provides an explanation for the implied volatility smile in currency options. It contains Garman-Kohlhagen formula as a special case. The analogy formula converges to Garman-Kohlhagen formula if there is no forward discount bias and there are no transaction costs. If the analogy formula is correct, then the forward discount bias and the implied volatility smile in currency options are the two sides of the same coin. It is interesting that the same idea of considering a call option to be a surrogate for the underlying asset provides explanations for diverse puzzles across very different markets (equity, commodity, and currency).

The central prediction of asset pricing theory is:

$$E[R_i] = R_F - \frac{1}{E[U'(c_{t+1})]} Cov[U'(c_{t+1}), R_i] \tag{0.1}$$

Where R_i and R_F denote the return on a risky asset and the return on the risk free asset respectively. Equation (0.1) shows that the return that a subjective expected utility maximizer expects from a risky asset depends on his belief about the covariance of the asset's return with his marginal utility of consumption.

An analogy maker is *a subjective expected utility maximizer who believes that the covariance of a currency call option's return with his marginal utility of consumption is equal to the covariance of the underlying currency's return with his marginal utility of consumption*. Consequently, the return he expects from a currency call option is equal to his subjective assessment of the expected return available on the underlying currency.

An analogy maker perceives the *quantity of risk* in a currency call option to be equivalent to the *quantity of risk* involved in investing in the underlying currency. One may re-write (0.1) as:

$$\frac{E[R_i]-R_F}{\beta_i} = \lambda \qquad (0.2)$$

Where $\beta_i = \frac{Cov\{U'(c_{t+1}),R_i\}}{Var(U'(c_{t+1}))}$ and $\lambda = -\frac{Var\left(U'(c_{t+1})\right)}{E[U'(c_{t+1})]}$

Clearly, for a given subjective expected utility maximizer, the (subjectively perceived) risk adjusted returns across all risky assets should be equal to λ, which is his (subjective) price of risk. An analogy maker equates the perceived *quantity of risk* in a currency call option (measured by β of call) with the perceived *quantity of risk* involved in buying the underlying currency outright (measured by β of the underlying stock).

Analogy making is related to the literature on mental accounting. Mental accounting, a term coined in Thaler (1980), is a broad concept. Thaler (1999) defines mental accounting as a set of cognitive operations used by individuals to organize, evaluate and keep track of financial activities. Analogy making is a special type of mental accounting in which a call option is classified in the same risk category as the underlying asset.

This paper is organized as follows: Section 1 develops the analogy based currency option pricing formula in continuous time. It shows that Garman-Kohlhagen formula is a special case corresponding to the absence of forward discount bias and transaction costs. Section 2 shows that if the forward discount bias exists in the presence of diverse beliefs, implied volatility smile arises in

currency options. Section 3 discusses the impact of analogy making on comparative statistics of interest for currency options. Section 4 concludes.

1. Analogy Making and the Value of Currency Options

An analogy maker equates the return he expects from a currency call option to his subjective assessment of the expected return from holding the underlying currency as he perceives their risks to be similar.

Suppose the marginal investor in a currency call option with a strike of K is an analogy maker. If C denotes the domestic currency price of a call option on one unit of foreign currency, μ is the expected percentage appreciation in the domestic currency price of foreign currency, and r_F is the foreign interest rate on a risk free bond, then over a time interval dt:

$$\frac{E[dC]}{C} = \mu + r_F \tag{1.1}$$

That is, the expected return from holding the foreign currency is μ, the drift of the exchange rate (domestic units per foreign unit), plus risk free interest rate earned from holding the foreign currency in a risk free asset such as foreign treasury notes.

All the assumptions of the Black Scholes (Garman-Kohlhagen) model are maintained except for one assumption. Here, the transaction costs are allowed whereas the Black Scholes model requires that transaction costs must be zero. As mentioned earlier, it is well known that the Black Scholes argument fails when transaction costs are allowed as the cost of replicating an option grows without bound. See Soner et al (1995).

If φ_F is the percentage transaction cost of investing in foreign currency, and φ_C is the percentage transaction cost of buying a currency call option, then (1.1) can be re-written as:

$$\frac{E[dC]}{C} - \varphi_C = \mu + r_F - \varphi_F \tag{1.2}$$

Foreign currency markets are perhaps the most liquid markets in the world with small transaction costs; however, currency option markets have transaction costs which are quite large. As

mentioned earlier, typically, the bid-ask spreads in currency options are of the order of 5 to 8% of option price in ATM options, and the spreads are around 10 to 15% of the option price in deep OTM options. Hence, φ_C is much larger than φ_F. For simplicity, I ignore φ_F. Hence, (1.2) can be written as:

$$\frac{E[dC]}{C} - \varphi_C = \mu + r_F \tag{1.2a}$$

I assume that the spot exchange rate (domestic units per foreign unit) follows geometric Brownian motion with drift μ and standard deviation equal to σ:

$$dS = \mu S dt + \sigma S dW \tag{1.3}$$

Equations (1.2a) and (1.3) are sufficient to arrive at a partial differential equation (PDE) that a currency call option must satisfy under analogy making. The associated PDE is described in proposition 1.

Proposition 1 ***If analogy making determines the price of a European currency call option in the presence of transaction costs, then its price must satisfy the following PDE:***

$$\boldsymbol{\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} = (\mu + r_F + \varphi_C) C} \tag{1.4}$$

With the boundary condition that at expiry, which is at time T, the value of a European call option with a strike of K is given by: $C = max(S - K, 0)$

Proof.

See Appendix A.

■

It is interesting to note that (1.4) is equal to the Garman-Kohlhagen PDE/Black Scholes PDE for currency options if there is no forward discount bias, and transaction costs are zero. To see this clearly, note that the forward exchange rate at time t for exchange at time $t + dt$ is given by:

$$F = Se^{(r_D - r_F)} \tag{1.5}$$

Where r_D the domestic risk free interest rate, and F denotes the forward exchange rate. If the forward rate is an unbiased predictor of future spot rate, that is, $F = E_t[S_{t+dt}]$, then one may write:

$$ln\big(E_t(S_{t+dt})\big) - ln(S_t) = \mu = r_D - r_F \tag{1.6}$$

Substituting (1.6) in (1.4) and equating φ_C to zero, one gets:

$$\frac{\partial C}{\partial t} + (r_D - r_F)S\frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} = r_D C \tag{1.7}$$

(1.7) is the Garman-Kohlhagen/Black Scholes PDE for currency call options.

By using a similar method to what is used in solving the Black Scholes PDE, the analogy based PDE given in (1.4) can be solved to recover an option pricing formula for a currency call option under analogy making. Proposition 2 presents the formula. By using put-call parity, the price of a currency put option is also obtained.

Proposition 2 *If analogy making determines the price of a currency call option in the presence of transaction costs, then the corresponding European call option pricing formula is:*

$$\boldsymbol{C = e^{-(r_F+\varphi_C)(T-t)}\{SN(d_1) - Ke^{-\mu(T-t)}N(d_2)\}} \tag{1.8}$$

Where $\boldsymbol{d_1 = \frac{ln\left(\frac{S}{K}\right)+\left(\mu+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}$ **and** $\boldsymbol{d_2 = \frac{ln\left(\frac{S}{K}\right)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}}$

Proof.

See Appendix B.

■

Corollary 2.1 **The price of a corresponding European put option under analogy making is given by:**

$$\boldsymbol{P = K\left\{e^{-(r_D)(T-t)} - e^{-(r_F+\varphi_C+\mu)(T-t)}N(d_2)\right\} - S\left\{e^{-r_F(T-t)} - e^{-(r_F+\varphi_C)(T-t)}N(d_1)\right\}} \qquad \textbf{(1.9)}$$

Proof.

Follows from put-call parity for exchange rate options.

■

It is easy to see that (1.8) and (1.9) contain the corresponding Garman-Kohlhagen formulas as a special case, which is obtained when $\mu = r_D - r_F$ and $\varphi_C = 0$.

1.1 Analogy Making and Arbitrage

Can arbitrage profits be made against analogy makers? In continuous time, the presence of transaction costs, no matter how small, makes the Black Scholes/Garman-Kohlhagen formula irrelevant as the cost of replication grows without bound (see Soner et al (1995)). Hence, analogy makers cannot be arbitraged away in this case.

In discrete time, the cost of replication is bounded, however, it grows very fast as the frequency of portfolio adjustments required for successful replication of an option increases. Add to this the comparatively large transaction costs involved in trading currency options, and the possibility of running a successful arbitrage scheme seems extremely unlikely if not impossible.

It's true the market of buying and selling currencies has one of the lowest transaction costs in financial markets; however, it's also true that, unlike currencies themselves, the transaction costs in the market for options on currencies are perhaps among the highest. As mentioned earlier, ATM currency options have bid-ask spreads in the range of 5 to 8% of option price, and OTM currency options have typical spreads of around 8 to 15% of the option price. These costs alone are large enough to make arbitraging quite difficult even if other transaction costs are ignored.

If the underlying currency does not follow geometric Brownian motion, then the portfolio replication argument fails. In that case, even if all transaction costs are assumed away, arbitrage profits cannot be made against analogy makers. Analogy making does not require geometric Brownian motion and works equally well with other possibilities such as jump diffusion or stochastic volatility. Geometric Brownian motion is assumed here because it's the simplest framework in which the influence of analogy making can be expressed. Analogy maker is a subjective expected utility maximizer who considers a call option to be a surrogate for the underlying asset irrespective of the stochastic process followed by the underlying asset. For an example of analogy making with jump diffusion and stochastic volatility, when the underlying is an equity index, see Siddiqi (2014), and for an example of analogy making with jump diffusion, when the underlying is a commodity, see Siddiqi (2015).

2. The Implied Volatility Smile

If exchange rate beliefs are heterogeneous and are in accordance with the forward discount bias, then the analogy formula generates the implied volatility smile. This is consistent with empirical evidence that heterogeneous exchange rate beliefs are an important determinant of the shape of the implied volatility smile. See Beber, A., Buraschi, A., and Breedon, F. (2010).

Survey evidence clearly suggests that people hold diverse beliefs regarding exchange rate movements (see the dataset in Beber et al (2010)). Furthermore, these diverse beliefs are biased as average expectation points to an appreciation when the forward rate suggests a depreciation and vice-versa. To take a concrete example, suppose the domestic interest rate is 1% per annum, and foreign interest rate is 3% per annum. The forward rate would suggest depreciation in the foreign currency of around 2% per annum to offset the arbitrage opportunity arising from the interest rate advantage in the foreign currency. However, actual average investor expectations would generally point to an appreciation of around 2%; hence, the expectations are biased in comparison with the forward rate.

Continuing with the above example, there are many ways in which biased beliefs relative to the forward rate can be distributed across marginal investors in call options as moneyness changes.

One expects more optimistic investors to self-select into higher strike calls. Two examples are shown in table 1:

Table 1

Moneyness (K/S)	**Low Belief Dispersion (% appreciation expected)**	**High Belief Dispersion (% appreciation expected)**
0.9	0.5%	0%
0.95	1%	0.5%
1.0	2%	1.0%
1.05	4%	4%
1.1	8%	9%

In the low dispersion example, as the name suggests, beliefs are comparatively less dispersed around the average belief, when compared with the high dispersion example (see Table 1), assuming that all beliefs have equal frequencies (20%). The average absolute deviation from the mean belief is 2.32% in the low dispersion example, whereas the corresponding deviation in the high dispersion example is 2.88%.

Continuing with the same example, the following parameter values are chosen for illustrative purposes:

$$S = 1.25, \sigma = 13\%, \varphi_C = 0, and\ \ T - t = 0.25\ year.$$

It does not matter what specific numbers are chosen here. As long as the belief of each marginal investor is biased in the direction of the forward discount bias, and more optimistic investors self-select into higher strike calls, the implied volatility smile is generated.

Figure 1 shows the implied volatility smile generated in this example.

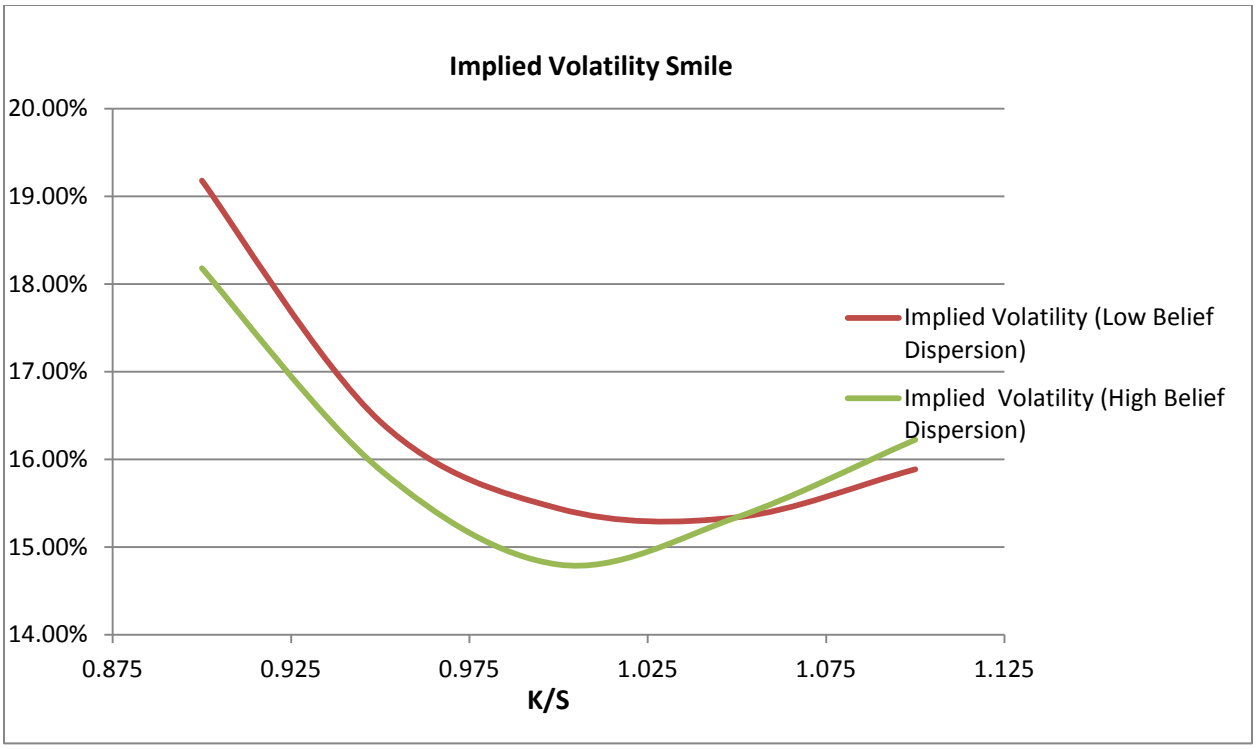

Figure 1

It is interesting to note two things in figure 1. Firstly, as belief dispersion increases, the curvature of the smile increases. Secondly, as belief dispersion increases, the slope of the smile increases as well. Both features are in line with empirical evidence on the impact of belief dispersion on currency smiles. See Beber et al (2010).

It is easy to construct the opposite example in which domestic interest rate is larger than foreign interest rate. Suppose the domestic interest rate is 3% and the foreign interest rate is 1%. The forward rate suggests an appreciation of 2% in the foreign currency. However, investors would expect depreciation in the foreign currency, making put options attractive. Appealing to foreign currency symmetries, a put option on foreign currency is equivalent to a call option on domestic currency, which is expected to appreciate. (1.8) can be used to price a call option in the domestic currency by redefining spot rate and strike appropriately, and using the domestic interest rate in place of foreign interest rate. As long as the beliefs are biased in the direction of the forward discount bias, and more optimistic investors self-select into higher strike calls, smiles similar to figure 1 are generated in this case.

It's easy to verify that a necessary condition for generating the smile is the existence of the forward discount bias. This necessary condition combined with the realization that more optimistic investors self-select into higher strike calls generates the smile. Hence, with analogy making, the smile is a manifestation of the forward discount bias in currency markets.

If beliefs are unbiased with respect to the forward rate, the analogy formula converges to the Garmen-Kohlhagen formula. Clearly, the smile disappears and is replaced by a flat line in this case. What happens when the beliefs are biased in the direction of the forward rate? That is, what happens when the forward rate suggests an appreciation, and beliefs are biased in the direction of the appreciation, instead of the being biased in the opposite direction as in figure 1? For the purpose of illustration, suppose the domestic interest rate is 3%, and the foreign interest rate is 1%. The forward rate suggests an appreciation of 2% in the foreign currency. Suppose beliefs are biased in the direction of appreciation. Specifically, consider the same beliefs as given in Table 1. Figure 2 plots the associated implied volatility given the beliefs in Table 1. The forward skew (implied volatility monotonically rises with strike) is seen, which does not conform to the observed typical smile in currency options.

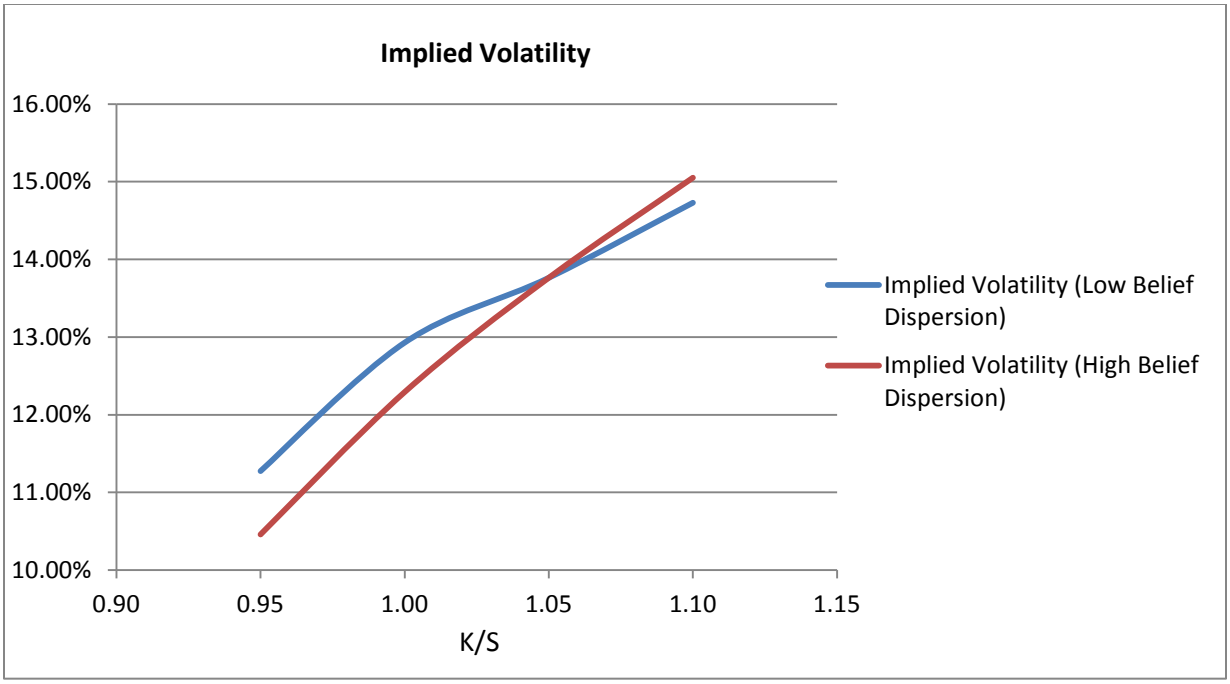

Figure 2

The existence of the forward discount bias is required to generate the smile. No bias means no smile, and bias in the same direction as the forward rate, generates forward skew.

The shape of the smile carries useful information about the distribution of the bias in investor population. As figure 1 shows, an increase in belief dispersion increases the curvature as well as the slope of the smile. Hence, curvature and slope carry useful information regarding how diverse investor beliefs are.

3. Comparative Statics

For the purpose of hedging risks arising from currency option exposure, partial derivatives of (1.8) and (1.9) are of special interest with 'hedge ratio' being of prime importance:

$$\frac{\partial C}{\partial S} = e^{-(r_F+\varphi_C)(T-t)}N(d_1) > 0 \tag{3.1}$$

As, exchange rate expectations are generally in the opposite direction to what the interest rate differential suggests, the Garman-Kohlhagen formula potentially under-estimates the hedge ratio when $r_D < r_F$, and it potentially over-estimates the hedge ratio when $r_D > r_F$. Getting the hedge ratio right is of key importance to currency option traders, and they are potentially exposed to large losses without it.

Some other partial derivatives are:

$$\frac{\partial C}{\partial K} = -e^{-(\mu+r_F+\varphi_C)(T-t)}N(d_2) < 0 \tag{3.2}$$

$$\frac{\partial C}{\partial \mu} = (T-t)e^{-(\mu+r_F+\varphi_C)(T-t)}KN(d_1) > 0 \tag{3.3}$$

$$\frac{\partial C}{\partial r_D} = 0 \tag{3.4}$$

$$\frac{\partial C}{\partial r_F} = -(T-t)e^{-(r_F+\varphi_C)(T-t)}SN(d_1) + (T-t)e^{-(\mu+r_F+\varphi_C)(T-t)}KN(d_2) \tag{3.5}$$

$$\frac{\partial C}{\partial (T-t)} = -(r_F+\varphi_C)e^{-(r_F+\varphi_C)(T-t)}SN(d_1) + (\mu+r_F+\varphi_C)e^{-(r_F+\varphi_C+\mu)(T-t)}KN(d_2) + \frac{e^{-(\mu+r_F+\varphi_C)}\sigma}{2\sqrt{T-t}}KN'(d_2) \tag{3.6}$$

To a speculator interested in taking directional bets on exchange rates, these partial derivatives are important as they constitute the risk constraints in a linear programming problem.[2] If prices are determined in accordance with the analogy formula, and Garman-Kohlhagen formula is used to estimate these partial derivatives, quite a few additional problems arise apart from getting the hedge ratio wrong as mentioned earlier.

Firstly, exchange rate expectations have no direct impact in Garman-Kohlhagen setting beyond their impact on spot rate, whereas under analogy making, an increase in expected appreciation directly increases the value of currency call options as (3.3) shows, over and above any indirect impact on the spot rate.

Secondly, in sharp contrast with Garman-Kohlhagen formula, an increase in domestic interest rate does not directly affect the call price. Of course, there are indirect effects due to impacts on exchange rate expectations as well as the spot rate. An increase in foreign interest rates has a negative impact on call price in Garman-Kohlhagen formula, whereas the impact is ambiguous under analogy making, and can be positive for out-of-the-money call options.

Thirdly, the impact of option life on call price is dependent on exchange rate expectations as (3.6) shows. If both foreign and domestic interest rates are zero, Garman-Kohlhagen formula predicts an increase in option life would increase call option value. Under analogy making, the impact depends on the direction of exchange rate expectations.

Last but not the least, the presence of transaction costs dampens the magnitude of nearly all partial derivatives. The transaction costs are smaller for ATM options when compared with OTM options. Hence, ignoring them has an asymmetric impact on the linear programming problem facing the speculator.

The partial derivatives of put option are similarly obtained from (1.9) and they have opposite signs.

[2] See Rendlemen (2002) (Chapter 7) for details on setting up linear programming problems associated with directional bets.

4. Conclusions

People choose which options to buy based on subjective reasons. This fact compels one to think that investor expectations matter for option demands and prices. Indeed, Bollen and Whaley (2004) find that demand pressures associated with various option series matter for implied volatility. However, theoretical option pricing models typically do not have any role for investor expectations, even though they provide an elegant framework of analysis. It seems that the challenge is to incorporate investor expectations in the elegant option pricing framework with the goal of increasing the explanatory power of the models while preserving the elegance of the framework. The idea of analogy making provides a tool to do just that.

Analogy making approach is inspired by the well-known practice of experienced market professionals to consider a call option to be a surrogate for the underlying asset. An analogy maker is a subjective expected utility maximize who considers a call option to be a surrogate for the underlying asset. Earlier research has shown that it provides a unified explanation for a number of puzzles in equity and commodity markets (see Siddiqi (2014), and Siddiqi (2015)).

In this article, the approach is extended to currency options. The approach leads to an analogy based currency option pricing formula which contains the Garman-Kohlhagen formula as a special case corresponding to the complete absence of the forward discount bias. That is, if investor expectations are random and in line with the forward rate, the Garman-Kohlhagen formula is obtained. However, if they are biased as suggested by the observed forward discount bias, the smile is generated. Hence, the analogy formula not only provides an explanation for the implied volatility smile in currency options, but also shows that the smile can be considered a manifestation of the forward discount bias in currency options. Ability to provide an explanation for a wide variety of phenomena across diverse markets is a testament to the power of the analogy approach.

References

Akerlof, G., and Shiller, R. (2009), "Animal Spirits", Princeton University Press, Princeton.

Barberis, N., and Thaler, R. (2002), "A Survey of Behavioral Finance", *NBER Working Paper No. 9222*.

Beber, A., Buraschi, A., and Breedon, F. (2010). "Differences in beliefs and currency risk premiums". *Journal of Financial Economics,* Vol. 98(3), pp. 415-438.

Bhattacharya, J., Goldman D., & Sood, N, (2009), "Market Evidence of Misperceived Mortality Risk", *Journal of Economic Behavior and Organization,* 72(1), pp. 452-461.

Black, F., and Scholes, M. (1973), "The pricing of options and corporate liabilities". *Journal of Political Economy* 81(3): pp. 637-65.

Bollen, N., and Whaley, R. (2004). "Does Net Buying Pressure Affect the Shape of Implied Volatility Functions?" *Journal of Finance* 59(2): 711–53

Engle, C. (1996), "The forward discount anomaly and the risk premium: A survey of recent evidence". *Journal of Empirical Finance*, 3, pp. 123-142.

Garman, M. B., and Kohlhagen, S. W., (1983), "Foreign Currency Option Values", *Journal of International Money Finance,* 2, pp. 231-237.

Hume, David (1748), "An Enquiry Concerning Human Understanding". E-version available at http://creativecommons.org/licenses/by-nc-sa/3.0/au/

Hofstadter, D., and Sander, E. (2013), "Surfaces and Essences: Analogy as the fuel and fire of thinking", Published by Basic Books, April.

Lewis, K.K., (1995), "Puzzles in International Financial Markets"*, Handbook of International Economics*, 3, Elsevier Science Publication Co., Amsterdam.

Mitchell, M. (2001). Analogy-making as a complex adaptive system. In L. Segel and I. Cohen (editors), *Design Principles for the Immune System and Other Distributed Autonomous Systems*. New York: Oxford University Press.

Rockenbach, B. (2004), "The Behavioral Relevance of Mental Accounting for the Pricing of Financial Options". *Journal of Economic Behavior and Organization,* Vol. 53, pp. 513-527.

Rotheli, T. F. (2010), "Causes of the Financial Crisis: Risk Misperception, Policy Mistakes, and Banks' Bounded Rationality", *Journal of Socio-Economics,* Vol. 39, Issue 2, pp. 119-126.

Siddiqi, H. (2015), "Analogy Based Valuation of Commodity Options".Available at SSRN: http://ssrn.com/abstract=2544347 or http://dx.doi.org/10.2139/ssrn.2544347

Siddiqi, H. (2015), "Analogy Making and the Structure of Implied Volatility Skew (January 1, 2015). Available at SSRN: http://ssrn.com/abstract=2465738 or http://dx.doi.org/10.2139/ssrn.2465738

Siddiqi, H. (2012), "The Relevance of Thinking by Analogy for Investors' Willingness to Pay: An Experimental Study", *Journal of Economic Psychology,* Vol. 33, Issue 1, pp. 19-29.

Siddiqi, H. (2011), "Does Coarse Thinking Matter for Option Pricing? Evidence from an Experiment" *IUP Journal of Behavioral Finance*, Vol. VIII, No.2. pp. 58-69

Siddiqi, H. (2009), "Is the Lure of Choice Reflected in Market Prices? Experimental Evidence based on the 4-Door Monty Hall Problem". *Journal of Economic Psychology, April.*

Soner, H.M., S. Shreve, and J. Cvitanic, (1995), "There is no nontrivial hedging portfolio for option pricing with transaction costs", *Annals of Applied Probability* 5, 327–355.

Thaler, R. (1999), "Mental Accounting Matters", *Journal of Behavioral Decision Making,* 12, pp. 183-206.

Thaler, R. (1980) "Toward a positive theory of consumer choice" *Journal of Economic Behavior and Organization*, 39-60 .

Appendix A

$$\frac{E[dC]}{C} - \varphi_C = \mu + r_F \tag{A1}$$

From Ito's Lemma:

$$E[dC] = \left(\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2}\right) dC \tag{A2}$$

Substituting (A2) in (A1) and simplifying leads to:

$$\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 C}{\partial S^2} = (\mu + r_F + \varphi_C)C \tag{A3}$$

(A3) is the analogy based PDE for a currency call option with the boundary condition:

$$C(S,T) = max(S - K, 0)$$

Appendix B

Start by making the following transformations in (A3):

$$\tau = \frac{\sigma^2}{2}(T - t)$$

$$x = ln\frac{S}{K} => S = Ke^x$$

$$C(S,t) = K \cdot\ c(x,\tau) = K \cdot c\left(ln\left(\frac{S}{K}\right), \frac{\sigma^2}{2}(T-t)\right)$$

It follows,

$$\frac{\partial C}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = K \cdot \frac{\partial c}{\partial \tau} \cdot \left(-\frac{\sigma^2}{2}\right)$$

$$\frac{\partial C}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{\partial x}{\partial S} = K \cdot \frac{\partial c}{\partial x} \cdot \frac{1}{S}$$

$$\frac{\partial^2 C}{\partial S^2} = K \cdot \frac{1}{S^2} \cdot \frac{\partial^2 C}{\partial x^2} -\ K \cdot \frac{1}{S^2}\frac{\partial C}{\partial x}$$

Plugging the above transformations into (A3) and writing $\tilde{r} = \frac{2(\mu)}{\sigma^2}$, $\widetilde{r_F} = \frac{2r_F}{\sigma^2}$, and $\widetilde{\varphi_C} = \frac{2\varphi_C}{\sigma^2}$we get:

$$\frac{\partial c}{\partial \tau} = \frac{\partial^2 c}{\partial x^2} + (\tilde{r} - 1)\frac{\partial c}{\partial x} - (\tilde{r} + \widetilde{r_F} + \widetilde{\varphi_C})c \qquad (B1)$$

With the boundary condition/initial condition:

$$C(S,T) = max\{S - K, 0\}\ becomes\ c(x,0) = max\{e^x - 1,0\}$$

To eliminate the last two terms in (B1), an additional transformation is made:

$$c(x,\tau) = e^{\alpha x + \beta\tau}u(x,\tau)$$

It follows,

$$\frac{\partial c}{\partial x} = \alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 c}{\partial x^2} = \alpha^2 e^{\alpha x+\beta\tau}u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial c}{\partial \tau} = \beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau}$$

Substituting the above transformations in (B1), we get:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\alpha^2 + \alpha(\tilde{r} - 1) - (\tilde{r} + \widetilde{r_F} + \widetilde{\varphi_C}) - \beta)u + \left(2\alpha + (\tilde{r} - 1)\right)\frac{\partial u}{\partial x} \tag{B2}$$

Choose $\alpha = -\frac{(\tilde{r}-1)}{2}$ and $\beta = -\frac{(\tilde{r}+1)^2}{4} - (\widetilde{r_F} + \widetilde{\varphi_C})$. (B2) simplifies to the Heat equation:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{B3}$$

With the initial condition:

$$u(x_0, 0) = max\left\{\left(e^{(1-\alpha)x_0} - e^{-\alpha x_0}\right), 0\right\} = max\left\{\left(e^{\left(\frac{\tilde{r}+1}{2}\right)x_0} - e^{\left(\frac{\tilde{r}-1}{2}\right)x_0}\right), 0\right\}$$

The solution to the Heat equation in our case is:

$$u(x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{4\tau}} u(x_0, 0) dx_0$$

Change variables: $= \frac{x_0 - x}{\sqrt{2\tau}}$, which means: $dz = \frac{dx_0}{\sqrt{2\tau}}$. Also, from the boundary condition, we know that $u > 0 \; iff \; x_0 > 0$. Hence, we can restrict the integration range to $z > \frac{-x}{\sqrt{2\tau}}$

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\pi}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}+1}{2}\right)(x + z\sqrt{2\tau})} dz - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\left(\frac{\tilde{r}-1}{2}\right)(x + z\sqrt{2\tau})} dz$$

$$=: H_1 - H_2$$

Complete the squares for the exponent in H_1:

$$\frac{\tilde{r}+1}{2}\left(x + z\sqrt{2\tau}\right) - \frac{z^2}{2} = -\frac{1}{2}\left(z - \frac{\sqrt{2\tau}(\tilde{r}+1)}{2}\right)^2 + \frac{\tilde{r}+1}{2}x + \tau\frac{(\tilde{r}+1)^2}{4}$$

$$=: -\frac{1}{2}y^2 + c$$

We can see that $dy = dz$ and c does not depend on z. Hence, we can write:

$$H_1 = \frac{e^c}{\sqrt{2\pi}} \int_{-x/\sqrt{2\pi}-\sqrt{\tau/2}(\tilde{r}+1)}^{\infty} e^{-\frac{y^2}{2}} dy$$

A normally distributed random variable has the following cumulative distribution function:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{y^2}{2}} dy$$

Hence, $H_1 = e^c N(d_1)$ where $d_1 = x/\sqrt{2\pi} + \sqrt{\tau/2}\,(\tilde{r}+1)$

Similarly, $H_2 = e^f N(d_2)$ where $d_2 = x/\sqrt{2\pi} + \sqrt{\tau/2}\,(\tilde{r}-1)$ and $f = \frac{\tilde{r}-1}{2}x + \tau\frac{(\tilde{r}-1)^2}{4}$

The analogy based European currency call pricing formula is obtained by recovering original variables:

$$C = e^{-(r_F+\varphi_C)}\{SN(d_1) - Ke^{-(\mu)(T-t)}N(d_2)\}$$

Where $\boldsymbol{d_1} = \frac{\boldsymbol{ln(S/K)+(\mu+\frac{\sigma^2}{2})(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$ *and* $\boldsymbol{d_2} = \frac{\boldsymbol{ln\left(\frac{S}{K}\right)+\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}}{\boldsymbol{\sigma\sqrt{T-t}}}$

PREVIOUS WORKING PAPERS IN THE SERIES

FINANCE

F12_1 Government Safety Net, Stock Market Participation and Asset Prices by Danilo Lopomo Beteto (2012).

F12_2 Government Induced Bubbles by Danilo Lopomo Beteto (2012).

F12_3 Government Intervention and Financial Fragility by Danilo Lopomo Beteto (2012).

F13_1 Analogy Making in Complete and Incomplete Markets: A New Model for Pricing Contingent Claims by Hammad Siddiqi (September, 2013).

F13_2 Managing Option Trading Risk with Greeks when Analogy Making Matters by Hammad Siddiqi (October, 2012).

F14_1 The Financial Market Consequences of Growing Awareness: The Cased of Implied Volatility Skew by Hammad Siddiqi (January, 2014).

F14_2 Mental Accounting: A New Behavioral Explanation of Covered Call Performance by Hammad Siddiqi (January, 2014).

F14_3 The Routes to Chaos in the Bitcoins Market by Hammad Siddiqi (February, 2014).

F14_4 Analogy Making and the Puzzles of Index Option Returns and Implied Volatility Skew: Theory and Empirical Evidence by Hammad Siddiqi (July, 2014).

F14_5 Network Formation and Financial Fragility by Danilo Lopomo Beteto Wegner (May 2014).

F14_6 A Reinterpretation of the Gordon and Barro Model in Terms of Financial Stability by Danilo Lopomo Beteto Wegner (August, 2014).

F14_7 Analogy Making and the Structure of Implied Volatility Skew by Hammad Siddiqi (October, 2014).

F15_1 Analogy based Valuation of Commodity Options by Hammad Siddiqi (January, 2015)