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PORTFOLIO CHOICES IN THE PRESENCE OF OTHER RISKS

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ABSTRACT

The effects of multivariate risk are examined in a model of portfolio choice. The conditions under which portfolio choices are separable from consumption decisions are derived. Unless the appropriate restrictions hold on investors' preferences or the probability distribution of risks, the optimal portfolio is affected by other risks. This requires generalizing the usual measures of risk aversion. With two risky assets, the choices made by investors coincide if and only if their generalized risk aversion measures are identical. An extension of Ross's notion of stronger risk aversion is used to characterize the effect of risk aversion on the investment level in the riskier asset. Finally, in a mean-variance model, the matrix measures of risk aversion are used to generalize the results of Arrow and Pratt concerning the effects of risk aversion and wealth on the optimal portfolio.

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Portfolio Choices in the Presence of Other Risks

1. Introduction

The effect of risk aversion on the level of investment in a risky asset is a well known result. An individual whose utility function defined over wealth is more concave than another's will pay a higher risk premium (Arrow (1965), Pratt (1964)) and will invest a smaller portion of his initial wealth in a risky asset (Pratt (1964)). If risk aversion decreases with wealth, then the larger is initial wealth, the larger is the investment in the risky asset (Arrow (1965)). If the choice set includes more than one risky asset, the Arrow-Pratt measure is not sufficient to rank investors' choices (Cass and Stiglitz (1972). In the absence of a risk-free asset and when the choice is between two risky assets, Ross (1981) suggested a stronger measure of risk aversion and showed that an investor who is strongly more averse to univariate risk than another would choose a lower level of investment in the more risky asset.

These studies make use of the expected utility hypothesis, in which an investor makes decisions so as to maximize the expected value of his utility function. An additional implicit assumption is that the utility function is defined only on the level of terminal wealth. The process by which wealth produces utility, presumably through the consumption of goods, is left implicit.

Assuming that the investor derives utility from the consumption of goods purchased from his wealth, there are many situations where his objective function is multivariate, defined on several random arguments, rather than a univariate function of wealth alone. A generalization of the Arrow-Pratt theory of behavior under uncertainty is then required, and indeed was carried out in a sequence of papers—Stiglitz (1969), Kihlstrom and Mirman (1974), Epstein (1975), Karni (1979) and others. In this paper, we apply some of the results from these papers to the analysis of portfolio choices.

In the presence of univariate risk, when the utility function is assumed to be defined on wealth alone (or if it is only wealth that is uncertain), the investment decision is not affected by preferences for consumption goods (ordinal preferences), but it is affected by the degree of risk aversion, which is a cardinal property of preferences. However, in the presence of multivariate risk, where the return on investment *and* other arguments of the utility function are uncertain, the portfolio choice is affected by both properties of preferences, and the familiar separability between consumption and investment decisions breaks down. In other words, the results in this paper show that the investment decision is affected by cardinal properties of the utility function representing the investor's ordinal preferences and the ordinal preferences themselves. This finding is analogous to the results of Finkelshtain and Chalfant (1989), who showed a similar non-separability with regard to production and consumption decisions in the presence of uncertain prices. These findings are examples of the general results of Kihlstrom and Mirman (1974) and Karni (1979)—ordinal preferences play an important role in agents' attitudes toward multivariate risks.

The first attempt to analyze the effect of multivariate risks on portfolio choices was made by Li and Ziemba (1989), who characterized the choices of investors with multi-attribute utility functions facing multivariate normal risks. Using a formula from Rubinstein (1973, 1976) for the covariance between two functions of normally distributed random variables, they showed that investors with identical multivariate Rubinstein measures of risk aversion will choose the same portfolios. In addition, they showed that if the vector of "attributes" of wealth follows a distribution that is independent of the asset returns, then the univariate Rubinstein measure is sufficient for characterization of portfolio choices. Below, we use multivariate generalizations of the Arrow-Pratt absolute and relative measures of risk aversion to generalize the first result to hold for arbitrary distributions, and show that the latter result need not hold without the normality assumption. Using the multivariate measures of risk aversion, we consider comparisons between investors with different risk attitudes. In addition, we show that Ross's (1981) measure of strong aversion to univariate wealth risk, surprisingly, is sufficient to characterize portfolio choices, even when the utility function is a multiattribute one. However, the multivariate case requires either a special case of stronger risk aversion or an additional restriction on the probability distribution.

The paper proceeds as follows. In section 2, we present a model of the investor's decision problem and discuss the cases in which his objective function is equivalent to the ordinary indirect utility function defined on wealth and prices. We then describe the conditions under which it will reduce to a function of wealth alone. In section 3, we derive necessary and sufficient conditions under which portfolio choices are separable from the consumption decisions made by the investor. It is shown that these conditions are very restrictive. However, if goods' prices are independent of asset returns, separability is restored with plausible assumptions about preferences. Following that, in section 4, we establish the main results of the paper concerning investment decisions under multivariate risk. We use a multivariate risk premium and corresponding measures of risk aversion to characterize the relationship between aversion to wealth risk and portfolio choices. Finally, we show that a version of Ross's measure of risk aversion is sufficient to indicate the relationship between the level of investment in the more risky asset and the investor's degree of risk aversion.

2. Modeling Investor Behavior Under Multivariate Uncertainty

In situations involving temporal risks, aggregation over goods may be misleading, as noted by Epstein (1975), who conducted a disaggregated analysis of consumer choices under risk. Li and Ziemba (1989) augmented the usual univariate objective function, utility defined on wealth, with a vector of "attributes" of wealth, denoted by F in their paper; as they noted, an obvious choice for an element of F is a measure of the real purchasing power of the nominal value of wealth. However, disaggregation suggests an important particular choice for the vector of attributes—the vector of prices of goods consumed by the investor. This case was considered by Finkelshtain and Chalfant (1989), who made use of the entire vector of prices of consumption goods, reflecting the fact that the Composite Commodity Theorem may not hold.

The model below follows this approach and allows the formal introduction of multivariate risks into the investor's maximization problem. The particular risks which are considered are wealth and price risks; the latter seem to be the most natural example of random attributes. However, the analysis in the reminder of the paper is conducted for arbitrary vectors of attributes, with prices presumed to make up a part of the vector.

We consider a one-period model of an investor engaged in both investment and consumption. The investor makes *ex ante* choices concerning his portfolio and some of his consumption goods and *ex post* choices of other goods. Hence, there are two types of goods; those which are precommitted before the realization of prices and returns on investment, and those which are chosen when all random variables are known. We denote the N_1 goods of the first type by $Z = (Z_1, Z_2, ..., Z_{N_1}) \geq 0$ and the N_2 goods of the second type by $Q = (Q_1, Q_2, ..., Q_{N_2}) \geq 0$. The prices of these goods are denoted by $\rho = (\rho_1, \rho_2, \ldots, \rho_{N_1})$ and by $p = (p_1, p_2, \ldots, p_{N_2})$, respectively. We assume that the investor's objective function is his Von Neumann-Morgenstern utility function $u(Z, Q)$, defined over both types of goods. $u(Z, Q)$ is assumed to be continuous in Z and Q, non-decreasing, and quasi-concave for $Z, Q \geq 0$.

The investor is subject to the following constraints. Initial wealth is W_0 (assumed to be non-random) and the budget constraint is given by

$$W = \rho' Z + p' Q,$$

where total wealth is given by

$$W = W_0[\alpha x + (1-\alpha) y] = W_0[\alpha(x-y)+y]$$

and α and $1-\alpha$ are the shares of initial wealth which are invested in the assets with returns x and y, respectively. Finally, we assume that the *ex ante* knowledge of the investor concerning ρ, p, x and y can be summarized by a subjective probability distribution function $F(\rho, p, x, y)$ with finite moments.

The investor must choose α and the levels of Z and Q so as to maximize the expected value of his utility function:

$$\max_{0 \leq \alpha \leq 1;\, Z, Q \geq 0} \int u(Z, Q)\, dF(\rho, p, x, y)$$

subject to the above constraints.[1] The maximization problem can be solved in two stages—maximization with respect to Q for a given realization of prices, returns and prior choices of Z and α, and then maximization with respect to Z and α. As Epstein (1975) argued, since consumption plans for Q can be revised when prices and asset returns are realized, the first maximization problem may be taken inside the integral to obtain a revised objective function, the variable indirect utility function[2] $g(Z,s,p)$, where s represents savings—the amount of total wealth available for consumption of the goods in Q—and is defined by

$$s(W,Z,\rho) = W(x,y,\alpha) - \rho' Z.$$

The objective becomes

$$\max_{0\le\alpha\le 1;\, Z\ge 0} E[g(Z,s(W,Z,\rho),p)] = \int_\rho \int_p \int_x \int_y g(Z,s(W,Z,\rho),p)\, dF(\rho,p,x,y).$$

Without additional restrictions, the objective function $g(\cdot)$ does not reduce to the traditional objective function of investors—utility defined on wealth. Both include wealth as an argument, but the former also includes the vector of consumption goods Z and the random vectors p and ρ. The maximization problem is therefore a multivariate risk problem; the utility function depends on more than one random argument.[3]

Two assumptions are required for $g(\cdot)$ to reduce to the univariate objective function defined on wealth alone. First, all consumption decisions must take place *ex post*, so that Z and ρ can be ignored. Second, the price vector p must be known when the investment decision is made. If the first assumption holds, $g(\cdot)$ reduces to the ordinary indirect utility function $V(W,p)$. If, in addition, p is known to be fixed at some level $\hat{p}$, then the objective

[1] To avoid the possibility of bankruptcy, we assume that the choice of Z is constrained by $Pr(\rho' Z \le W) = 1$ (see also Epstein (1975)).

[2] The variable indirect utility function was investigated and used by Epstein (1975), who established its duality with the direct utility function u.

[3] Randomness of attributes other than prices would have exactly the same effect on the objective function, as is easily seen by adding other random variables of interest to the functions g and F.

function is $V(W, \hat{p})$. This, of course, may be treated as a function of wealth alone.

The second assumption seems unrealistic, since it is hard to imagine that the prices of all consumption goods (or other attributes) are known in advance. The first assumption is more plausible, since situations which involve *ex ante* consumption decisions are relatively rare. Below, we adopt the first assumption and consider the effects of relaxing the second.

In order to keep the notation close to that of Li and Ziemba (1989), we use $U(W,C)$ to denote the investor's indirect utility function defined over wealth and a vector C, of M random attributes.[4] As noted earlier, goods' prices are assumed to be included in C.[5]

3. Separability of Portfolio Choices from Consumption Decisions

We showed above that, in the presence of price risk or other random attributes, the investor's objective function need not reduce to the traditional objective function, utility defined over wealth alone. However, with certain types of preferences and/or probability distributions, the investor's portfolio choice is not affected by the introduction of the additional risks, *i.e.* the levels of α in the two environments coincide. In this section, we derive necessary and sufficient restrictions on the form of the utility function under which portfolio choices are identical under univariate risk (wealth risk alone) and under multivariate risk. Characterizing these conditions is of interest, since they imply that the familiar separability between the portfolio choice and preferences for consumption goods or other attributes, which holds under univariate risk, is preserved under multivariate risk.

The investor facing univariate wealth risk is assumed to maximize the expectation, over the joint distribution of asset returns x and y, of a utility function defined over wealth and a

[4] U is assumed to be differentiable, and we assume the existence of unique interior solutions for the investor's problem, both in the univariate and multivariate models, as well as for the choice of consumption goods.

[5] Note that it would be straightforward to include the goods in Z in the vector of attributes C, should interest be in the effects of precommitted consumption quantities, but this is beyond the scope of the paper and does not affect the results we establish.

fixed vector $\overline{C}$, taken to be the mean of the marginal distribution of C when it is random (the multivariate case). Under multivariate risk, the investor's problem is

$$\max_{0 \le \alpha \le 1} \int_x \int_y \int_C U(W, C)\, dF(C, x, y)$$

where $F(C, x, y)$ is the joint probability distribution function of C and the returns on assets. Restrictions on the form of the objective function are needed for equality between the levels of α chosen under univariate and multivariate uncertainty. This is shown in Proposition I.

Proposition I: Denote the optimal levels of α under multivariate and univariate risk as α^m and α^u, respectively. Then $\alpha^m = \alpha^u$ for all joint probability distributions of returns and attributes if and only if

$$U(W, C) \equiv U^1(W) + U^2(C).$$

The proof is presented in the Appendix. ■

For portfolio choices to be unaffected by the presence of the additional uncertainty resulting from the randomness in C, the utility function must be additively separable in W and C. If the vector C consists only of the prices of consumption goods, U reduces to the ordinary consumer's indirect utility function defined on wealth (income) and prices alone. In this case, Finkelshtain and Chalfant (1989) showed that additive separability of U, of the form in Proposition I, implies that

$$U(W, p) = log(W) - log[G(p)],$$

where $G(p)$ is linearly homogeneous in p. This form places strong restrictions on preferences. The coefficient of relative risk aversion and the income elasticities of demand for each good must all equal 1, thus excluding the cases of risk neutrality or risk seeking. It also severely limits the nature of preferences for goods.[6]

[6] If the set of probability distributions under consideration is such that some prices are deterministic, then the restrictions on preferences are somewhat less restrictive. The term for wealth or income in the indirect utility function must be separable only from the *random* prices, and income elasticities must equal R only for the goods whose prices are random.

Only investors with these preferences will choose $\alpha^m = \alpha^u$ for *all* probability distributions of returns and attributes. When the returns on assets are statistically independent from the random attributes in C, however, one is tempted to conclude that the additional risk will not affect the portfolio choice. Indeed, when these random variables have a multivariate normal distribution, Li and Ziemba (1989) showed that this is the case. However, if the normality assumption is relaxed, then α^m and α^u may differ, even under independence. Under independence, the joint density of returns and attributes equals the product of the two marginal densities (of returns and of C), so the investor solves

$$\max_{0 \le \alpha \le 1} \int_x \int_y \int_C U(W, C) g_1(C) g_2(x,y)\, dC\, dy\, dx = \max_{0 \le \alpha \le 1} \int_x \int_y E_c[U(W, C)]\, g_2(x,y)\, dy\, dx.$$

Only for a utility function $U(\cdot)$ which is linear in each element of C (*i.e.*, the investor is indifferent about instability in the attributes), can the expectation over C in the first integral yield the function the investor would maximize in the case of univariate uncertainty, $U(W, E(C))$ or $U(W, \bar{C})$. However, even if $U(\cdot)$ is nonlinear in C, so that the investor facing multivariate uncertainty has a different objective, the choice of portfolio could still be the same. This requires not only independence, but a particular form for the utility function, special cases of which are linear in attributes or wealth.

Proposition II: Given independence between C and the returns on assets, $\alpha^m = \alpha^u$ if and only if the utility function takes the form

$$U(W, C) = A(C) + B(C)D(W)$$

or

$$U(W, C) = L(W) + K(C) + \frac{1}{H(W)} \sum_{i=1}^{N} b_i C_i,$$

where each b_i is an arbitrary constant. The proof is provided in the Appendix.[7] ■

[7] Again, if C is assumed to coincide with p, more specific forms for the utility function may be derived. These are given in Finkelshtain and Chalfant (1989), along with the implications of these forms for both ordinal preferences and the corresponding cardinal properties.

The results established in this section illustrate that the presence of multivariate risks changes the optimal portfolio unless restrictions are imposed on preferences and/or the joint distribution of asset returns and attributes. Only under these restrictions is there separability between the portfolio choice and preferences for consumption goods. We turn now to characterizing portfolio choices in the presence of wealth and attribute risks, in cases where separability does not hold.

4. Portfolio Choices Under Multivariate Uncertainty

In this section, we examine in more detail the portfolio choice in the presence of randomness in wealth and attributes. The analysis requires generalization of the Arrow-Pratt notions of the risk premium and the measure of risk aversion, as well as an extension of the analogous Ross (1981) notion. After describing these measures, we turn to comparisons between investors.

4.1: A Multivariate Risk Premium and Measures of Aversion to Risk

We begin with the generalization of the Arrow-Pratt risk premium and the correspondings measure of risk aversion. Based on Karni (1979), Finkelshtain and Chalfant (1988) developed a generalization of the univariate Arrow-Pratt risk premium. They defined the wealth risk premium as the maximum amount that an individual would be willing to pay to stabilize wealth with the prices of consumption goods random. An analogous notion may be defined when other attributes are random, as well. In that case, the multivariate risk premium is denoted by $\Pi(W_0, F)$ and is defined by

$$EU(W, C) = EU(\overline{W} - \Pi, C).$$

Π depends on the particular risk, F, and on W_0 since the individual's willingness to pay might depend on initial wealth.[8]

[8] If the individual can affect the probability distribution of wealth, as in the case of the investor who chooses α, then the risk premium depends not only on F, but also on the choice variable.

The interpretation of Π as a wealth risk premium under multivariate risk is analogous to that of the regular Arrow-Pratt risk premium, which appears as a special case of Π, when goods' prices and other attributes are fixed. This can be illustrated nicely for small risks.[9] As shown in the Appendix, a Taylor approximation of the above expression yields

$$\Pi = -\frac{1}{2}\sigma^2{}_W \frac{U_{WW}}{U_W} - \sum_{i=1}^{M} \sigma_{c_i W} \frac{U_{Wc_i}}{U_W}$$

where $\sigma^2{}_W$ is the variance of W, $\sigma_{c_i W}$ is the covariance between wealth and the $i^{\underline{th}}$ element in C, and subscripts of U denote partial derivatives. The first term in this expression is the Arrow-Pratt risk premium, the amount that the investor would be willing to pay to stabilize wealth when prices and other attributes are fixed. The second term can be thought of as a monetary measure of the investor's aversion to the stochastic interaction between attribute risk and wealth risk. If C is fixed, the second term vanishes, and Π reduces to the Arrow-Pratt (univariate) risk premium. However, Π need not even be of the same sign as the Arrow-Pratt risk premium when C is random, due to the second term. Note that the covariance matrix of C does not affect Π—only terms involving wealth affect the willingness to pay for stabilizing it.

The risk premium Π suggests the following matrix measure of absolute risk aversion, when the elements in C are random:

[9] Following Karni (1979), we define small risks as those risks such that $Pr[(c_1, \ldots, c_M, W) \in b] = 1$ where b is an $M+1$–dimensional ball centered at $(\overline{c}_1, \ldots, \overline{c}_M, \overline{W})$, with radius ε which is arbitrarily close to zero. Note that, in the investor's case, the distribution of W is determined by the distribution of (x, y). For that problem, then, (x, y) will replace W in the above statement.

$$H = -\begin{bmatrix} \frac{U_{WW}}{U_W} & \frac{U_{Wc_1}}{U_W} & \cdots & \frac{U_{Wc_M}}{U_W} \\ \frac{U_{Wc_1}}{U_W} & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{U_{Wc_M}}{U_W} & 0 & \cdots & 0 \end{bmatrix}$$

H is constructed from the first row and column of Karni's (1979) matrix measure of multivariate risk aversion.[10] For small risks, the diagonal element is twice the risk premium per unit of variance of wealth, while the i^{th} off-diagonal element is the risk premium per unit of covariance between wealth and the i^{th} attribute. Every term of the form

$$\frac{U_{c_i c_j}}{U_W}$$

is replaced with 0; such terms do not affect the risk premium, and, as is illustrated below, they also do not affect the portfolio choice.

Finkelshtain and Chalfant [1988] showed that $\Pi \geq 0$ for all risks if and only if H is positive semi-definite, which implies that the utility function is additively separable in C and W, as in Proposition I. Moreover, if i and j denote two investors, then under small risks,[11] a positive semi-definite matrix $D = H^i - H^j$ is both necessary and sufficient for $\Pi^i \geq \Pi^j$ for all risks. We state this as Proposition III.

Proposition III: A positive semi-definite (p.s.d.) difference $D = H^i(\overline{W},\overline{C}) - H^j(\overline{W},\overline{C})$ is necessary and sufficient for $\Pi^i \geq \Pi^j$ for all small risks with mean $(\overline{W},\overline{C})$.

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[10] Karni's definition was specified in terms of the standard indirect utility function, but easily generalizes to $U(W,C)$, with attributes other than prices included.

[11] Any of the results we obtain for small risks would be exact for quadratic utility functions. However, we refer only to small risks, given the undesirable properties of such utility functions.

Proof: From the expression derived for Π and the definition of H, it can be verified that for a given small risk with mean $(\overline{W},\overline{C})$,

$$\Pi^i - \Pi^j = \frac{1}{2} tr\,[D\,\Omega]$$

where Ω is the covariance matrix of wealth and attributes. Sufficiency follows from the fact that the covariance matrix is p.s.d. and the product of p.s.d. matrices has a non-negative trace. Thus, a p.s.d. D is sufficient for a non-negative $\Pi^i - \Pi^j$.

To prove necessity, we choose the probability distribution so that the correlation coefficients corresponding to Ω are either 1 or -1. Under this assumption

$$\Pi^i - \Pi^j \geq 0$$

implies that

$$tr\,[D\,\Omega] = tr\,[Dvv'] = tr\,[v'Dv] = v'Dv \geq 0,$$

where v is a vector of standard deviations, premultiplied by either 1 or -1, depending on the signs of the covariances of the specific argument with the rest of the arguments. Since the elements in v can take any values,[12] the above inequality holds for all vectors v and thus implies that D is p.s.d.[13] ■

In cases where a riskless asset does not exist, the Arrow-Pratt measures are not sufficient to rank choices. Ross (1981) showed that a stronger notion of risk aversion is required to characterize investors' choices in such cases. By his definition, a univariate utility function $A(W)$ is *strongly* more averse to wealth risk than a utility function $B(W)$ if and only if there is a $\lambda \geq 0$ such that, for any two levels of wealth W_1 and W_2,

$$\frac{A''(W_1)}{B''(W_1)} \geq \lambda \geq \frac{A'(W_2)}{B'(W_2)},$$

[12] Some of its elements can be equal to zero if the corresponding random variable has a degenerate distribution.

[13] Note that a positive semi-definite difference is a strong condition—the off-diagonal terms in the two matrices must be equal and the first diagonal terms (the Arrow-Pratt measures) differ by a positive amount.

where $'$ and $''$ denote first and second derivatives. Ross showed that this property is equivalent to

$$A(W) = \lambda B(W) + G(W),$$

where $\lambda>0$, $G' \leq 0$, and $G'' \leq 0$ (Ross (1981, p. 626)).

This definition can be extended naturally to multiattribute utility functions in the following manner. An individual with utility function $U(W,C)$ is said to be strongly more averse to wealth risk than one with utility function $V(W,C)$, *for any level of* C, if and only if there is a $\lambda > 0$ such that, for every W_1, W_2, and C,

$$\frac{U_{WW}(W_1,C)}{V_{WW}(W_1,C)} \geq \lambda \geq \frac{U_W(W_2,C)}{V_W(W_2,C)}.$$

This relationship holds if and only if

$$U(W,C) = \lambda V(W,C) + G(W,C),$$

and again, $G_W \leq 0$, and $G_{WW} \leq 0$. It can be shown that for every level of C, U would be willing to pay a higher amount to stabilize wealth risk than V, and if $E(x-y) \geq 0$, U will invest less in the more risky asset.

By fixing C, the problem remains one of univariate risk. However, if interest is in comparisons when C is stochastic, so that the investor faces multivariate risk, the above definition must be modified or the set of probability distributions under consideration must be further restricted. In the next section we show that, in the presence of both wealth and attribute risk, the form of the above relationship which permits comparisons with no further restrictions on the distribution is the case where G is additively separable in W and C. This leads us to the following definition.

Definition: We say that U is strongly more averse to wealth risk than V and has the same attitude toward covariation of W and C (in the sense that the partial derivatives U_{Wc_i} and V_{Wc_i} are proportional for each attribute C_i) if and only if

$$U(W,C)=\lambda V(W,C)+G(W)+H(C),$$

with $G' \leq 0$, $G'' \leq 0$. We now turn to using these definitions in comparing the portfolio choices of investors facing multivariate risk.

4.2 Comparisons Between Investors: General Preferences and Distributions

The necessary condition for the investor's optimal choice of α is

$$E[U_W(x-y)]=0.$$

The sufficient condition holds for the case of an individual who is averse to univariate wealth risk, since $U_{WW} < 0$ implies that

$$E[U_{WW}(x-y)^2]<0.$$

Therefore, in the remainder of the analysis, we assume that the investor is averse to wealth risk in the univariate sense (*i.e.* $U_{WW} < 0$).

We begin with a generalization of the Li and Ziemba (1989) result concerning individuals with identical measures of risk aversion facing multivariate normal risks. In Proposition IV, we show that the measure of aversion to multivariate wealth risk defined in section 4.1 reveals when two individuals facing the same arbitrary multivariate risks will choose the same optimal portfolio.

Proposition IV: Let H^i and H^j be the risk aversion measures defined in section 4.1, of individuals i and j, respectively, and assume that the following conditions hold:

(i) The two investors face the same joint distribution of returns and attributes.

(ii) The two investors have the same levels of initial wealth.

Then, $H_i = H_j$ globally (for all values of W and C) is both necessary and sufficient for

$$\alpha_i^* = \alpha_j^*$$

for all distributions of prices and attributes, where α_i^* and α_j^* are the optimal levels of α of investors i and j, respectively. The proof is presented in the Appendix. ■

An interesting observation is that, for $H^i \equiv H^j$, the two investors need not have the same ordinal preferences.[14] This is demonstrated easily by noting that, for the preferences used to prove Proposition IV, Roy's Identity implies that the two investors need not have the same demand functions. Thus, only the attitude toward wealth variation and the covariation between wealth and attributes affects the portfolio choice, while the attitude toward variation in consumption prices or other attributes alone is not relevant. The latter is determined by the sub-matrix of second partial derivatives of the indirect utility function with respect to attributes. This sub-matrix is part of Karni's (1979) measure of multivariate risk aversion, but does not affect the choice of portfolio, thus illustrating why it is omitted from our multivariate measure of aversion to wealth risk.

We turn now to comparisons between individuals with different attitudes towards multivariate risk. We derive a Ross (1981) type result concerning the relationship between two investors' utility functions and their corresponding levels of α. In the next section, analogous results are derived under mean-variance analysis, using the measure H. Following Ross (1981), we reformulate the portfolio problem as a problem of investment in one risky asset, $x - y \equiv z$. Recall that

$$W = W_0[\alpha x + (1-\alpha)y] = W_0 y + \alpha W_0(x-y).$$

The choice of α can be thought of as the choice to move a portion of wealth from an asset paying random amount y to one yielding $(x-y)$. $W_0 y$ is random initial wealth, then, in the sense that it is the uncertain amount carried forward if α equals zero. Following Ross (1981), we assume that $E(z \mid W_0 y) \geq 0$, *i.e.* the expected return of the asset, given any level of initial wealth, is positive.

[14] On the other hand, if the assumptions in Proposition IV are strengthened, so that comparisons are restricted to investors who have the same ordinal preferences, then equality of the univariate Arrow-Pratt measures of aversion to risk is sufficient to ensure that two investors will choose the same portfolio, even under multivariate risk. This is a particular case of the above Proposition, where one utility function is an affine transformation of the other.

Proposition V: Let U^i and U^j be utility functions such that U^i is strongly more risk averse than U^j, while both have the same attitude toward covariation of wealth and attributes:

$$U^i(W,C) = \lambda U^j(W,C) + G(W) + H(C), \qquad G' \leq 0, G'' \leq 0.$$

If $E(z \mid W_0 y) \geq 0$, then $\alpha^i \leq \alpha^j$ for all risks.

Proof: The proof is the same as in Ross (1981), and is repeated for completeness. Evaluate the first order condition for investor i at the optimal value of α of investor j:

$$E[U_I{}^i \cdot z] = E\{[\lambda U_I{}^j + G'] \cdot z\} = E[G' \cdot z],$$

where the last equality holds since, at the optimal level of α for investor j, $E[U_I{}^j \cdot z] = 0$. To see that the last expression is negative, note that

$$E[(G' \cdot z) \mid W_0 y] = Cov[G', z \mid W_0 y] + E[G' \mid W_0 y] \cdot E[z \mid W_0 y].$$

The product of the two conditional expectations on the right hand side is clearly negative. For a given y, z is random only through x, but G' is decreasing in x. Hence, the conditional covariance is negative. Therefore, the conditional expectation

$$E[(G' \cdot z) \mid W_0 y]$$

is negative for any y. This implies that the unconditional expectation

$$E[G' \cdot z]$$

is negative. The first order condition is decreasing in α (an immediate implication of the second order condition), so it must be the case that $\alpha^i \leq \alpha^j$, because investor i has an incentive to reduce α at the point α_j. ■

In Proposition V it was shown that, with the assumption that $E(x-y) \geq 0$, investors that are strongly more averse to wealth risk will choose a lower α. However, this result requires that the two investors have the same attitude toward covariation of wealth and other attributes. To avoid this, one can further restrict the set of probability distributions under consideration. A natural extension of Ross's assumption about the distribution of (x,y) is that $E(x-y \mid y, C) \geq 0$, *i.e.*, the expected return of the asset z, given any level of initial wealth and attributes, is positive. Under this assumption, an analogous proposition may be derived and the assumption that G is a function of wealth alone can be relaxed. This, of course, will allow comparisons over a larger set of preferences.

4.3. Comparisons Between Investors: Mean-Variance Analysis

Li and Ziemba (1989) derived their useful results under the assumption that the distribution of attributes and returns is multivariate normal. They noted that a possible extension of their study is its application to other situations where mean-variance analysis is valid. An example is where the investor faces small risk, which may be plausible if investors have the flexibility to rebalance their portfolios frequently enough. Under this assumption, a second order Taylor series expansion of expected utility is exact and facilitates the analysis below. If the normality assumption is valid, then analogous results can be derived using the Rubinstein measure of multivariate risk aversion (Li and Ziemba (1989)), instead of the corresponding multivariate Arrow-Pratt measure.

The portfolio problem to be analyzed is the choice between one risky and one risk-free asset. In this case, we show that either the Rubinstein multivariate measure which was introduced by Li and Ziemba (1989) or the one in section 4.1 is sufficient to characterize the optimal portfolios of investors with different attitudes toward wealth risk, under multivariate risk.[15]

Consider the following portfolio problem of an investor:

$$\max_{0 \le \alpha \le 1} \; E\{U[W_0(\alpha(x-r)+r)), C]\},$$

where α is the share of initial wealth (W_0) allocated to the risky investment with return x, and $1-\alpha$ is the share allocated to the risk-free investment with return r.

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[15] A second problem for which the same approach can be applied is where the investor must choose between two risky assets and there is no risk-free asset. However, ordering the level of investment in the more risky asset according to the risk aversion measure requires additional restrictions on the joint distribution of returns and attributes. An example is that $Cov(x,y) \ge Var(y)$.

Proposition VI: Let H^i and H^j be the multivariate measures of risk aversion defined in section 4.1 of investors i and j. Assume that two investors face the same joint distribution of x and C, that they have the same levels of initial wealth, and that $E(x) \geq r$. Then a positive semi-definite (p.s.d) difference

$$H^i(\overline{W},\overline{C}) - H^j(\overline{W},\overline{C})$$

implies the following equivalent statements:

(i) $\alpha^i \leq \alpha^j$ for all small risks with mean $(\overline{x},\overline{C})$,

where α^i and α^j denote the optimal levels of α of investors i and j;

(ii) $E(P^i) \leq E(P^j)$

where P denotes the return on the optimal portfolio for each investor; and

(iii) $Var(P^i) \leq Var(P^j)$.

Proof: Using the assumption of small risks, a Taylor expansion of the first-order conditions for the above portfolio problem yields

$$\frac{1}{2} tr(\hat{H}\Omega) = (\overline{x} - r),$$

where Ω is the covariance matrix of (x, C) and $\hat{H}$ is identical to H except that the first diagonal element is multiplied by $2W_0\alpha$. Note that, from the definition of H, it follows that a p.s.d. difference $H^i - H^j$ is equivalent to p.s.d. $\hat{H}^i - \hat{H}^j$. We now evaluate the first-order condition of investor i at α^j:

$$-\frac{1}{2} tr[\hat{H}^i(\alpha^j)\Omega] + (\overline{x} - r) = -\frac{1}{2} tr\{[\hat{H}^j(\alpha^j) + D]\Omega\} + (\overline{x} - r),$$

where D is p.s.d. by assumption. Since both Ω and D are p.s.d., the trace of their product must be non-negative. The remaining terms in the above expression are identical to the first-order condition for investor j and hence vanish. Therefore, the first order condition of investor i evaluated at α^j is negative and (i) implies (ii). Since $E(x) \geq r$, (ii) and (iii) are equivalent. Moreover, since r is risk-free, it is easy to see that (ii) is equivalent to (iv). □

Propositions III and VI established that an investor who is more averse to wealth risk in the present of attribute risk will pay a higher risk premium to stabilize this risk and will invest a smaller amount in the risky asset. These results generalize Pratt's (1964) result, which was obtained under wealth risk alone. However, in the multivariate case, unlike Pratt's univariate case the relationship "more risk averse" seems to be sufficient but not necessary to ensure that one investor will invest less in the risky asset.

Propositions III to VI compared the behavior of two investors. A second approach that has been taken in the literature (Arrow (1964), Li and Ziemba (1989)) is comparing the positions of the same individual at different levels of initial wealth. Below we use the portfolio problem of Proposition VI to examine the relationship between how initial wealth changes the measure H and how it changes the level of investment in the risky asset.

Definition: The multivariate measure of absolute risk aversion is decreasing (constant, increasing) in W_0 at (W_0, C_0) if the matrix

$$\frac{\partial H(W_0, C_0)}{\partial W_0}$$

is negative semi-definite (the null matrix, positive semi-definite).

Proposition VII: Recall the portfolio problem of Proposition VI. Let a denote the absolute amount invested in the risky asset ($a \equiv \alpha W_0$). A decreasing [constant] [increasing] multivariate measure of absolute risk aversion implies the following equivalent statements:

(i) $\dfrac{\partial a}{\partial W_0} \geq [=][\leq] 0$

(ii) $\dfrac{\partial E(P)}{\partial W_0} \geq [=][\leq] 0,$

where P is the return of the optimal portfolio;

(iii) $\dfrac{\partial Var(P)}{\partial W_0} \geq [=][\leq] 0.$

Proof: Using the Taylor expansion of the first order condition we obtain

$$\frac{\partial a}{\partial W_0} = \frac{\frac{1}{2} tr\left[\frac{\partial \overline{H}}{\partial W_0}\Omega\right]}{d},$$

where d is the second order condition, negative by assumption, and $\overline{H}$ is identical to H, except that the first diagonal element is premultiplied by a. A decreasing (constant, increasing) measure of multivariate risk aversion is sufficient to ensure that the numerator is negative for any small risk, and, hence, is sufficient for α to increase [constant] [decrease] with the level of initial wealth. Since $E(x) \geq r$ and r is risk-free, the equivalence of (i) with (ii) and (iii) is trivial.[16] ■

6. Conclusions

This paper has examined portfolio choices in the presence of randomness of other attributes of the utility function. Investors were assumed to face multivariate risk consisting of an uncertain wealth, uncertain prices of goods consumed, and possibly other risky attributes. For the case where all of the risky attributes are prices, we formally introduced multivariate risk into the investor's maximization problem and showed that the indirect utility function of the investor replaces the typical investor's objective function defined on wealth alone.

Necessary and sufficient conditions were derived under which the portfolio is not affected by the introduction of the additional attribute risk. It was shown that the assumption of statistically independent attributes and returns is not sufficient to ensure this result. Additional restrictions on the utility function are needed. If these do not hold, the presence of attribute risk affects the portfolio choice and the separability between investment behavior and preferences for goods breaks down.

[16] An analogous proposition may be derived concerning the proportion of initial wealth which is invested in the risky asset. This requires analogous assumptions about the multivariate measure of relative risk aversion which is identical to H except that the diagonal element is premultiplied by W_0.

A multivariate risk premium and corresponding measures of risk aversion were used to characterize the optimal portfolios of investors. Only under the special assumptions of additively separable objective functions or investors with the same ordinal preferences for goods and attributes will the Arrow-Pratt risk aversion measures be sufficient to determine the relationship between the levels of investment in the risky asset and the degree of aversion to risk, when multivariate risks are present. Yet, the multivariate generalizations of these measures are useful for characterizing optimal portfolios and making comparisons between investors without these restrictions. Investors with identical multivariate measures of risk aversion will choose identical portfolios, no matter what is the joint distribution of returns and attributes. If mean-variance analysis is valid, a positive semi-definite difference between investors' multivariate measures of risk aversion implies that the investor with the "larger" measure would invest less in the risky asset. In addition, if the multivariate measure of absolute risk aversion is decreasing with wealth, the amount which is invested in the risky asset increases with wealth. Finally, if choices are between two risky assets, then an investor with a larger strong measure of risk aversion than the corresponding measure of a second investor, but with the same attitude toward covariation of wealth and attributes, will choose a less risky portfolio.

APPENDIX

Proof of Proposition I: (Sufficient) Let $f(C,x,y)$ denote the joint probability density function of C, x, and y, and let $f_i(\cdot)$ denote the corresponding marginal densities (of returns and of attributes). If the restriction on preferences holds, the maximization problem becomes[17]

$$\max_{0\le\alpha\le 1} \iiint_{xyC} [U^1(W) + U^2(C)] f(C,y,x)\, dC\, dy\, dx \;=\; \max_{0\le\alpha\le 1} E_{x,y}[U^1(W)] + E_c[U^2(C)] .$$

The solution to the above maximization is identical to that of the univariate problem

$$\max_{0\le\alpha\le 1} E_{x,y}[U^1(W)] \;+\; U^2(\bar{C}),$$

since the objective functions differ only by terms that are constant with respect to α.

(Necessary) We assume two particular distributions, G^1 and G^2. Let G^1 be given by

$$(x,y) = (x^0,y^0) \text{ and } C=C^0 \quad \text{with probability} \quad ½$$

and

$$(x,y) = (x^1,y^1) \text{ and } C=C^1 \quad \text{with probability} \quad ½,$$

where C^0 and C^1 differ only by the fact that a particular element of C takes on values C_i^0 or C_i^1, respectively. Let each 0 superscript denote a low value and each 1 superscript denote a high value. Also, assume that the two values for C_i are each $\Delta/2$ away from its mean, given by the $i^{\underline{th}}$ element in $\bar{C}$. Let the second distribution, G^2, be given by

$$(x,y) = (x^0,y^0) \text{ and } C=C^1 \quad \text{with probability} \quad ½$$

and

$$(x,y) = (x^1,y^1) \text{ and } C=C^0 \quad \text{with probability} \quad ½,$$

For both G^1 and G^2, all attributes except C_i are assumed to remain fixed at the levels defined

[17] The proof of this part is given for the case where the relevant distribution function is assumed to have proper densities. The proof of the discrete case would be similar. This comment is applicable to proofs of other propositions as well.

by $\overline{C}$. When (x, y) is low, wealth equals $W^0 = W(x^0, y^0)$ and when it is high, wealth is $W^1 = W(x^1, y^1)$. The necessary conditions for maximization of expected utility under G^1 and G^2 are

$$U_W(W^0, C^0)\,(x^0 - y^0) \;+\; U_W(W^1, C^1)\,(x^1 - y^1) = 0$$

and

$$U_W(W^0, C^1)\,(x^0 - y^0) \;+\; U_W(W^1, C^0)\,(x^1 - y^1) = 0$$

respectively. Assuming that $U_W(\cdot) > 0$, it follows from these that $(x^0 - y^0)$ and $(x^1 - y^1)$ have different signs. Subtracting the second condition from the first and rearranging, we find that

$$(x^0 - y^0)[U_W(W^0, \overline{C}_i + \frac{\Delta}{2}, \overline{C}_{j\neq i}) \;-\; U_W(W^0, \overline{C}_i - \frac{\Delta}{2}, \overline{C}_{j\neq i})]$$

$$= \;(x^1 - y^1)[U_W(W^1, \overline{C}_i + \frac{\Delta}{2}, \overline{C}_{j\neq i}) \;-\; U_W(W^1, \overline{C}_i - \frac{\Delta}{2}, \overline{C}_{j\neq i})]$$

where $\overline{C}_{j\neq i}$ denotes the attributes in $\overline{C}$ that are assumed constant. Dividing both sides of the equation by Δ and letting it approach 0, we obtain the following partial derivatives as the limits of both sides of the above equation:

$$U_{Wc_i}(W^0, \overline{C})\,(x^0 - y^0) = U_{Wc_i}(W^1, \overline{C})\,(x^1 - y^0).$$

Recalling that $sign\,(x^0 - y^0) \neq sign\,(x^1 - y^1)$, this result implies that either the two cross derivatives have different signs, or that they are equal to zero for every value of W^0 and W^1. The former could not occur for choices of W^0 and W^1 arbitrarily close together, given that U_W is continuous. Thus, the latter alternative must hold, and $U_{Wc_i}(\cdot) \equiv 0$. We can repeat the above argument for each $i = 1, ..., N$, hence $U_{Wc_i}(\cdot) \equiv 0$ for each i, which implies the separable form above. ■

Proof of Proposition II: (Sufficient) Using the first form for U and the independence of C and (x, y), the investor's maximization problem is

$$\max_{0 \leq \alpha \leq 1} \quad E[A(C)] + E[B(C)]\, E[D(W(\alpha, x, y))].$$

Monotonicity of U in W, for every C, implies that the sign of $B(C)$ is the same for every vector C, and hence, the maximization problem above is equivalent to the univariate problem

$$\max_{0 \leq \alpha \leq 1} \quad [A(\bar{C})] + [B(\bar{C})]\, E[D(W(\alpha, x, y))].$$

The proof of the sufficiency of the second form is similar.

(Necessary) We establish this using a particular distribution for attributes and returns. Let (x, y) equal either $(x^l - y^l)$ or $(x^h - y^h)$, each occurring with probability 1/2. Wealth then will be either $W^l \equiv W(\alpha, (x^l - y^l))$ or $W^h \equiv W(\alpha, (x^h - y^h))$. Similarly, let C equal either C^l or C^h, each occurring with probability 1/2, where *all* attributes are now assumed to vary between low and high levels.

The necessary conditions for the multivariate and univariate maximization problems are:

$$\frac{(x^l - y^l)}{(x^h - y^h)} = - \frac{U_W(W^h, C^h) + U_W(W^h, C^l)}{U_W(W^l, C^h) + U_W(W^l, C^l)}$$

and

$$\frac{(x^l - y^l)}{(x^h - y^h)} = - \frac{U_W(W^h, \bar{C})}{U_W(W^l, \bar{C})}.$$

If the levels of α are equal, then the left-hand sides of the two conditions are the same, so we can rearrange the two right-hand sides, to obtain

$$\frac{U_W(W^h, C^l) + U_W(W^h, C^h)}{U_W(W^h, mean(C^h, C^l))} = \frac{U_W(W^l, C^l) + U_W(W^l, C^h)}{U_W(W^l, mean(C^h, C^l))}.$$

Note that these ratios, which we designate $r(W\ , C^l, C^h)$, must not depend on W, since for any arbitrary choice of W^h and W^l, $r(W^h, C^l, C^h) = r(W^l, C^l, C^h)$. If r does not depend on W, while its components, the derivatives of U do so,[18] then the numerator and

[18] It might be that $U(W, C)$ is linear in W, in which case U_W is independent of W, so r itself is independent of W. However, this is simply a special case of the form $A(C) + B(C) \cdot D(W)$, where $D(W) \equiv W$.

denominator must have a common factor, such that any term involving W cancels:

$$r(W, C^l, C^h) = \frac{\kappa(W, C^l, C^h)\,[\lambda(C^l) + \zeta(C^h)]}{\kappa(W, C^l, C^h)\,[\phi(C^l, C^h)]}$$

where

(i) $\kappa(W, C^l, C^h)\phi(C^l, C^h) = U_W(W, mean(C^h, C^l))$,

(ii) $\kappa(W, C^l, C^h)\lambda(C^l) = U_W(W, C^l)$,

and

(iii) $\kappa(W, C^l, C^h)\zeta(C^h) = U_W(W, C^h)$.

Condition (ii) implies that κ does not contain C^h, while condition (iii) implies that it does not contain C^l, and hence from (i) it follows that U_W is of the form $B(C) \cdot D'(W)$. Integrating with respect to W yields the form

$$U(W,C) = A(C) + B(C) \cdot D(W). \ \blacksquare$$

Derivation of Π: Π is defined by

$$E[U(W, C)] = E[U(\overline{W} - \Pi, C)],$$

where $\overline{W}$ is the expected value of the wealth. By a second-order Taylor expansion of the left hand side of the equation around the point $(\overline{W}, \overline{C})$,

$$EU(W, C) \ = \ U(\overline{W}, \overline{C}) \ + \ \frac{1}{2}\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\sigma_{ij}U_{ij} \ + \ o(tr\Psi)$$

where Ψ is the covariance matrix of the arguments of U. A second order approximation of the right hand side yields

$$EU(\overline{W} - \Pi, C) \ = \ U(\overline{W}, \overline{C}) \ - \ \Pi \cdot U_1(\overline{W}, \overline{C}) + \frac{1}{2}\sum_{i=2}^{N+1}\sum_{j=2}^{N+1}\sigma_{ij}U_{ij} + o(tr\Psi^1)$$

Another possibility is that U_W is linear in C, which is the second form that was given in the Proposition.

where Ψ^1 is the covariance matrix of the attributes alone. We ignore terms that contain Π^2 since those are of the same order as the remainders. Now, by the assumption that the risk is small,[19] we can ignore the reminder terms, and by setting the two expressions equal, we can solve for the required representation of Π. ■

Proof of Proposition IV: (Sufficient) It is easy to see that $H_i = H_j$, globally, if and only if the corresponding utility functions are of the forms

$$U^i(W, C) = \Phi(W, C) + a^i(C)$$

$$U^j(W, C) = \lambda\Phi(W, C) + a^j(C)$$

Assuming that both i and j have the same initial wealth, the two would have an identical first order condition and hence would choose the same α.

(Necessary) Necessity can be proved as follows. First, note that a particular case of a joint distribution of (x, y, C) is the case where the vector C is constant. A necessary condition for the two investors to choose the same levels of α for any such risk is that their Arrow-Pratt measures will coincide (Pratt (1965)). For this to hold for any level of W and C, the corresponding wealth derivatives of the utility functions must be proportional. Thus,

$$U_W{}^i(W, C) = \lambda U_W{}^j(W, C).$$

Integration of the above equation with respect to W yields the required result. ■

[19] See Karni's definition above.

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