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An Animated Instructional Module for Teaching Production Economics with the Aid of 3-D Graphics

by

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An Animated Instructional Module for Teaching

Production Economics with the Aid of 3-D Graphics

Research involving computer graphics has suggested considerable potential in instructional applications. Despite this, computer graphics has not been widely employed in instruction. Over a decade ago, Debertin, Pagoulatos and Bradford (1977a and 1977b) illustrated a variety of production functions using three-dimensional computer graphics. Bay and Schoney applied three-dimensional graphics to an empirical data set. Three-dimensional computer graphics appear in texts by Debertin and by Beattie and Taylor and in recent articles by Fawson, Shumway and Basmann and by Gustafson. Debertin (1985) used the graphics capabilities of the Statistical Analysis System (*SAS*) to derive two-dimensional graphics of total and marginal product and cost curves, and illustrated the "dual" linkages between production and cost. Recently, Debertin and Jones explained how computer graphics could be employed to teach a freshman-level agricultural economics course. Debertin, Pagoulatos and Bradford (1991) showed how three-dimensional computer graphics could be used to illustrate a variety of constrained optimization problems.

This paper pushes beyond past efforts in employing computer graphics in a number of ways. In this paper, the capabilities of the *SAS* for drawing surfaces and contours of three-dimensional functional forms are combined with the drawing, annotation and sequential animation features of Harvard Graphics in order to obtain animated sequences containing both three-dimensional production surfaces and two-dimensional contour maps. In essence, key features of the two software packages have been combined in order to obtain graphic images not possible with either of the two software packages alone and to generate animated sequences of graphics images that were previously impossible to do. Harvard Graphics also permits merger of two or more of the 3-D images generated by *SAS* into the same drawing (for example, a production surface and a price hyperplane) and therefore makes it possible to illustrate concepts such as ridge lines, pseudo-scale lines, and high-profit points using 2- and 3-D graphics.

Unlike the presentations developed in Debertin and Jones that were designed for a freshman-level course, this instructional module is intended to supplement instruction in courses in production economics at the advanced undergraduate and beginning graduate level. The instructional module is based upon what is usually regarded as the "classical" (or perhaps *neo*-classical) presentation of the two-factor, one-output model, employing a mathematical production function largely consistent with diagrams appearing in textbook presentations of the two-factor model. [Textbook diagrams drawn by technical artists, although in general agreement with regard to the basic features to be included in the graphical presentation of the two-input, one-output, factor-factor model, differ in numerous details. Compare illustrations contained in Heady, Chapter 6; Debertin,

Chapter 5; Mansfield, pp. 154-8; Beattie and Taylor, pp 31-2; Doll and Orazem; and Leftwich in chapters dealing with production economics in various editions].

PROC G3D and PROC GCONTOUR in *SAS* provide excellent capabilities for plotting three-dimensional surfaces and for generating two-dimensional contour maps for illustrating economic concepts such as isoquants and indifference curves. However *SAS* is limited in its ability to modify the illustrations, and even simple annotations are difficult to do. Harvard Graphics has no capabilities for illustrating 3-D surfaces or drawing contour maps for two-input functions. However, it can modify a drawing if the drawing containing the file is compatible, and it is very flexible in permitting annotations of drawings to highlight key elements of production theory.

Further, *SAS* makes no provision for overlaying more then one surface on the same axis, but techniques in Harvard Graphics permit more than one surface to be overlaid on the same diagram. This provision is critical for locating a point of profit maximization on the production surface.

Technical Considerations

The key to utilizing more than one graphics software package in combination is that the graphics files be compatible between programs. There are two basic types of graphics files--bit map files and vector files.

Bit-map files, sometimes also called raster files, consist of the information the computer needs to recreate a video screen using pixels (rectangles) of colors consistent with the chosen resolution level. For example, at the VGA level of resolution, a bit map file contains the information needed to produce an image 480 pixels high by 640 pixels long. The EGA image is 350 x 640; XVGA (in early versions referred to as 8514/A) is 1024 x 768, and so on. One problem with bit mapped images is that, when printed, they are still at the screen resolution, far courser than the 300 dots per inch resolution of office laser printers. Another problem with bit-map files is that they can get extremely large, up to a megabyte or more at high resolution levels. This puts extraordinary demands on hard disk storage space and otherwise limits their usefulness in developing sequences of graphs for instructional purposes.

Modern graphics programs usually instead rely on vector-based graphics languages for creating images. Instead of storing information as a bit-mapped file, coordinates and other information about the graphics image to be displayed, plotted or printed are stored. When the image is brought to a screen or is printed, the program containing the coordinate information is executed, and the image generated. The size of the vector-based files depends primarily on the complexity of the graph being created, but vector-based files suitable for representing economic relationships are usually somewhat smaller than are needed to store the equivalent bit-map image. There are several other advantages to vector-based files. First, the vectors can be manipulated on-screen as

individual entities, making possible simple modifications of entities such as color changes or line widths. But the most important advantage is that hard-copy images can be produced limited only by the resolution of the printer or plotter being driven. This results in high-quality graphics output, much higher than can be obtained from bit-map files, even those generated at high resolutions.

The primary disadvantage of vector-based files is that there are many different file formats and languages and there has been little standardization across software packages. Furthermore, individual software developers often make modifications to the basic vector languages to suit their own needs. Harvard Graphics uses a vector language called CGM, which is one of the older and more widely used vector-based graphics languages. But Harvard Graphics has also made modifications to the CGM language. Version 2.3 (and later versions) of Harvard Graphics is capable of importing what they term a CGM metafile, a file containing CGM commands generated by another software program.

Fortunately, *SAS* is also capable of generating a CGM file compatible with Harvard Graphics. In versions 6.04 and later of PC-*SAS*, a *SAS*-supplied output device driver called CGMHG writes to a disk a CGM file that can be imported into Harvard Graphics. This is the key element that makes it possible to combine multiple 3-D graphs generated with *SAS* and permits extensive modification of vectors and annotations to the graphs generated by this technology. A file called GRAPH.GSF is written to disk each time a program using the CGMHG device driver is run. This file can then be renamed to {filename}.CGM in DOS, and the *SAS* program rerun to generate additional graphs for a sequence. The size of GRAPH.GSF cannot exceed 32 K. This limits the grid for surface plots, but for most illustrations of production surfaces, a 40 x 40 grid generates adequate detail in the surface and a file size considerably less than 32 K.

With the exception of Figure 1, graphs contained in this paper were generated using the combination of software packages described above. First, a *SAS* program for generating a 3-D graph or 2-D contour plot is run using CGMHG as the output device driver. Then the renamed file is imported into Harvard graphics as a CGM metafile. This file is then saved as a Harvard Graphics chart. The chart thus created can be modified and annotated with either the draw-annotate provisions of Harvard Graphics or the facilities of the Drawpartner module that runs in conjunction with Harvard Graphics v. 2.3 (and is integrated into later versions). Finally, sequences of graphs are developed to construct the animated instructional module using an approach similar to that described in Debertin and Jones. A copy of the instructional module described in this paper for display in color on an IBM-compatible computer with an EGA or VGA card on a high density (1.44 Mb) 3 1/2 inch disk is available at no charge by contacting the author.

The 3-D Production Function

To illustrate the application of the technologies outlined above, a third-degree polynomial with parameters for a three-stage production function (consistent with the basic geometry of the "neoclassical" textbook two-factor model was chosen. The function is

$$y = x_1 + 1.5x_1^2 - 0.05x_1^3 + x_2 + 1.5x_2^2 - 0.05x_2^3 + 0.4x_1x_2,$$

where y is an output; x_1 and x_2 are inputs (Debertin, Pagoulatos and Bradford, 1977, p. 46). Griffin, Montgomery and Rister discuss selection criteria for choosing an appropriate functional form for representing production relationships. Nearly any production (cost, or revenue) function of interest to the instructor could be used. This one was chosen because it broadly conforms to "textbook" illustrations and also possess a region near the origin where isoquants are concave, not convex to the origin (Debertin, Pagoulatos and Bradford, 1991) yielding some excellent opportunities for illustrating the differences between minimization and maximization under constrained optimization. A numerical solution (employing PC-*SAS*) reveals that this function possesses a finite maximum output of 237.21 corresponding to input levels for both x_1 and x_2 of 16.41, where ridge lines intersect and the marginal products for both inputs are zero.

A Solid (Bit-Map) Rendering of the Production Surface

Figure 1 illustrates the surface of this function as viewed from the horizontal (x_1) axis using technologies employed for bit-map display of three-dimensional solids in a Computer Assisted Design (CAD) Package. The CAD package employed was Personal Designer. This package is designed to be run on computers running at XVGA (8514/A) resolution levels of 1024 x 768 or higher, and this image was generated on a computer capable of the 1024 x 768 resolution. Graphics boards capable of generating high resolution levels usually are also capable of displaying large numbers of colors simultaneously, and these subtle color shadings in large measure make the surface appear to be 3-D. The results when displayed on a high-resolution monitor are far more impressive that is possible to illustrate on paper.

In an instructional setting there remain a number of difficulties in working with solid renderings of 3-D surfaces at high resolution. First, CAD programs capable of such renderings are not user-friendly. Furthermore, since the images must be bit-map, not vector files, in order to achieve subtle shadings for a 3-D effect, they tend to be large files. File size increases (approximately) with the product (not the sum) of the resolution values. The surface illustrated here at 1024 x 768 is a file of approximately 800 K. Experiments run at the 1280 x 1024 resolution level often generate bit-map image files of a single surface in excess of 2 megabytes. These large file sizes present few problems if but one or two images are to be displayed, but would present major storage problems if many images are to be used in a sequence.

The second problem is with the display and hard copy recovery of such images. While high resolution graphics cards and monitors have declined somewhat in price, they are still expensive. A more serious difficulty is with the hard-copy recovery of such images. The computer program Pizazz +Plus was used to recover the 1024 x 768 (8514/A-XVGA) screen file using a gray scale assignment and printed on a laser printer to generate Figure 1. (Pizazz recovers 1024 x 768 images, but I have found no PC-based computer program capable of recovering and printing the screen images at the 1280 x 1024 resolution levels.) The slight "patchiness" in the rendering is not visible on screen, and is probably due to limitations of current-generation, 300 dots-per-inch laser printers, (Laser printers capable of textbook-quality 1,500 dots-per-inch still cost over $10,000. I have also achieved moderately good results using a simple 24 pin color dot matrix printer, and expect that one of the new color printers based on thermal wax technologies would do an even better job.) Furthermore, annotation of such high-resolution bit-map images, even the simple labeling of axes, poses additional problems. Finally, commonly used programs for the display of animated screenshows generally operate at EGA or VGA resolution levels, making it impossible to incorporate the higher-resolution output into computer slide sequences for classroom display.

3-D Animations Employing Vector-Based Graphics

The remaining graphics sequences used vector-based files generated by *SAS* in combination with the vector manipulation and drawing capabilities of the "Draw/Annotate" and "Drawpartner" components of Harvard Graphics. Drawpartner functions like a simple vector-based CAD program for manipulating vectors created by *SAS* or Harvard Graphics and can be otherwise used enhance the appearance of a vector-based illustration.

Horizontal Slices

The first step was to generate a 3-D graphic of the production surface in *SAS* and then slice it horizontally to uncover the shape of various isoquants. These surfaces were generated with a simple *SAS* program (Table 1). Figure 2 illustrates the sequence. To highlight the isoquants, Drawpartner was used to generate a thick line (red against a green surface in the instructional module) representing the isoquant at each level. That isoquants near the origin are concave is apparent.

A Basic Isoquant Map

A 2-D plot of the isoquants for this production function (a "top" view of the surface) with PROC GCONTOUR in *SAS* was generated using the same program (Table 1), imported the resulting contour (isoquant) map into Harvard Graphics, and then used Drawpartner to add isocost lines, the expansion path, ridge lines, pseudo scale lines (Debertin, 1986, p. 124-127) and identify key points of economic interest such as the global points of output and profit maximization. These are done sequentially in the instructional module and are illustrated in the diagrams comprising Figure 2. Notice that

several isoquants near the origin are concave and one has a nearly constant slope, violating second order conditions for constrained output maximization.

The expansion path begins at the first convex isoquant and follows the convention of extending through the global point of output maximization, the widely accepted diagram, although the expansion path should stop at the global point of profit maximization where the pseudo scale lines converge. The final slides in the sequence (the second and third panels of Figure 2) illustrate the implications of changes in input prices. In the second panel, the price of x_2 has risen, causing the pseudo scale line for x_2 to shift away from ridge line 2, and the expansion path to curve away from the x_2 axis. (Note the shape of the expansion path under this set of prices. Clearly this production function is non-homothetic.) Notice that the position of pseudo scale line 1 is independent of the price of x_2 and remains in its old position. The third panel illustrates an instance in which the price of x_2 decreases. Pseudo scale line 2 shifts toward ridge line 2 and the expansion path favors the use of the second, now relatively less expensive, input.

Vertical Slices

The next sequence of graphs (Figure 4) illustrate a series of vertical cuts in the production surface made at the angle represented by the budget constraint (first panel, figure 2) following procedures suggested in Debertin, Pagoulatos and Bradford (1991). This series of images was constructed by running multiple *SAS* GRAPH jobs using G3D (Table 1). The budget constraints set at various levels were incorporated directly into the *SAS* programs to generate the vertically-cut surface. The series of charts was imported into Harvard Graphics for annotation and other modifications. For example, in the instructional module, modifications make the cut appear in red while the remainder of the surface is in yellow. Points A, B, and C were added and a curve drawn between the points and a line representing the budget constraint was added.

The key element to note here is that when the isoquants are concave, point B lies below A and C and the objective function ABC in the constrained optimization problem is convex to the horizontal plane and therefore the solution is output-minimization (at the point of tangency between the budget constraint and the isoquant, the isoquant is concave with an increasing MRS). As the budget constraint shifts outward the objective function ABC is nearly horizontal (corresponding to an isoquant with a constant slope and a constant MRS). As the budget constraint shifts further outward, B rises above A and C, the objective function ABC is therefore concave, and the "standard" constrained optimization solution for convex isoquants is obtained. This solution fulfills both first and second order conditions for a maximum.

Gross and Net Revenue Functions

Precisely locating pseudo scale lines is critical in locating the global point of profit maximization along the expansion path, but positioning these "high-profit" lines on a diagram has often involved approximations as was the case in positioning them in figure 2. Computer graphics can be used to precisely locate pseudo scale lines and the global point of profit maximization. I illustrate three different techniques for doing so. Each, I believe, is of instructional value.

Option I: The Input-Price Hyperplane

A basic first order condition for profit maximization in the single input case is $p_y MPP_x = p_x$, where p_y is the price of the output (y), $MPP_x = dy/dx$ and p_x is the price of the input (x). Rearranging this equation yields $MPP_x = p_x/p_y$. The point of profit maximization in the single input case is the point where the MPP of x equals the factor/product price ratio. One need merely to locate the point on the production function with the slope p_x/p_y to locate the profit-maximizing input and output level. Generally, in introductory courses, a line of slope p_x/p_y is drawn a diagram of the production function, and this line is shifted upward to locate the point on the production function with slope p_x/p_y. In the 3-D sequence, this same procedure is followed using the two-input polynomial. First, the production surface for the polynomial is generated in *SAS*. Then a 3-D hyperplane is generated representing $(p_1+p_2)/p_y$, the two input analog to p_x/p_y. I assumed the price of y to be \$1.00 per unit, the price of x_1 to be \$3.00 Per unit, and the price of x_2 to be \$1.50 per unit. Both graphs were then imported into Harvard Graphics, and positioned the production surface over the price hyperplane. The hyperplane was then shifted upward to become tangent to points on the production surface, and the pseudo scale lines and global output maximization point located. This is illustrated in Figure 5.

Option 2: Gross and Net Revenue Surfaces

For any production function, there is a corresponding net revenue, or profit function. For the production function $y = x_1 + 1.5x_1^2 - 0.05x_1^3 + x_2 + 1.5x_2^2 - 0.05x_2^3 + 0.4x_1x_2$ with the aforementioned prices, the profit function is $\$1\ (x_1 + 1.5x_1^2 - 0.05x_1^3 + x_2 + 1.5x_2^2 - 0.05x_2^3 + 0.4x_1x_2) - \$3\ x_1 - \$1.50\ x_2$. A numerical solution locates the global point of profit maximization where the pseudo scale lines intersect at a profit level of \$165.65 with a value for x_1 of 15.21 and x_2 of 15.71, thus favoring use of the relatively less expensive input. Total cost represented by the budget constraint intersecting this point is \$69.20, and total revenue at the profit-maximizing input level is therefore \$234.85, compared with \$237.21 at the revenue- (output-) maximizing level.

Just as the surface of the production function can be illustrated, so also can the surface of the profit function be illustrated. Furthermore, the two surfaces can be placed on the same graph. In Figure 6, the production surface (and gross revenue surface, since

p_y is \$1.00; changing the product price merely shifts the units on the axes without repositioning the surface) appears in dotted lines, while the profit surface is represented by solid lines. Ridge lines defining points of maximum (or perhaps minimum) output and gross revenue intersect at the global point of output and gross-revenue maximization. Since profit (but not output, or gross revenue) can be negative, the profit surface for input x_1 drops below 0 at high levels of x_1 use. The global point of profit maximization where the pseudo scale lines intersect can be located at the high point on the profit surface. The pseudo scale lines can then readily be located as the ridge lines on the profit surface, representing maximum (or perhaps minimum) profit for one input holding the other input constant at various levels.

Option 3: Isorevenue and Isoprofit Contours

Contour lines for the gross-revenue and profit surfaces can also be generated and superimposed on each other. In Figure 7, the contour lines for the gross-revenue surface appear as dotted lines while the profit surface, contour lines appear as solids. Ridge lines can be positioned as lines connecting points of zero or infinite slope on the gross revenue contour map. Pseudo scale lines can be positioned as lines connecting points of zero or infinite slope on the contour lines of the profit function. In essence, they are ridge lines for the isoprofit contours. The global point of output maximization occurs at the same level of input use that maximizes gross returns. The global point of profit maximization occurs at the center of the series of concentric rings representing the isoprofit contours.

Concluding Comments

Debertin and Jones have shown that computer-based instructional aids for teaching undergraduate economics meet with favorable responses from students and may have a positive impact on student learning. Thus, there is reason to believe that students will also be favorably impressed with such learning aids in more advanced courses. The first use of this instructional module, in a graduate-level production economics class, will be made in the Fall semester of 1991. Moreover, the *SAS* program presented in Table 1 (without transferring files to Harvard Graphics) is simple enough that graduate students might modify by varying functional forms, parameters, constraints, or prices to generate surfaces and contour maps of specific interest to them.

Although the technologies presented here have been demonstrated using a third-degree polynomial function, they have a much broader potential application in that they could be applied to any function in which the three-dimensional surface is of interest to the researcher or instructor. Not only could other production functions be used, but the technology could be applied to functions in which a single input was used to produce two outputs. Some potential product-space functional forms are suggested in Debertin, Pagoulatos and Bradford (1991). Another application involves the analysis of flexible functional forms after empirical results are obtained (Diewert and Wales). Finally, there are many applications in a duality context. For example, there are a number of duality

theorems related to cost as a function of prices for two inputs--for example cost functions employed in duality theory must be concave with respect to any single input price, but homogeneous of degree 1 with respect to all input prices. Technologies presented here could be employed to verify these and other duality theorems for specific functions.

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Table 1. A Simple PC-SAS program for Generating Production, Profit and Revenue Surfaces and Contour Lines With and Without Constraints.

```
*A simple SAS program for 3-D surfaces and contour maps;
data;
*These loops create a 40 x 40 grid with x1 and x2 incremented
from 0 to 20;
do x1 = 0 to 20 by .5;
do x2 = 0 to 20 by .5;
*The production function;
y = x1+x1**2-.05*x1**3+x2+x2**2-.05*x2**3+.4*x1*x2;
*A gross revenue function for PY = $1.00;
reven = 1*y;
*A budget constraint P1=$1.50, P2=$3.00;
cost = 1.5*x1+3*x2;
*A profit function;
profit = reven-cost;
*The following statement (with * removed), slices the surface
vertically consistent with a $30 budget constraint;
*if cost < 30  then y=0;
*The following statement, (with * removed), slices the output surface
horizontally at 180 units of output (below the maximum of the function);
*if y > 180 then y=180;
*The following statements close the do loops;
output;end;end;
*The following statements create the 3-D Surface plot for y in yellow
tilted at 70 degrees, rotated to 45 degrees (use also for cost or reven);
proc g3d;
plot x1*x2 = y/ tilt =70 rotate=45 ctop=yellow
XTICKNUM=11 YTICKNUM=11 side zmax=250;
*The following statement displays to a color display (EGA 350 x 640)
consistent with bit-map file Harvard Graphics can capture (see text);
goptions device=egal;
*Use device driver CGMHG to create vector-based CGM file as GRAPH.GSF
for importing into Harvard Graphics (see text);
*goptions device=cgmhg;
run;
*The following statements generate the contour plot of the surface;
*Ten uniformly spaced isoquants are created in this example;
proc gcontour;
plot x1*x2 = y/levels=0 to 237 by 23.7
XTICKNUM=11 YTICKNUM=11 nolegend;
goptions device=egal;
*Device driver CGMHG can again be used for importing the contours into
Harvard Graphics;
run;
```

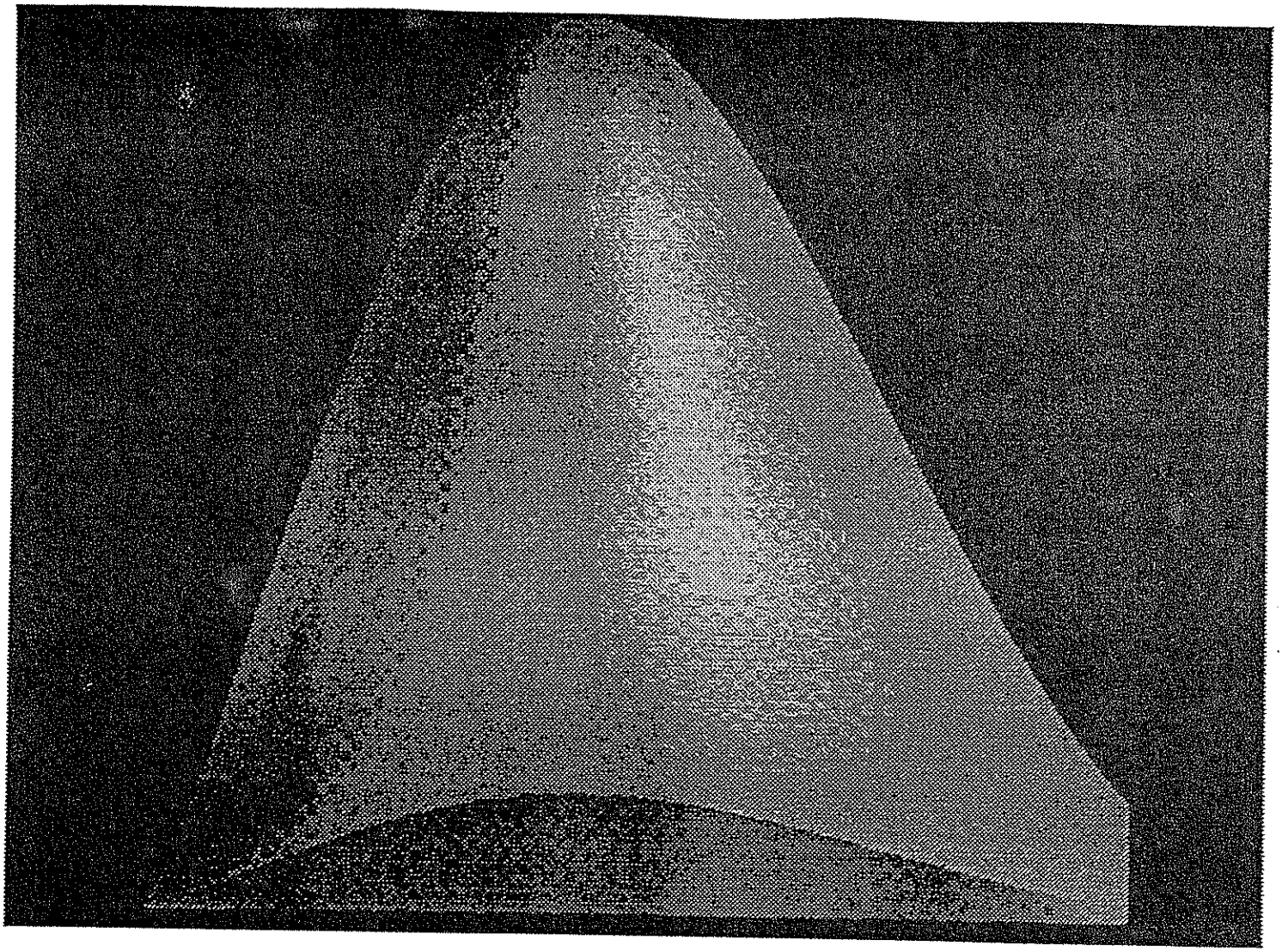
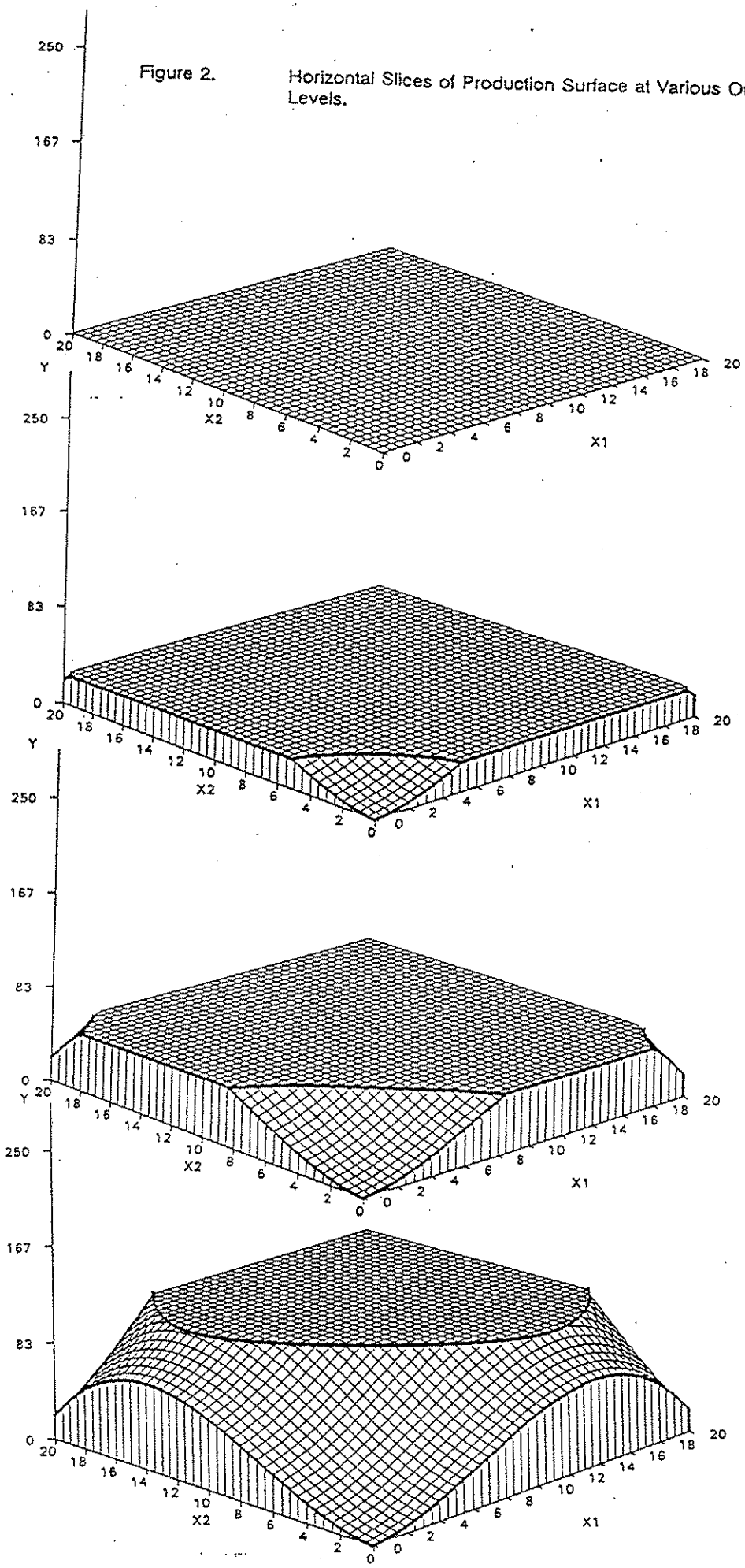
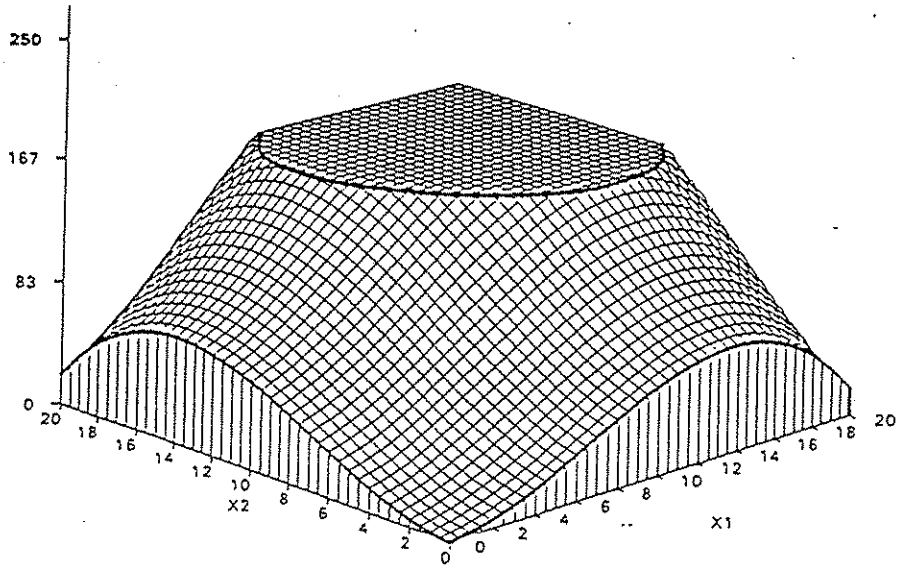
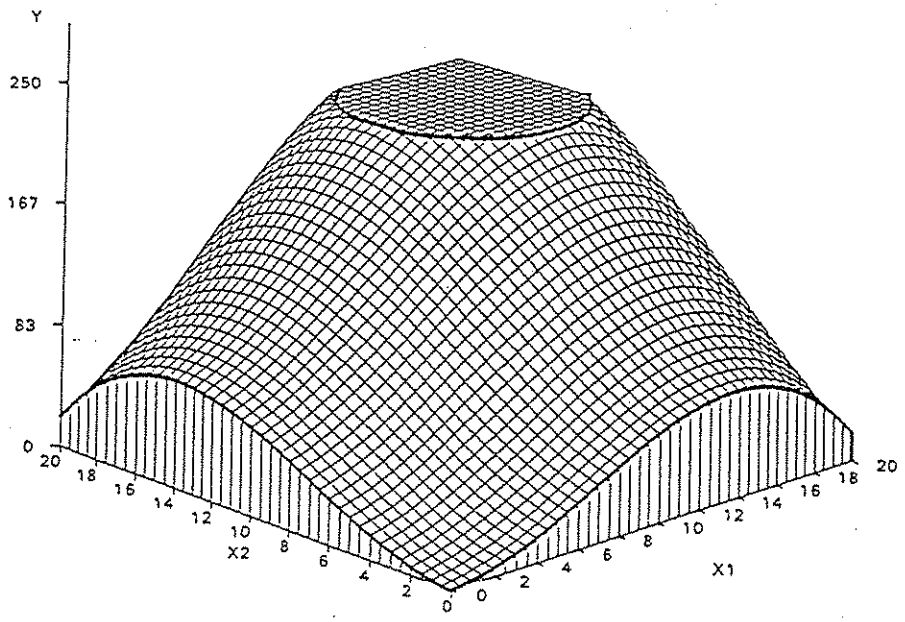


Figure 1. 3-D Surface Rendering of the Polynomial Using A CAD-Generated Bit-Map File

Figure 2. Horizontal Slices of Production Surface at Various Output Levels.




250
167
83
0
20 18 16 14 12 10 8 6 4 2 0
X2
0 2 4 6 8 10 12 14 16 18 20
X1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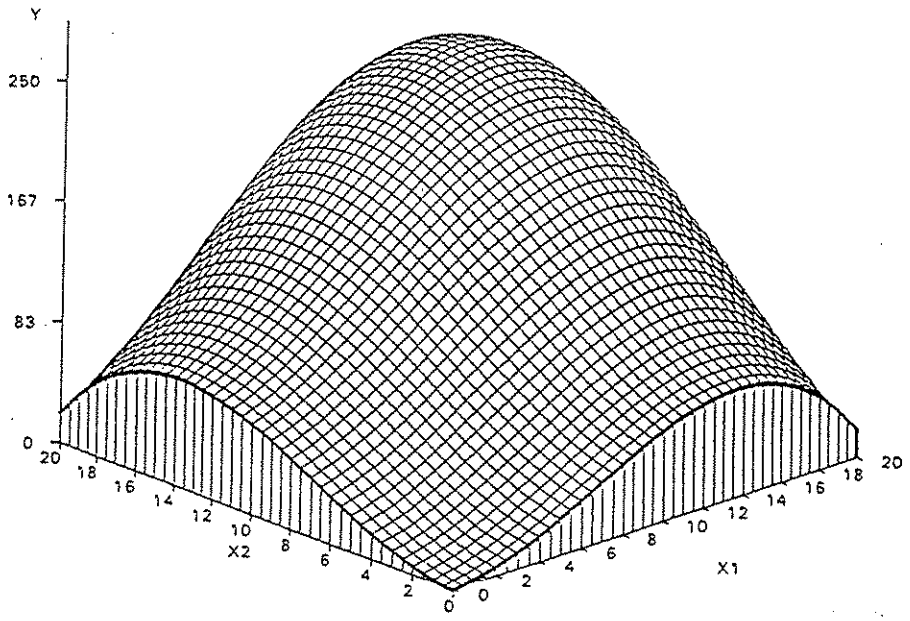
X1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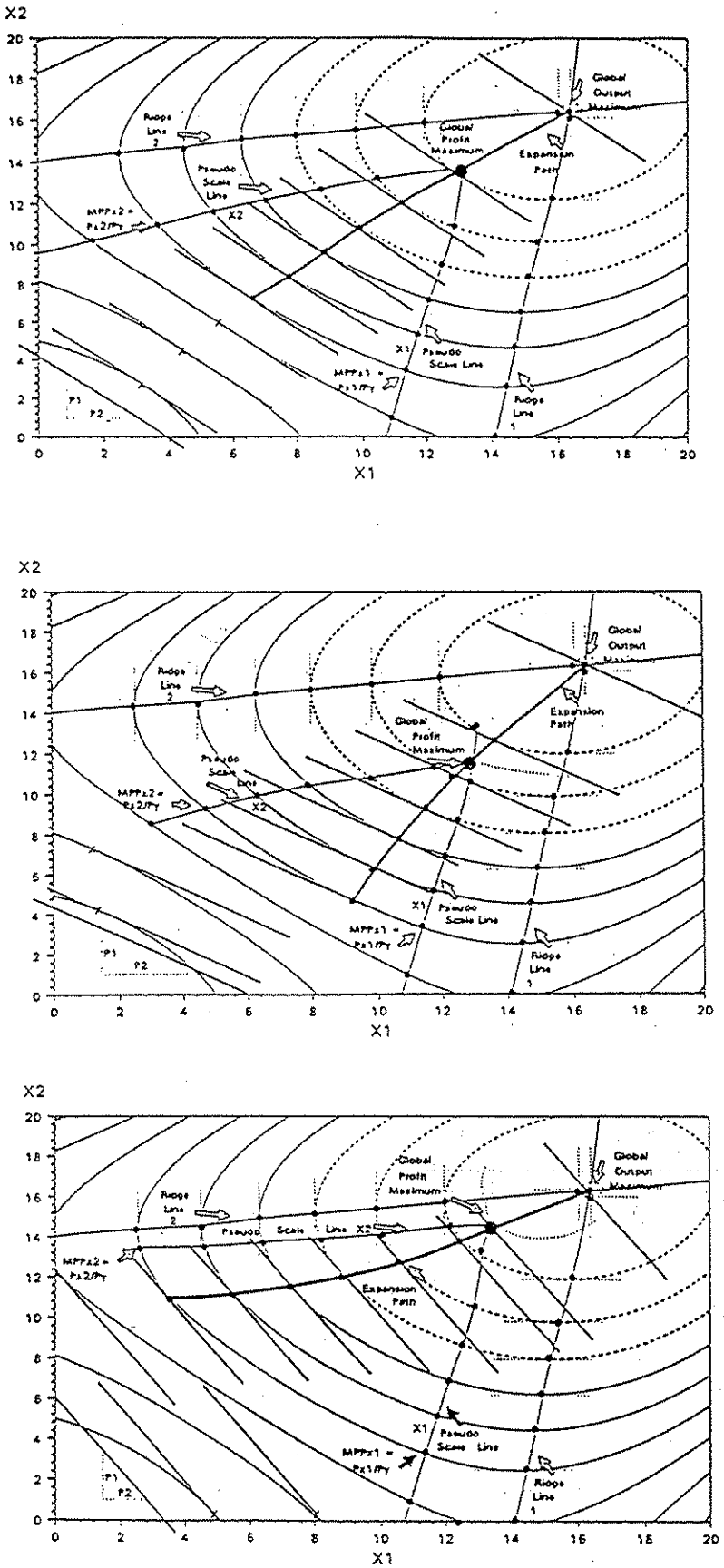


Figure 3. Isoquants, Ridge Lines, Pseudo Scale Lines, Budget Constraints and Expansion Paths for Three Price Ratios.

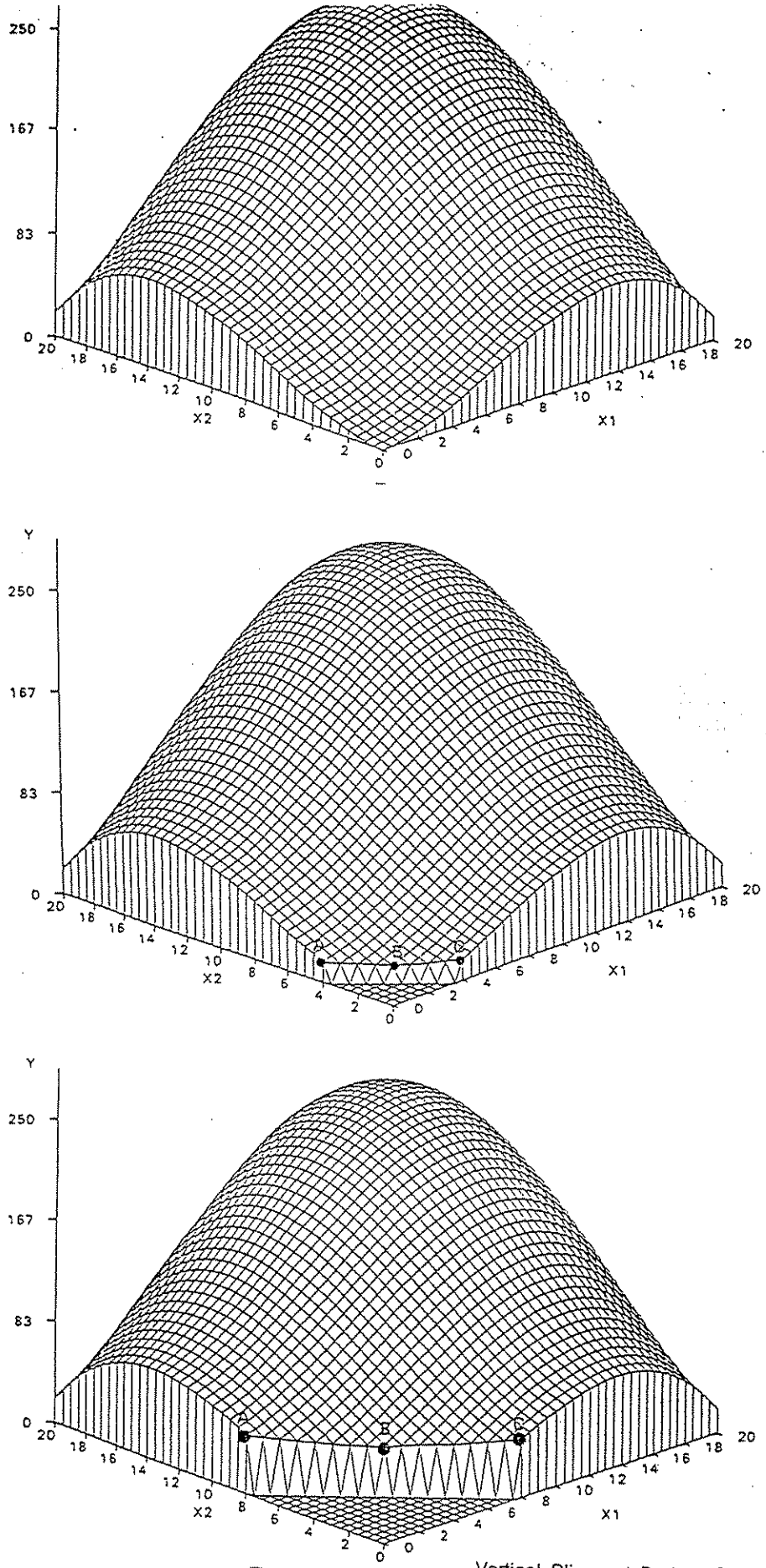
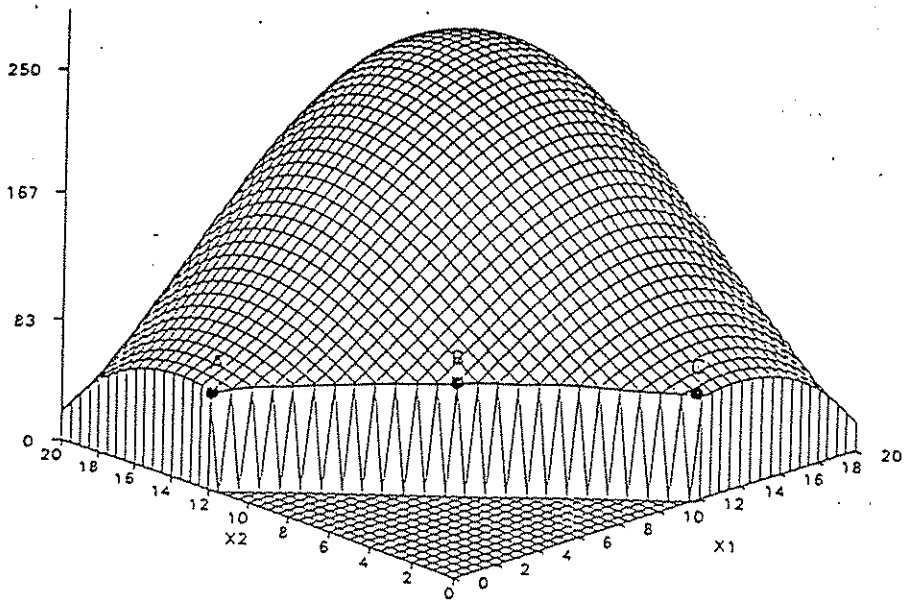


Figure 4. Vertical Slices at Budget Constraint Angle, Various Cost Outlays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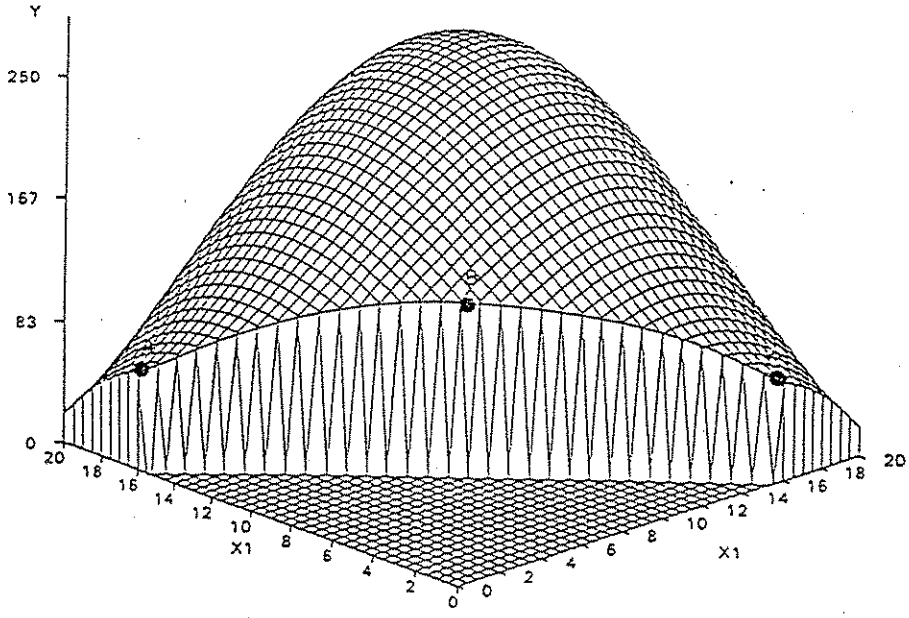
0 2 4 6 8 10 12 14 16 18 20
X1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X1
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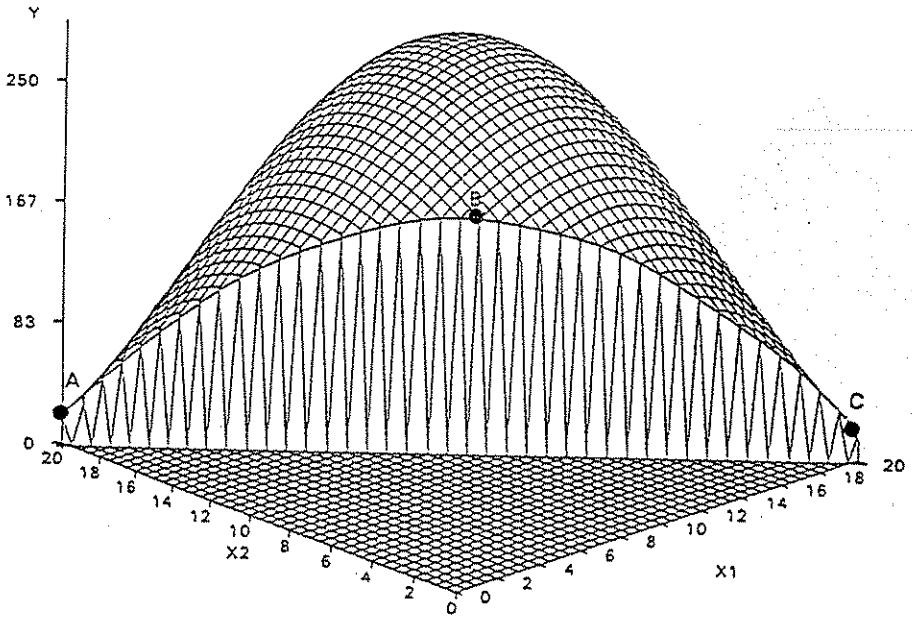
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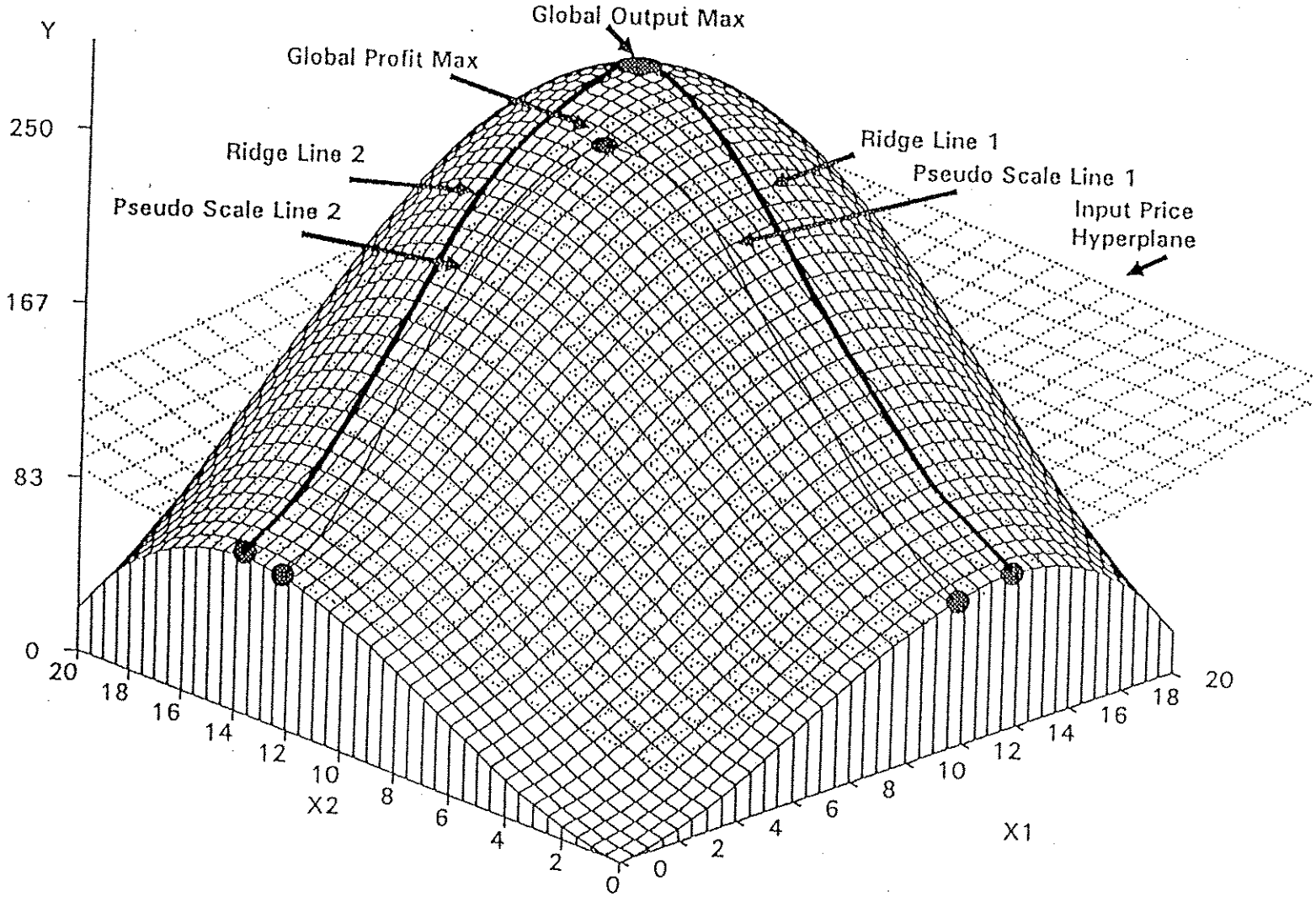


Figure 5. Locating the Global Point of Profit Maximization Using an Input Price Hyperplane.

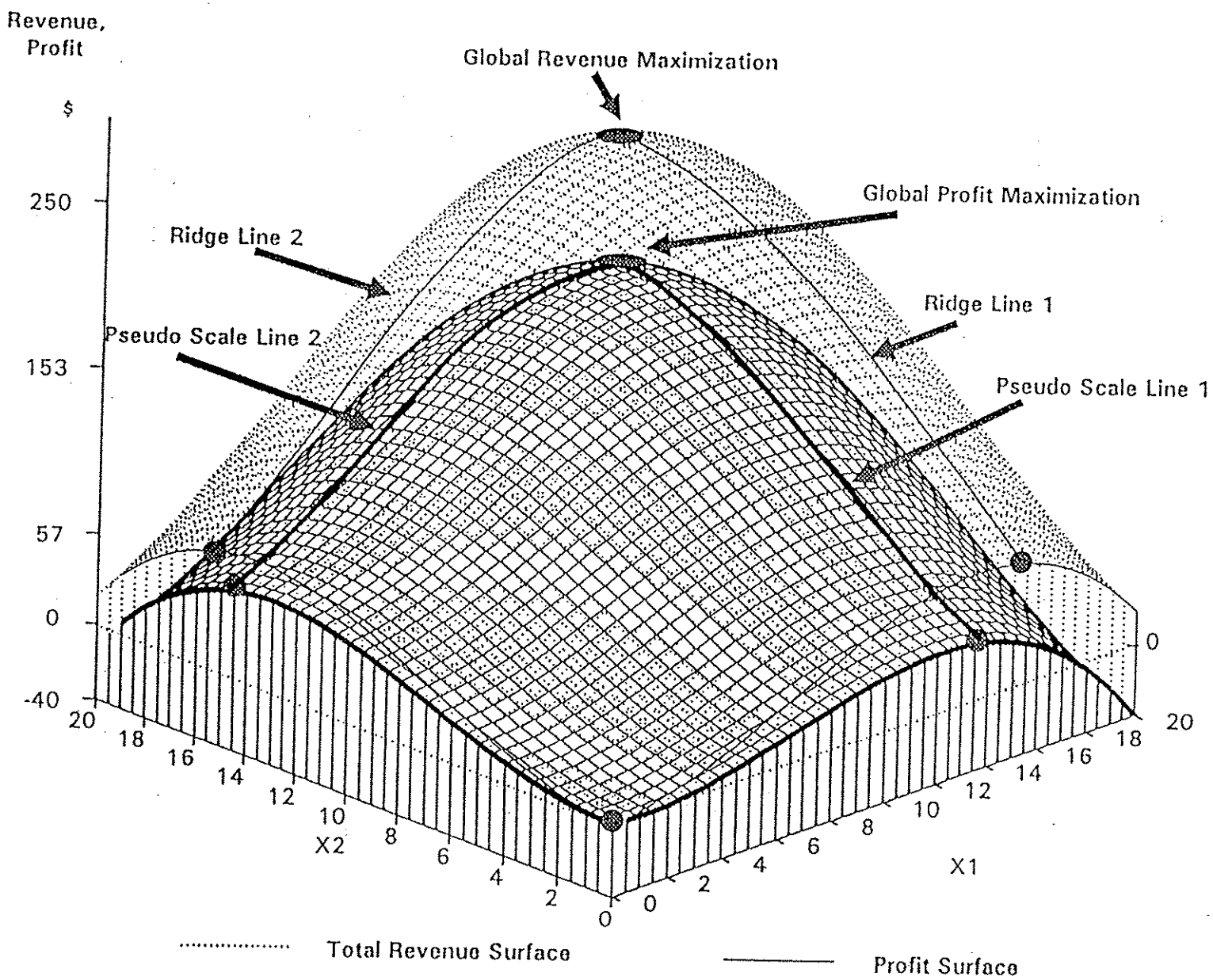


Figure 6. Locating the Global Point of Profit Maximization Using Revenue and Profit Surfaces.

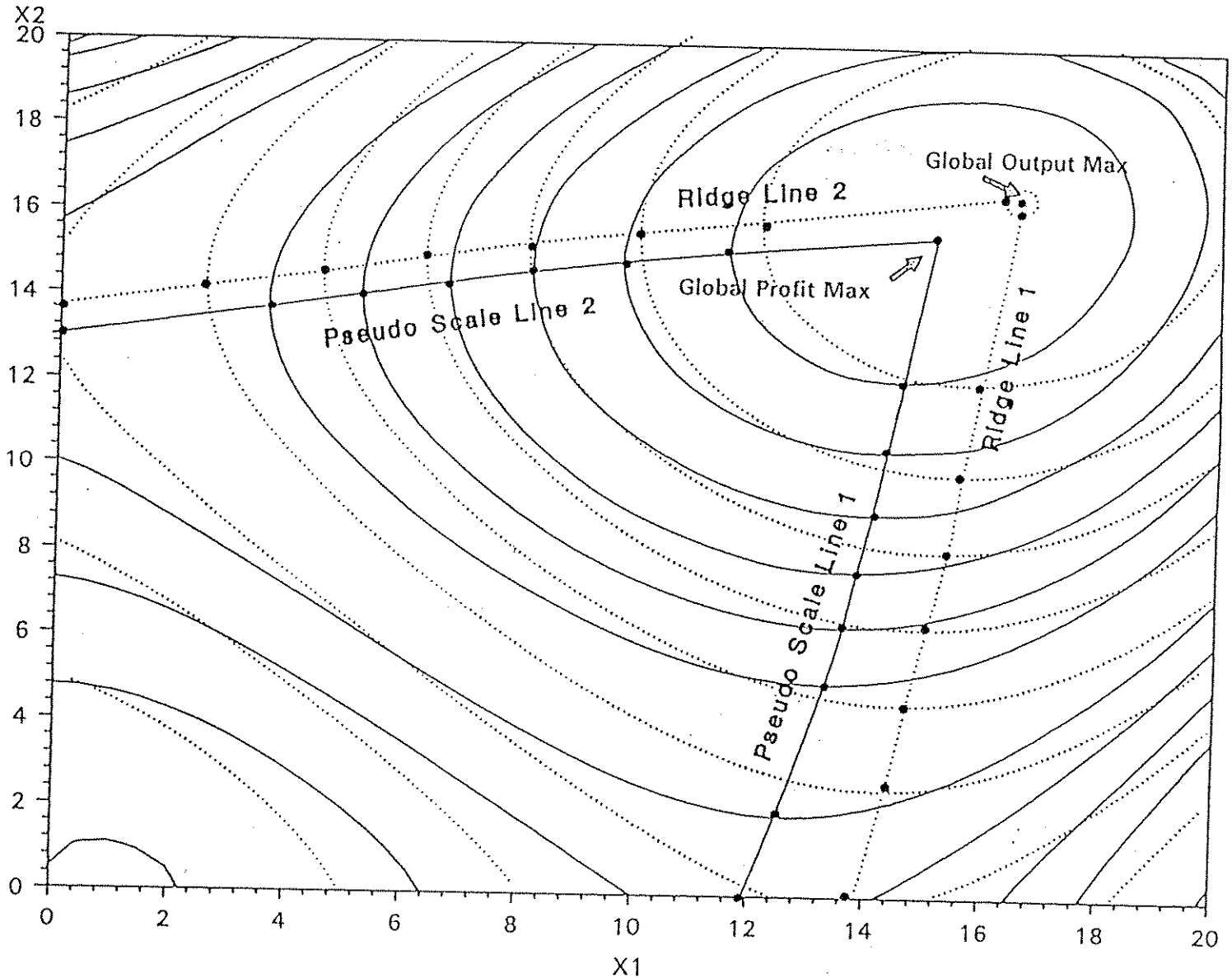
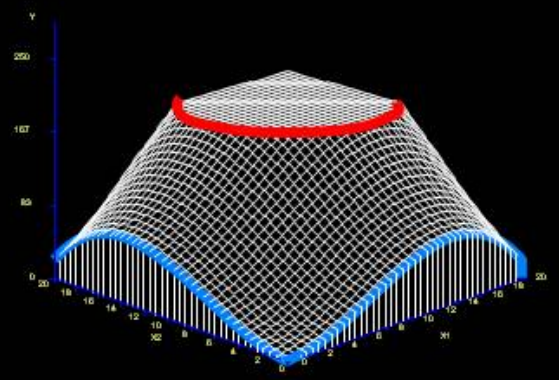


Figure 7. Locating the Global point of Profit Maximization Using Isorevenue and Isoprofit Contours.

An Animated Instructional Module for Teaching Production Economics with the Aid of Three-Dimensional Graphics

David L. Debertin
University of Kentucky



This instructional module is based on the polynomial production function

$$y = x_1 + x_1^2 - 0.05\,x_1^3 + x_2 + x_2^2 - 0.05\,x_2^3 + 0.4\,x_1 x_2.$$

This production function was chosen for a number of reasons.

1. It possesses a region of increasing marginal returns, a region of diminishing marginal returns and a region of negative marginal returns to the variable inputs

x_1 and x_2.

$$\frac{dy}{dx_1} = 1 + 2x_1 - 0.15x_1^2 + 0.4x_2.$$

$$\frac{dy}{dx_2} = 1 + 2x_2 - 0.15x_2^2 + 0.4x_1.$$

2. Since the function has a finite maximum,

there are "ring" isoquants, centered at the maximum output level.

At the maximum output level,

$\frac{dy}{dx_1}$ and $\frac{dy}{dx_2}$ are both zero.

Maximum output is 237.21 units corresponding to

$x_1 = x_2 = 16.41$ units

3. There is a global point of profit maximization.

This point occurs at an output level less than the global point of profit maximization.

For example, if the price of the product (y) is \$1.00 per unit, the price of x_1 is \$3.00 per unit, and the price of x_2 is \$1.50 per unit, then profit is maximum at x_1=15.21 and x_2 = 15.71 units. Total Revenue is \$234.85; Total Cost for x_1 and x_2 is \$69.20;

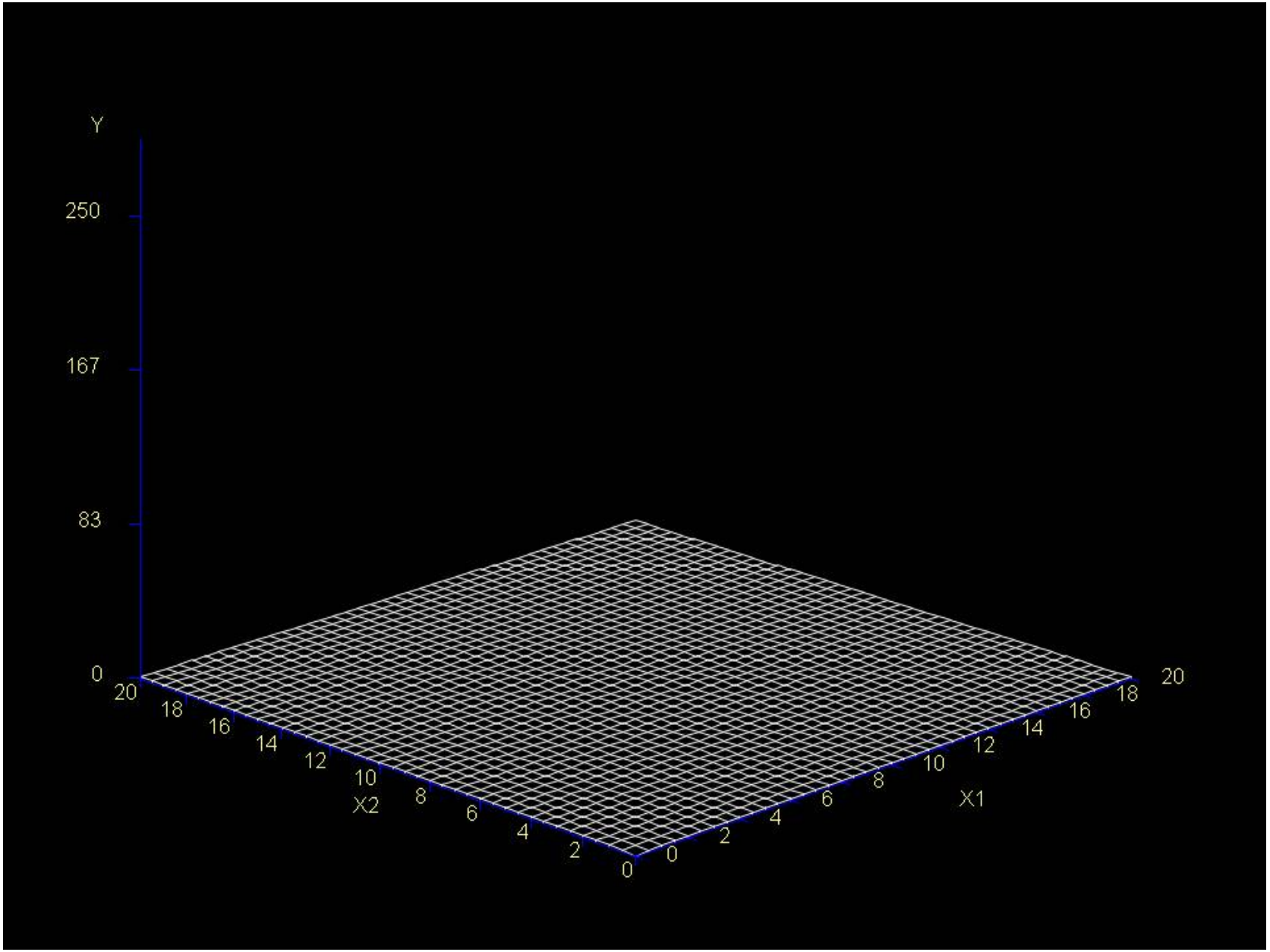
Profit is Total Revenue - Total Cost = \$165.20

In the following sequence, the production surface for the polynomial is sliced horizontally at various levels.

The isoquant at each output level appears in red.

Note that isoquants at low output levels are concave to the origin, but as the output level increases, the isoquants become convex to the origin.

You are looking at the 3-D surface from the origin. x_1 is at your right; x_2 at your left. Output (y) is measured on the vertical axis.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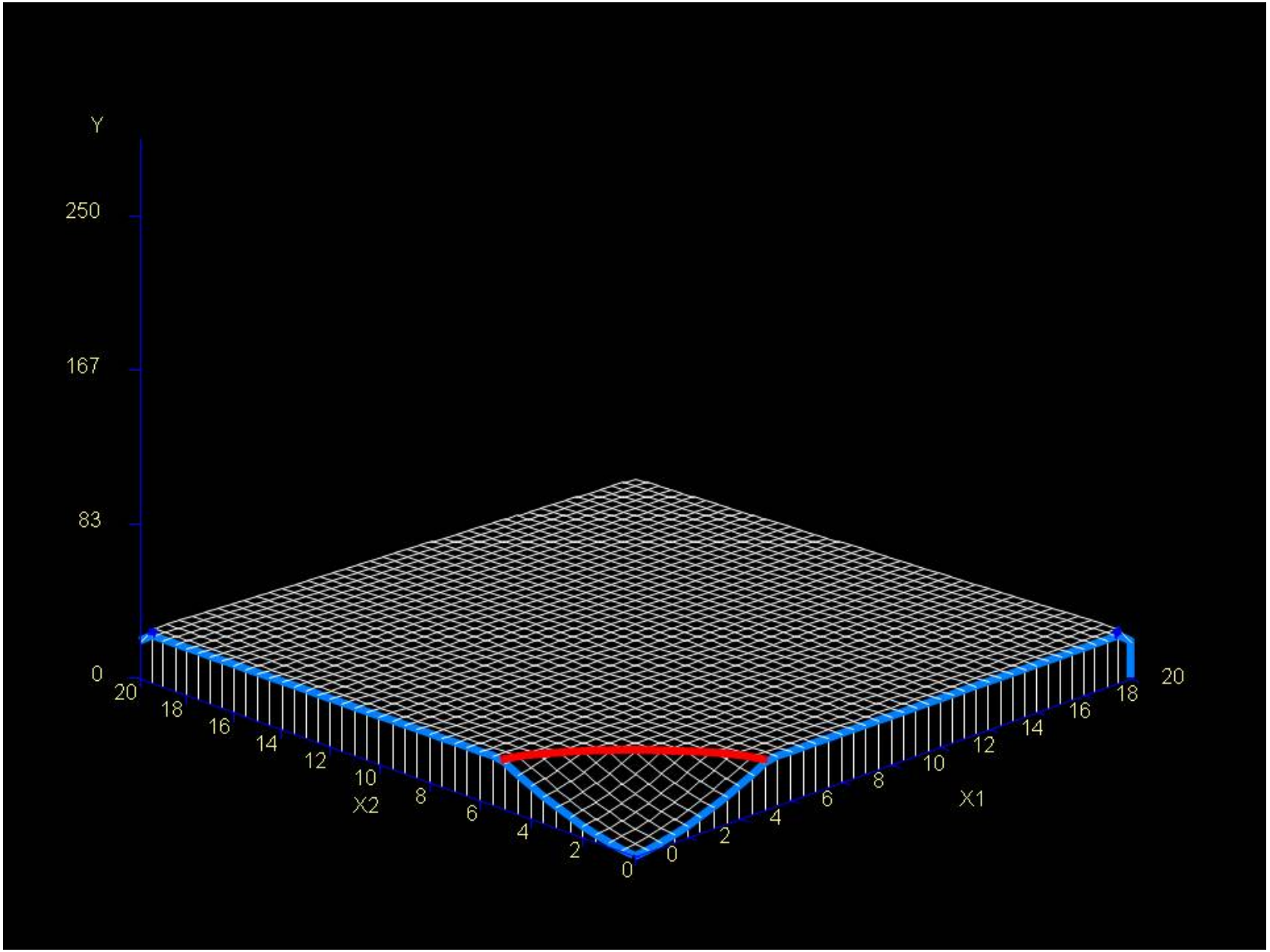
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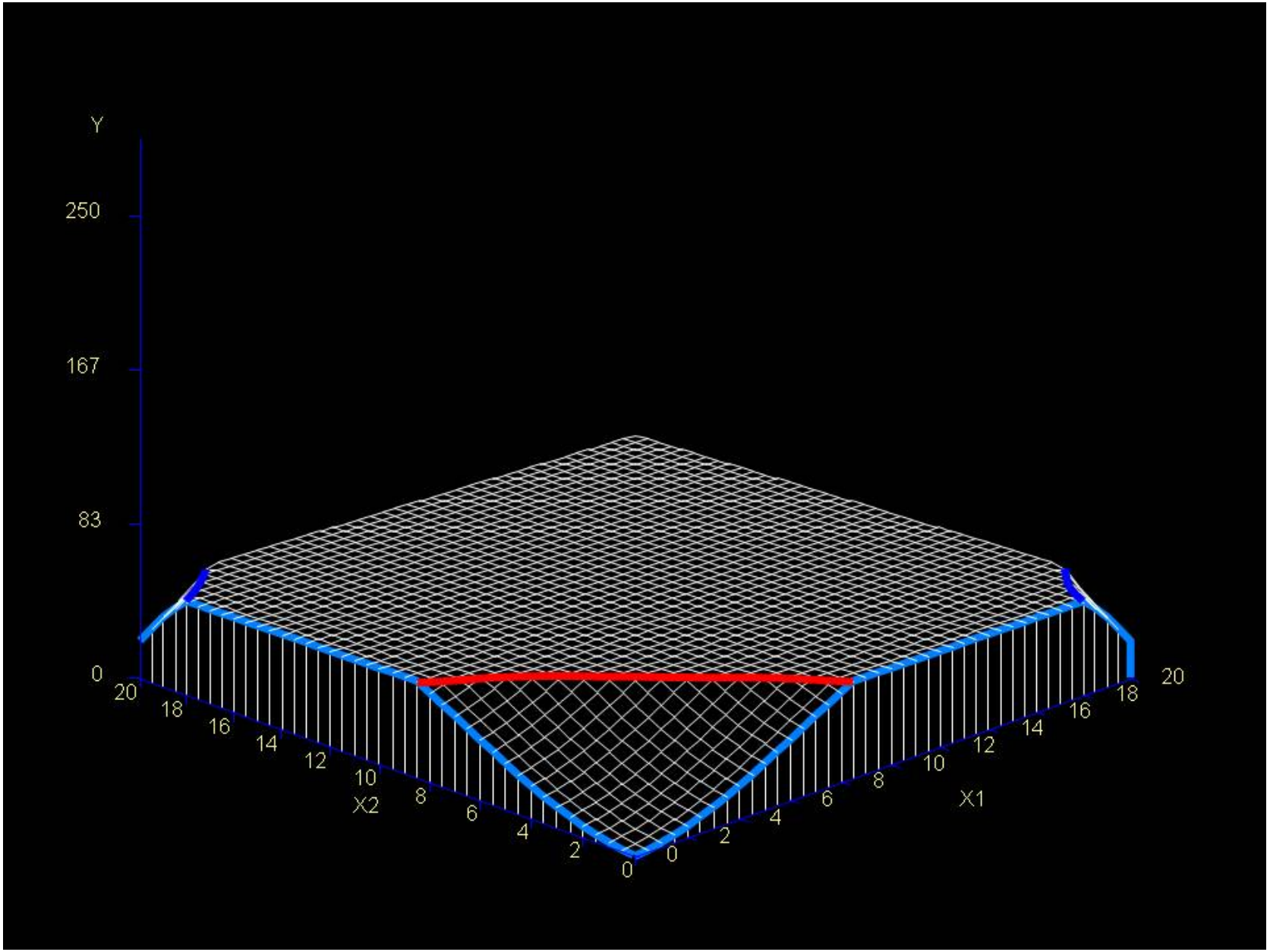
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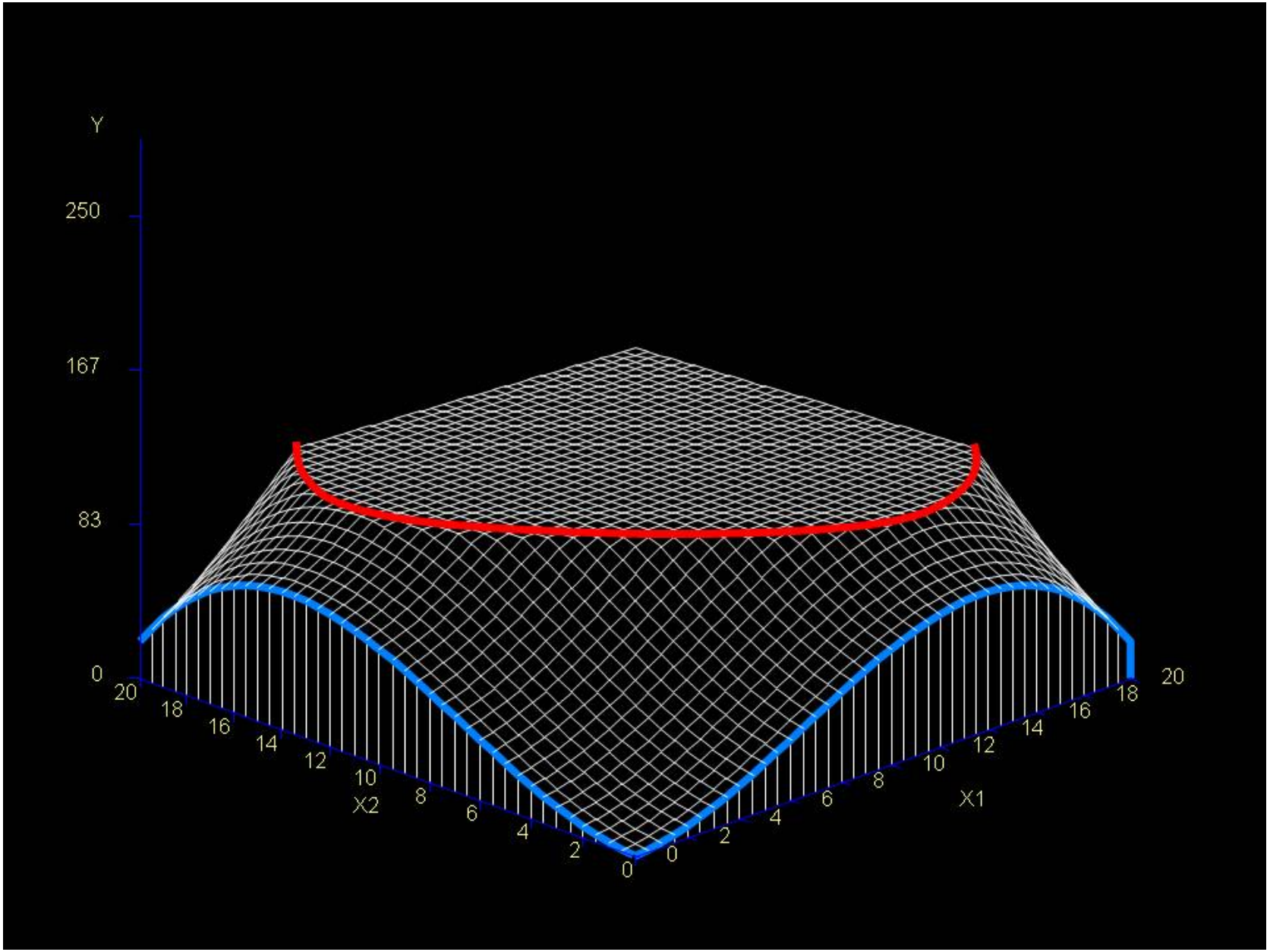
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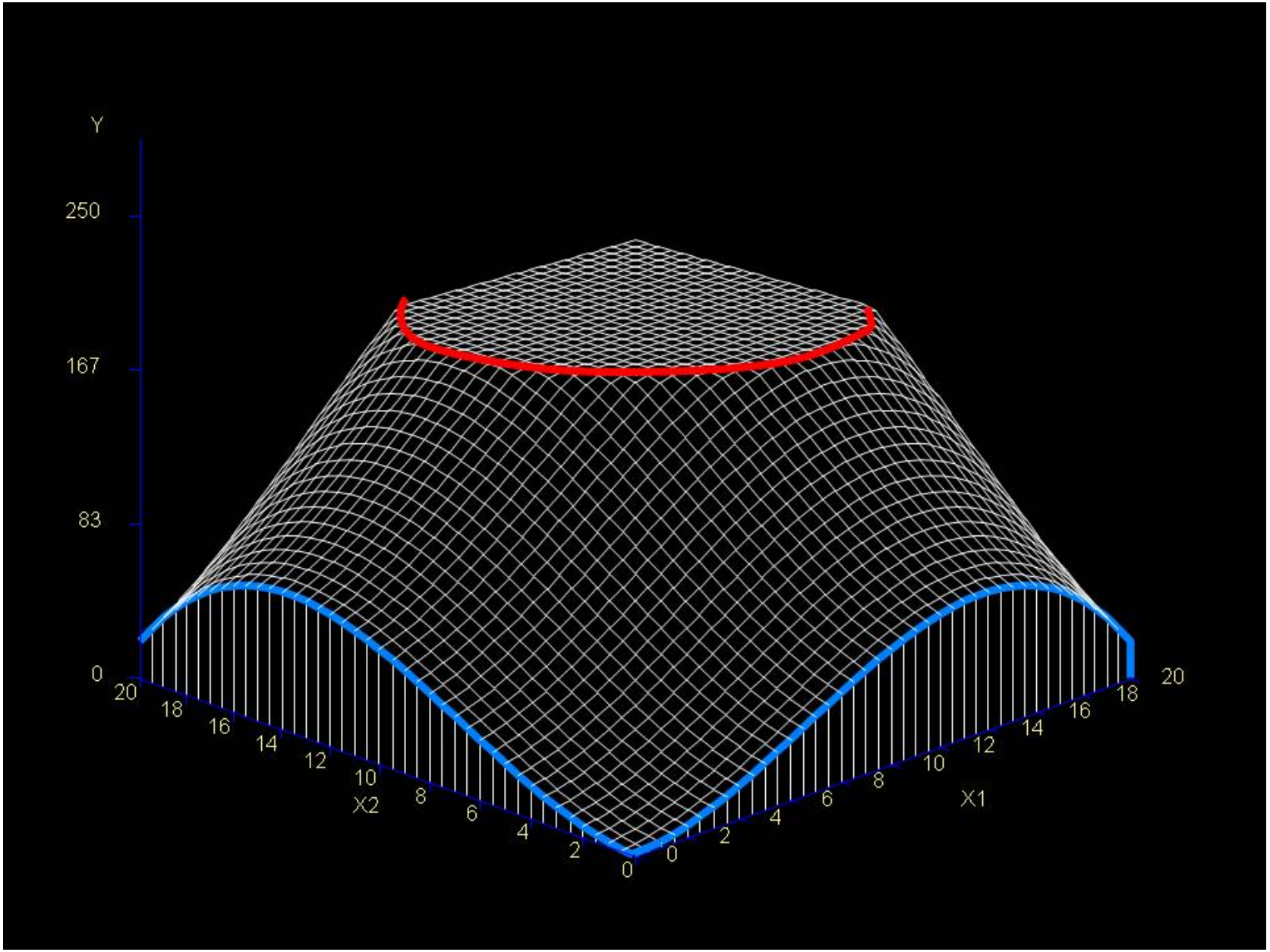
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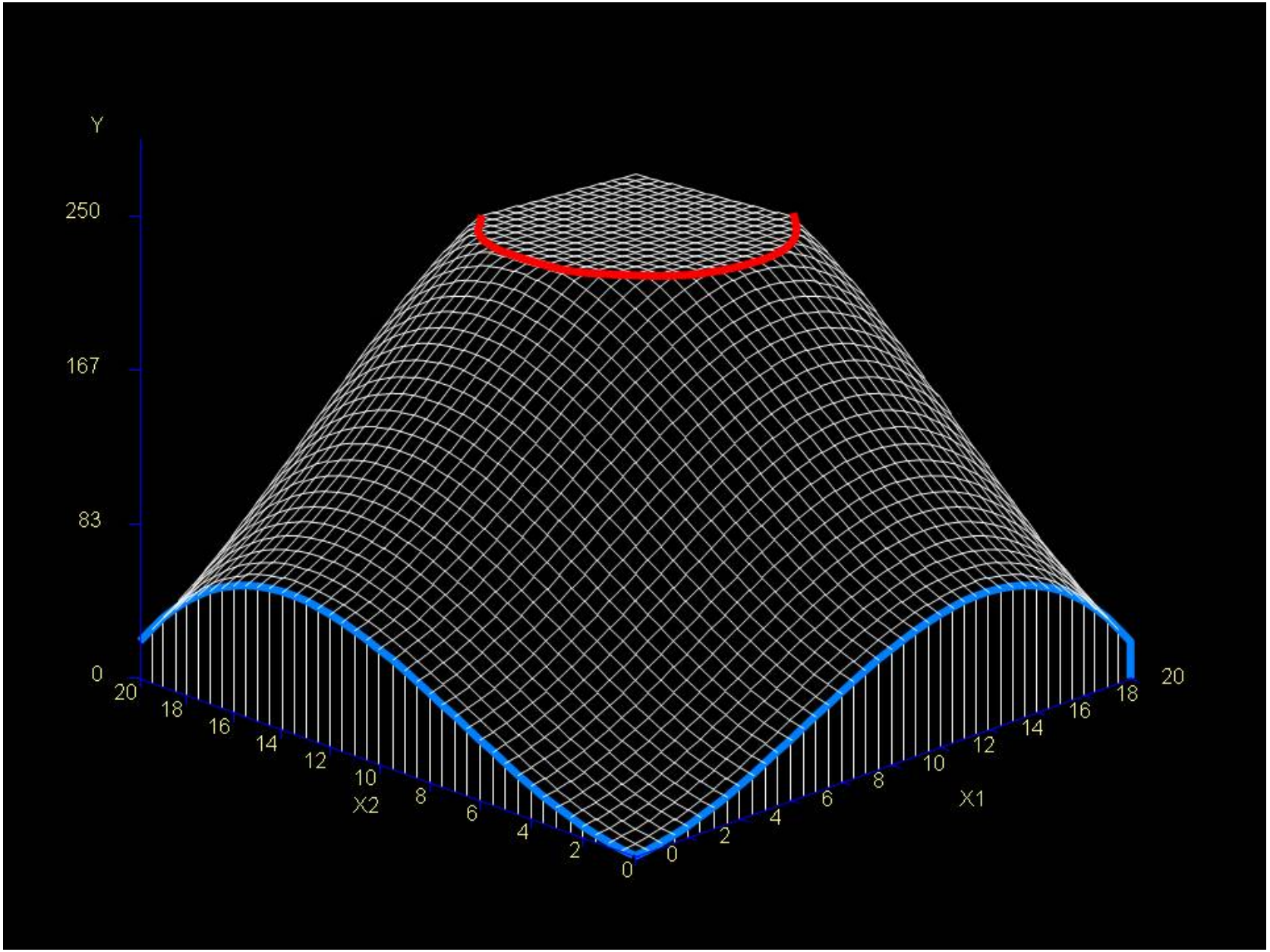
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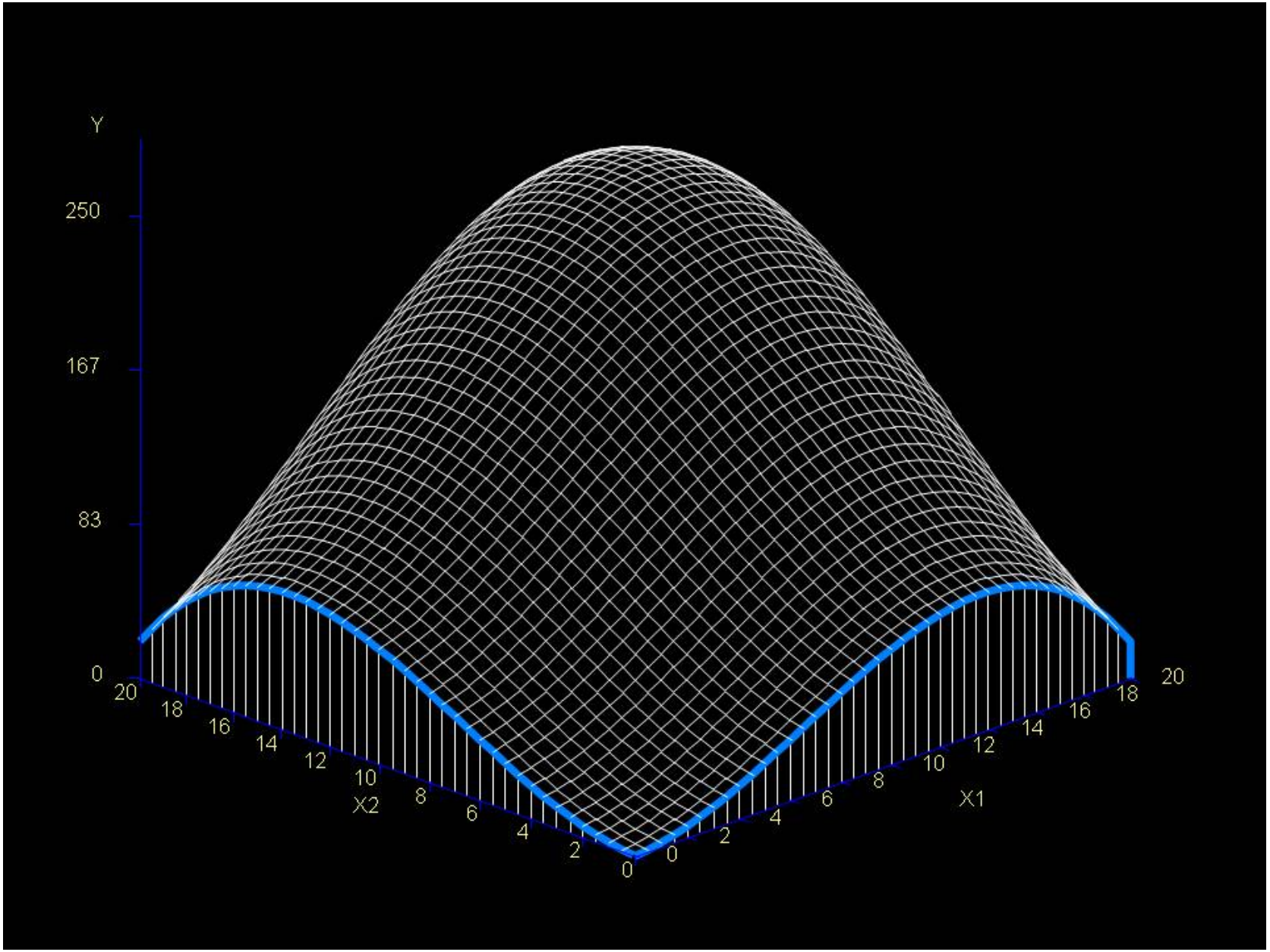
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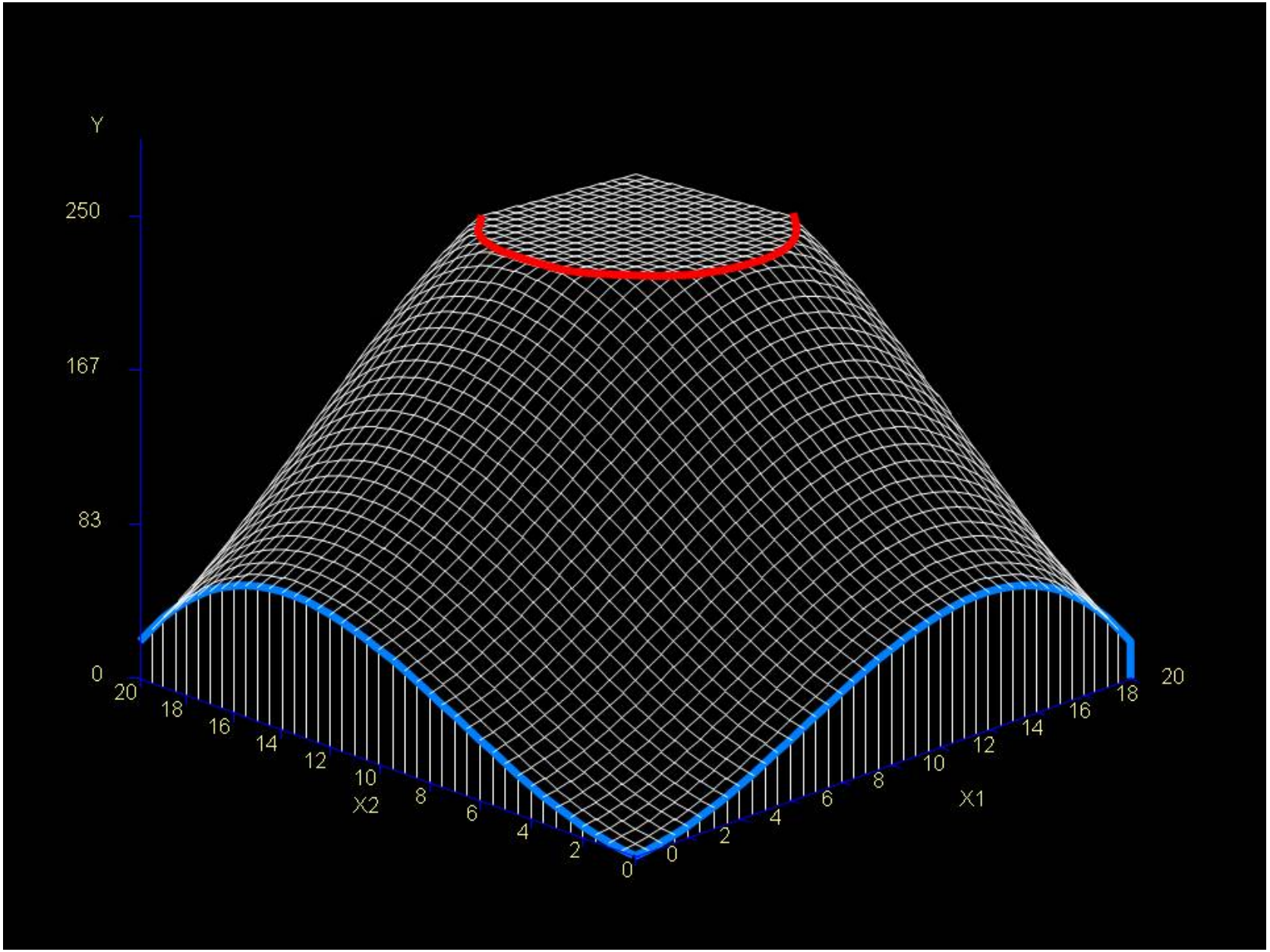
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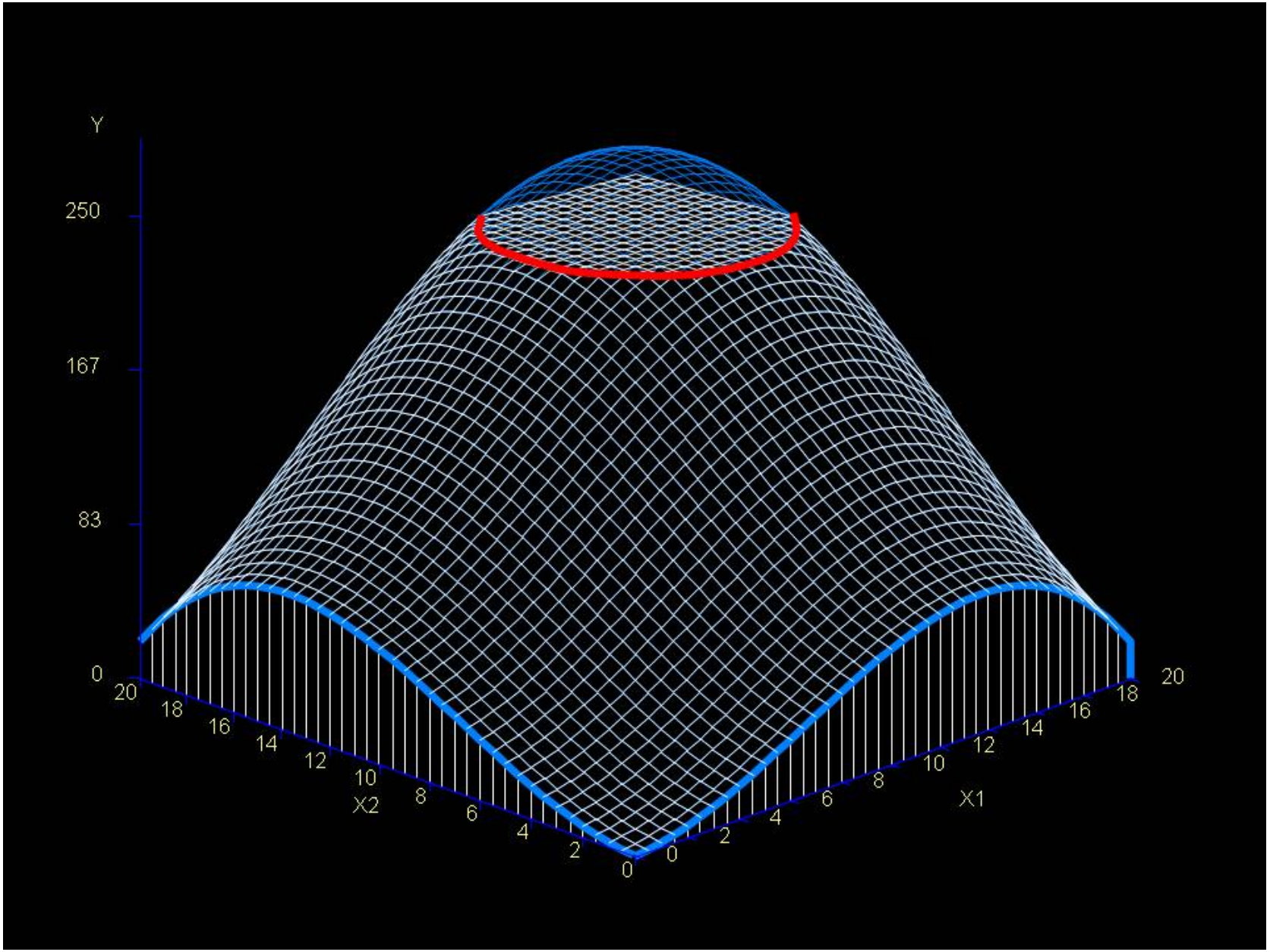
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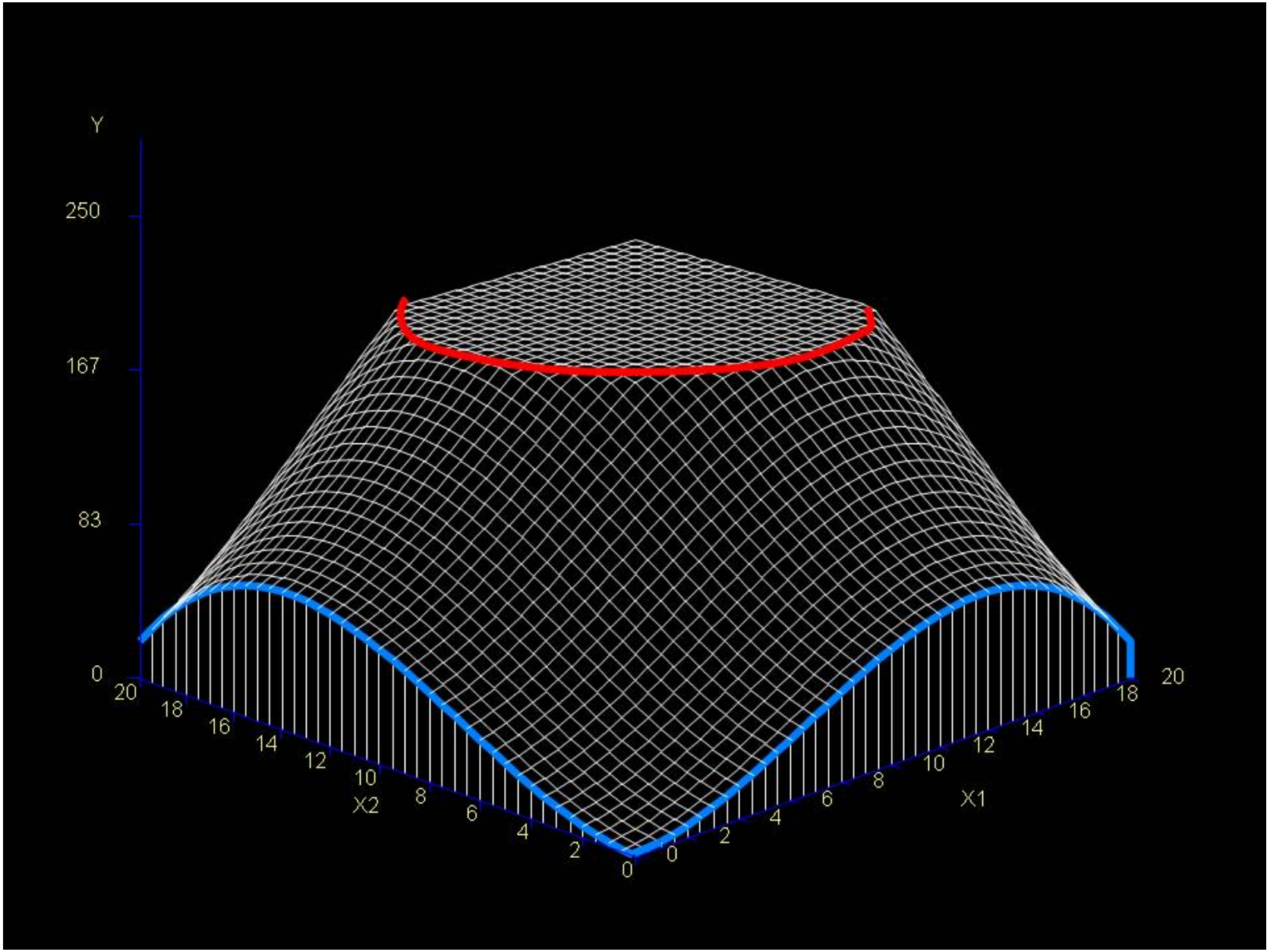
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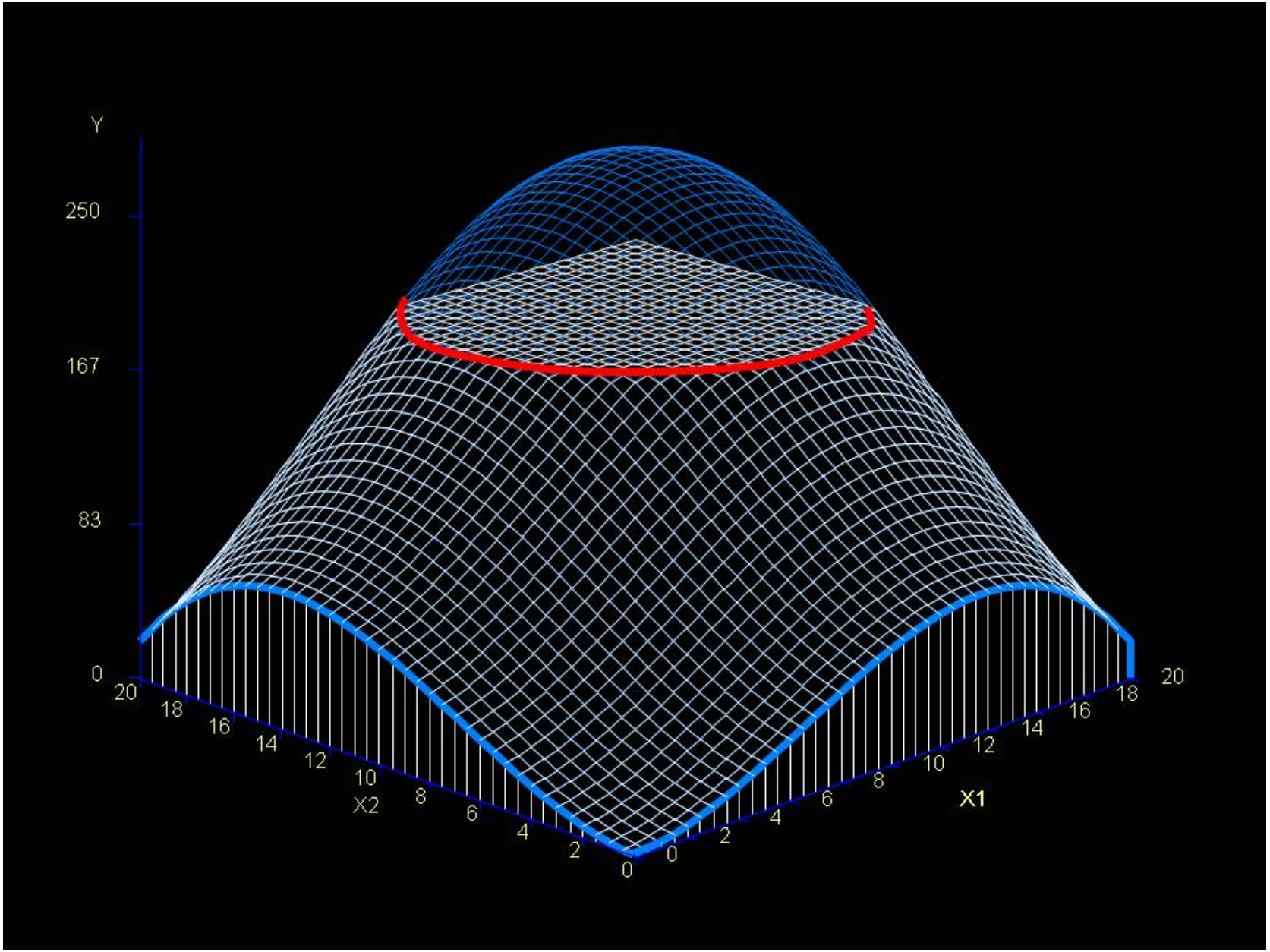
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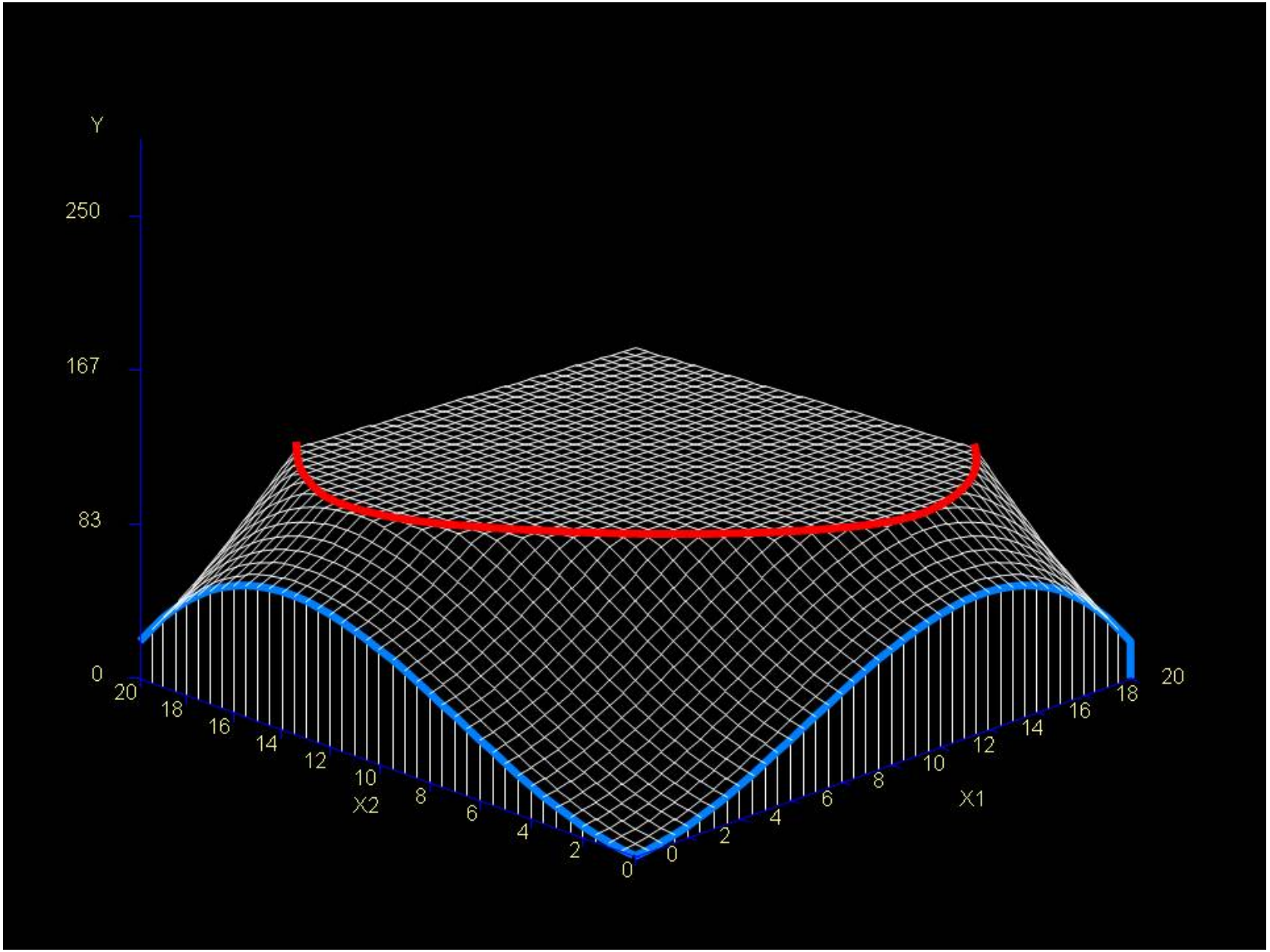
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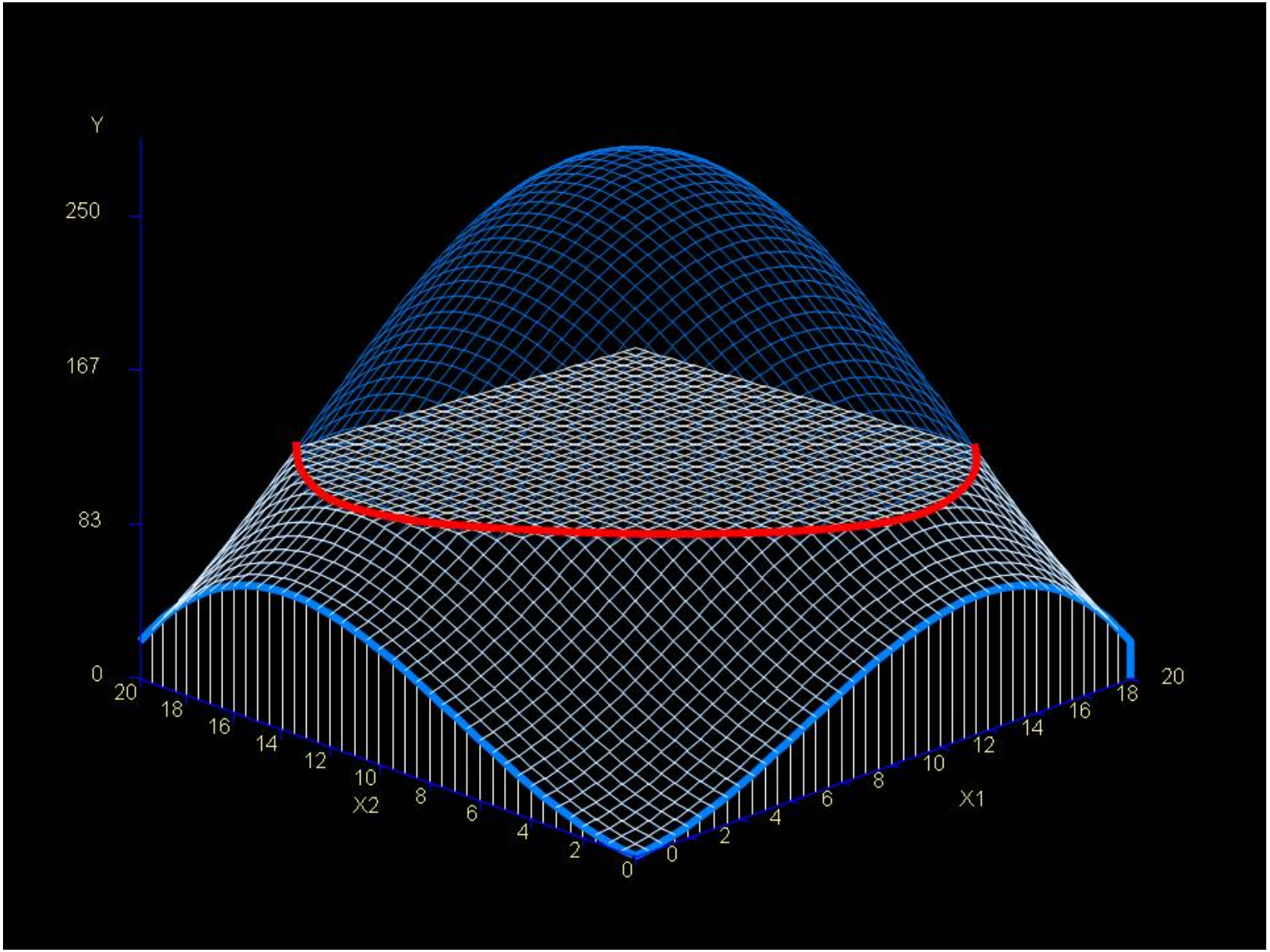
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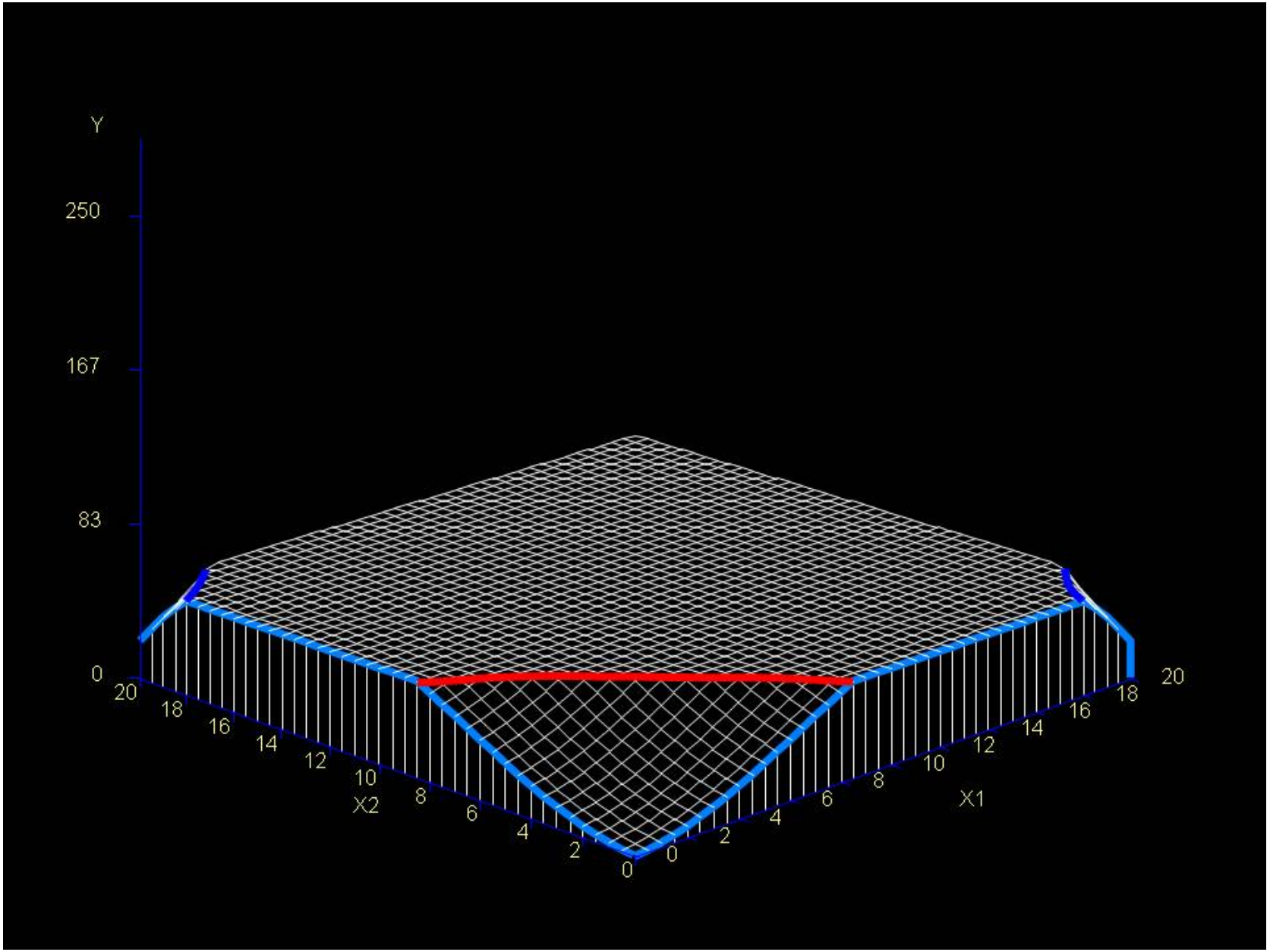
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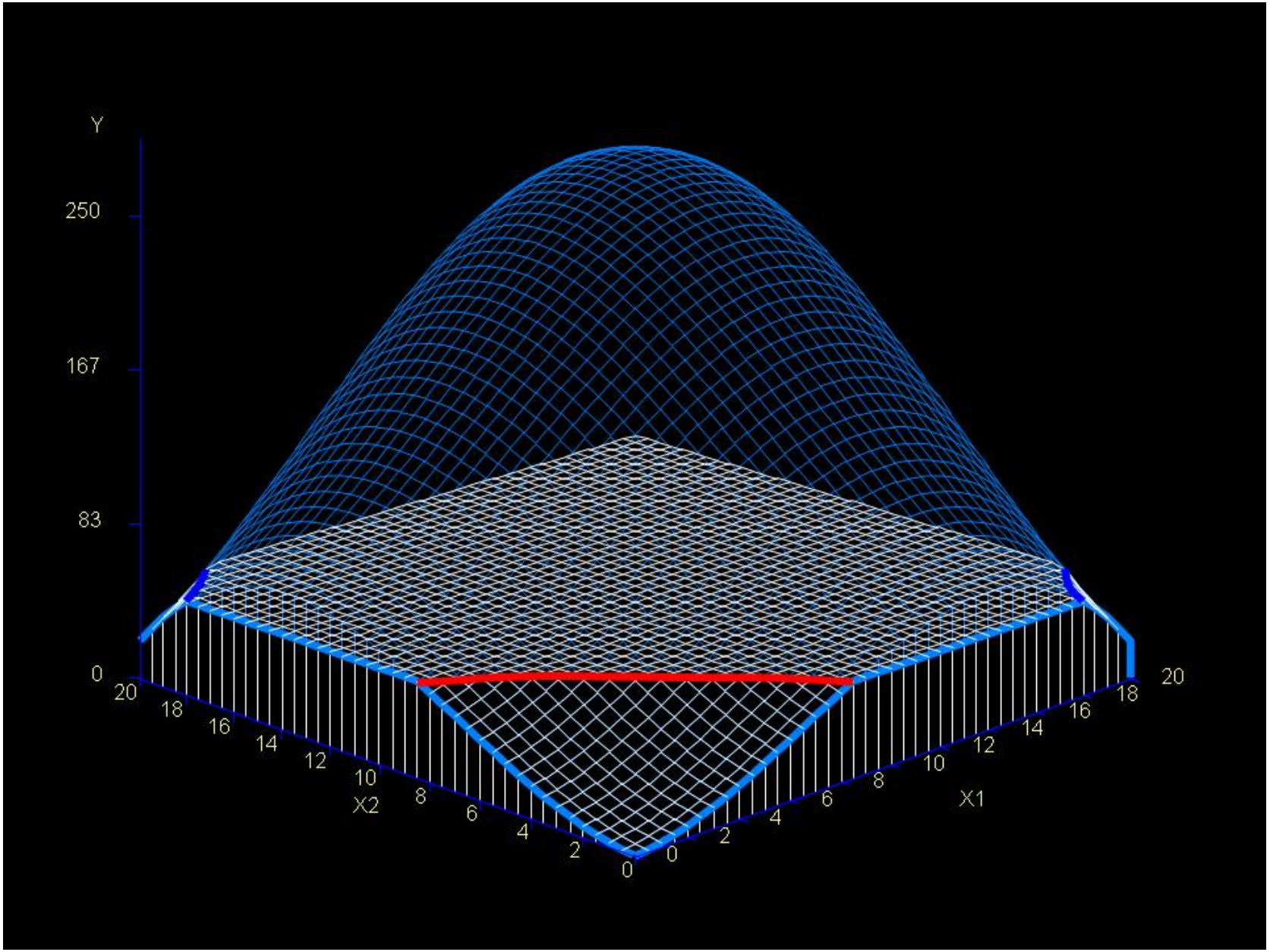
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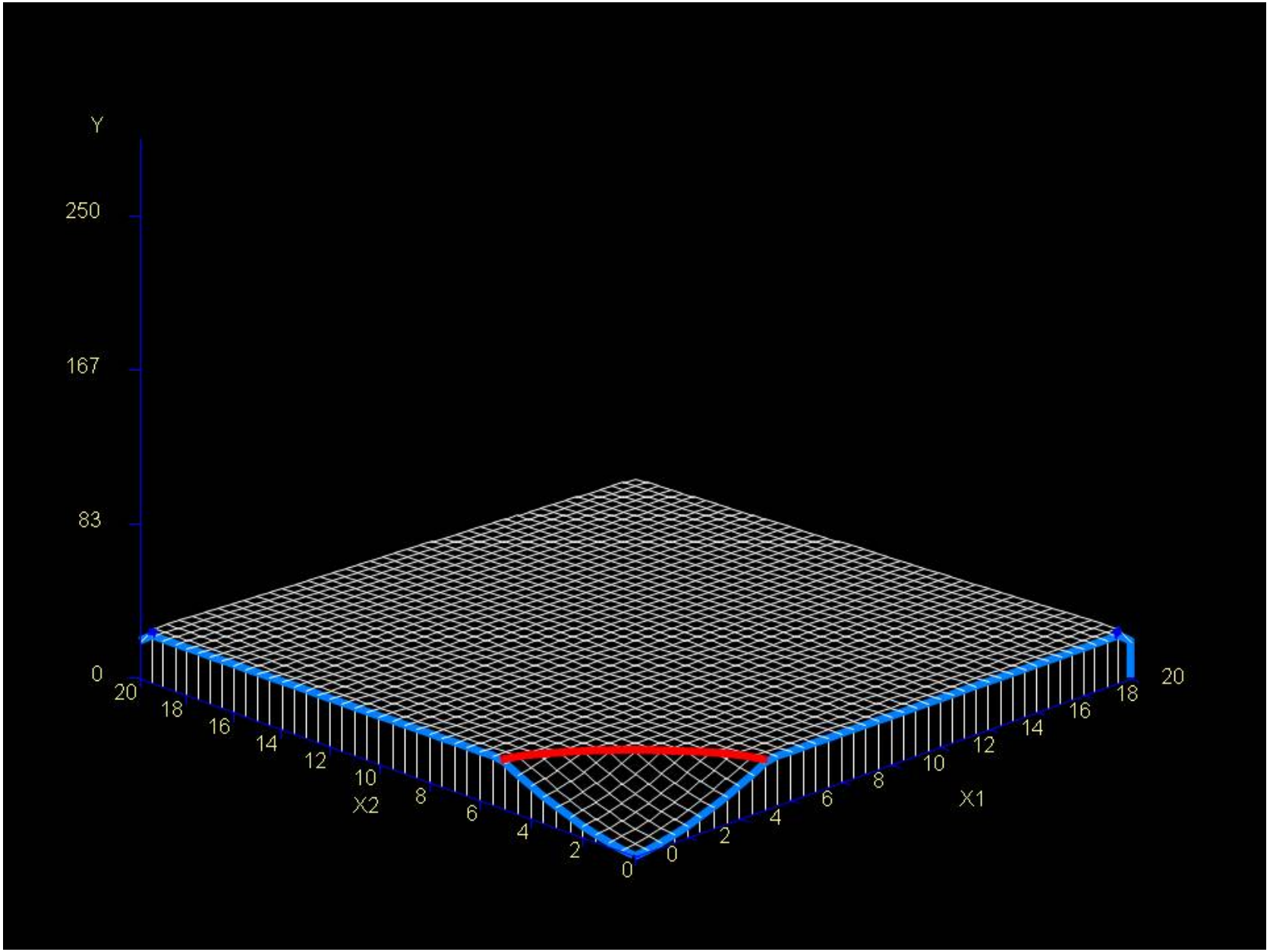
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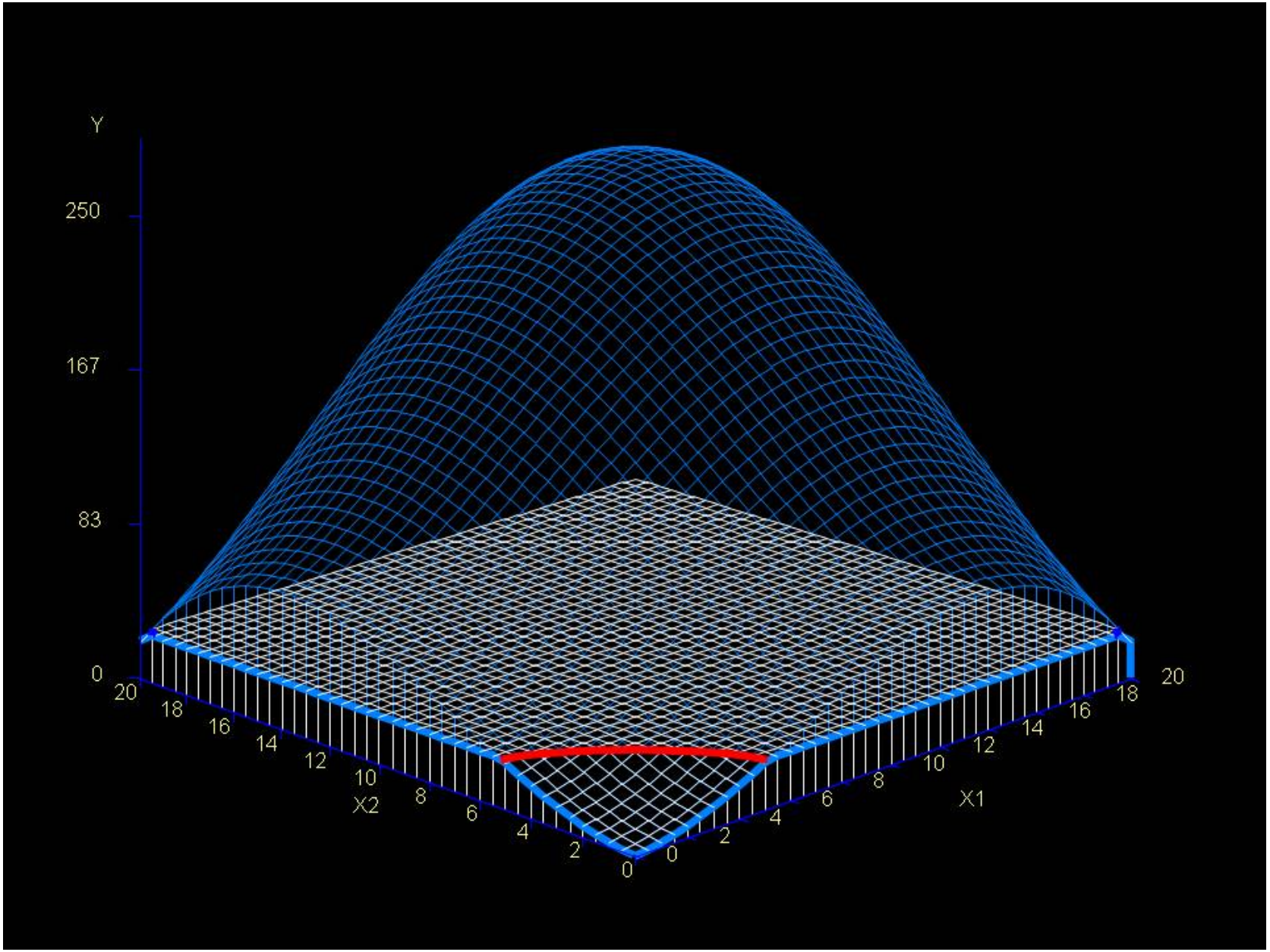
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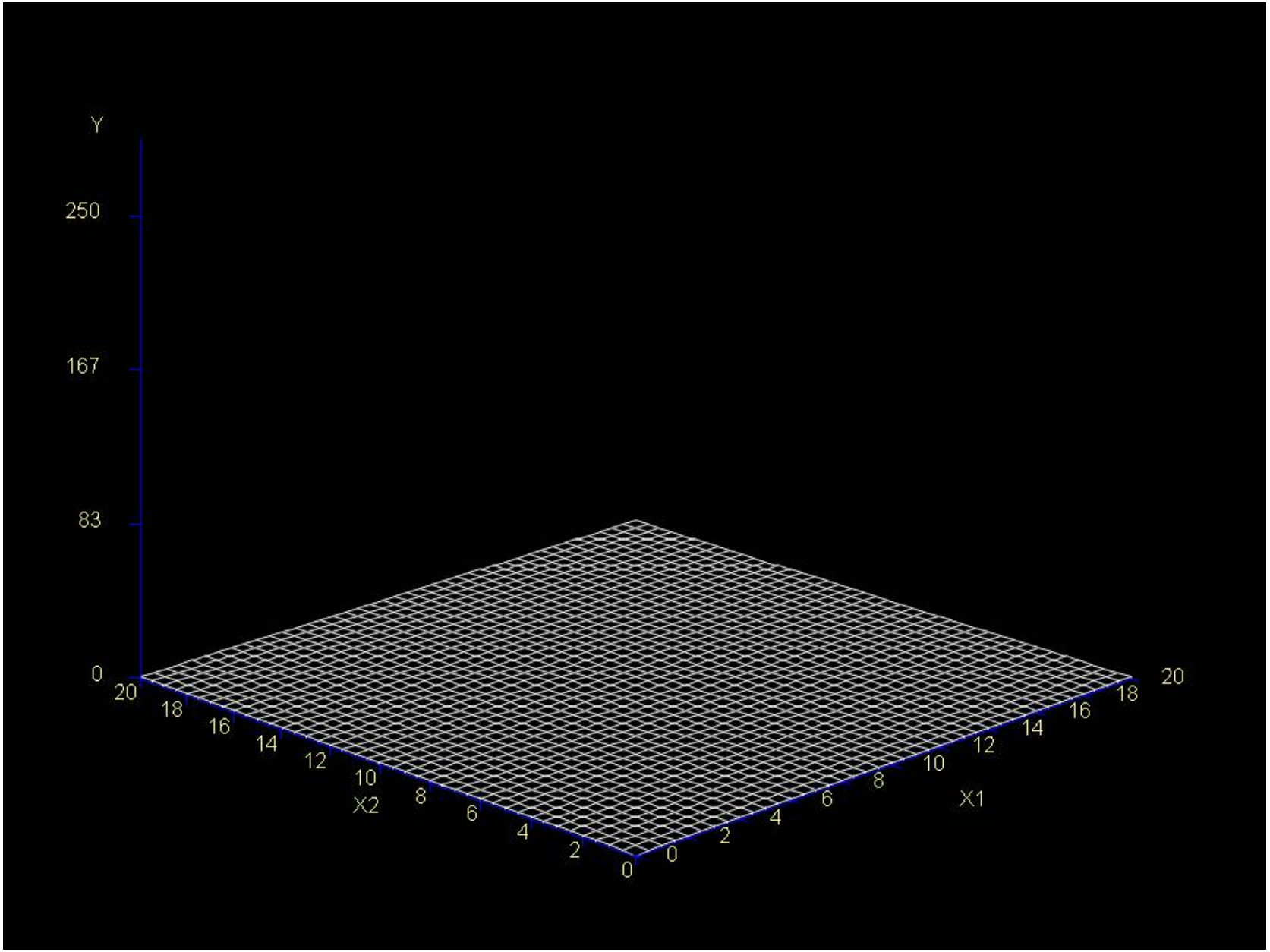
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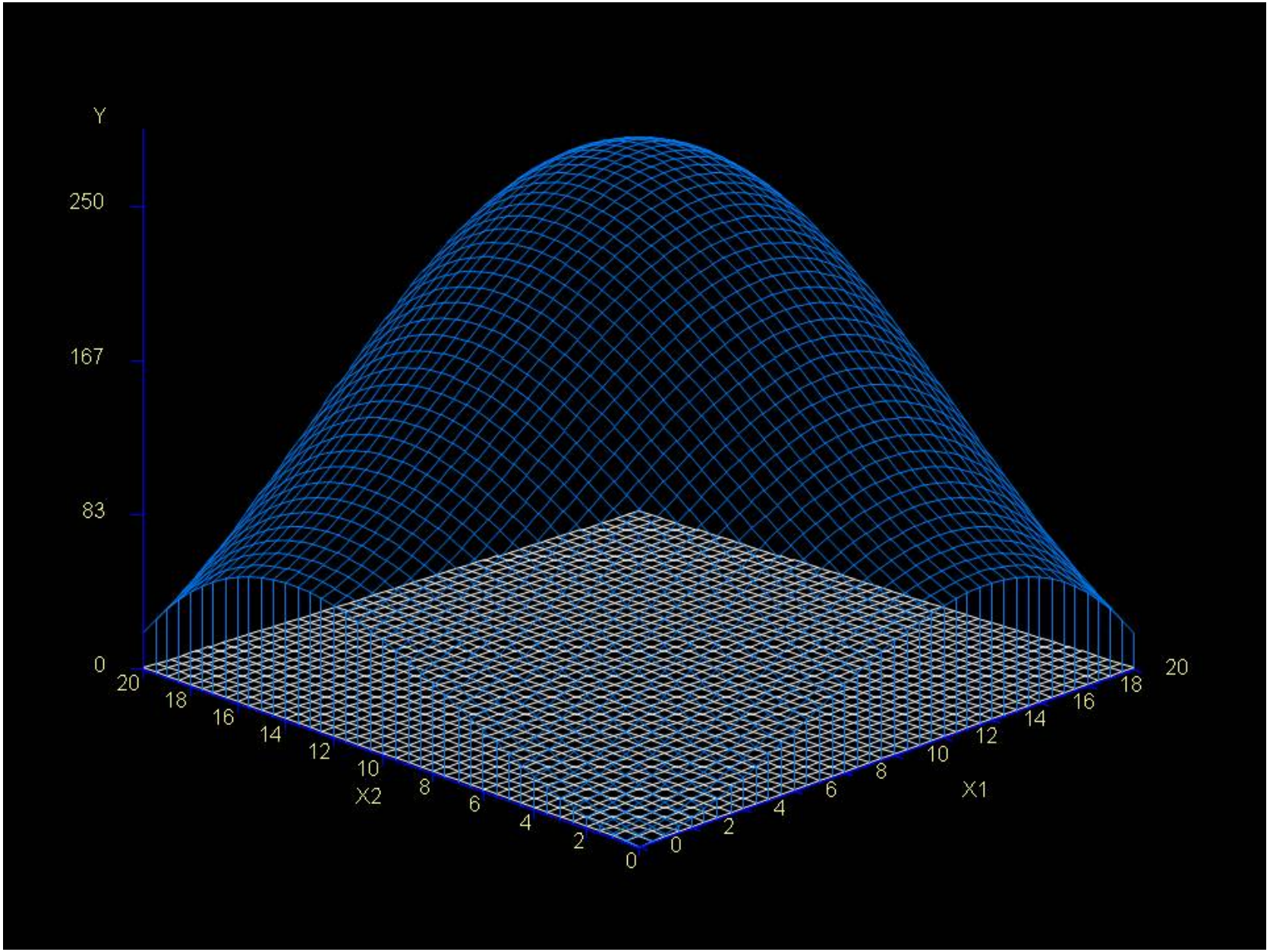
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In the following sequence, isoquants representing various output levels appear in different colors and the output level represented by each isoquant corresponds with the key at the bottom of the chart.

The budget constraint is represented by red lines of constant slope P1/P2 where P1 is the price of input x_1, and P2 is the price of input x_2.

Increases in the amount of money available for the purchase of inputs shift the budget constraint outward.

Each budget constraint is tangent (just touches) an isoquant.

These points, here marked by blue circles, are where P1/P2 = the Marginal Rate of Substitution of x_1 for x_2.

Valid constrained output maximization points are only those on isoquants that are convex to the origin of the graph. Points on concave isoquants are constrained output minimization points, and are marked with a yellow X.

The expansion path connects valid points of constrained output maximization, and is shown in green.

Ridge line 1 connects all points of zero slope on the isoquants.

Ridge line 2 connects all points of infinite slope on the isoquants.

Ridge lines, shown here in blue-green intersect at the global point of output maximization.

This occurs at $x_1 = x_2 = 16.41$ and $y = 237.21$.

Pseudo Scale Line 1 connects profit maximization points for x_1 holding x_2 constant.

Each point on Pseudo Scale Line 1 is defined by MPP x_1 = P1/Py, where P1 is the price of x_1 and Py is the output price.

Pseudo Scale Line 2 connects profit maximization points for x_2 holding x_1 constant.

Each point on Pseudo Scale Line 2 is defined by MPP x_2 = P2/Py, where P2 is the price of x_2 and Py is the output price.

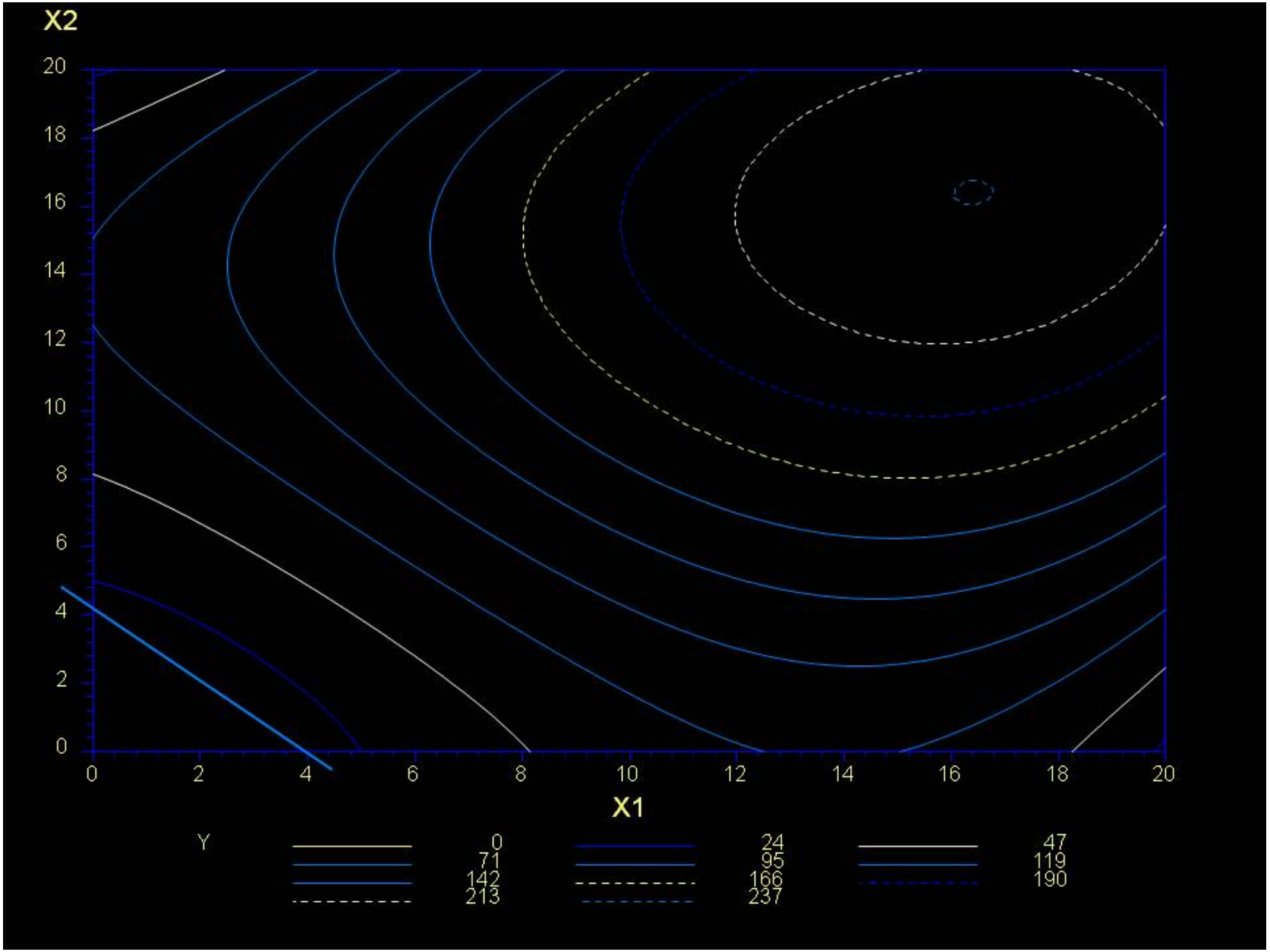
Pseudo Scale Lines, shown here in orange, Converge at the point of global profit maximization.

What happens to Pseudo Scale Lines and the position of the Expansion Path when one of the input prices changes is also shown.

First, P2 (the price of x_2) is increased.
Pseudo Scale Line 2 moves in from Ridge Line 2, and the Expansion Path now favors the use of the now relatively cheaper input x_1.

Then P2 (the price of x_2) is decreased.
Pseudo Scale Line 2 moves toward Ridge Line 2, and the Expansion Path now favors the use of the now relatively cheaper input x_2.

Pseudo scale line 1 has not moved in either case as the price of x_1 has not changed.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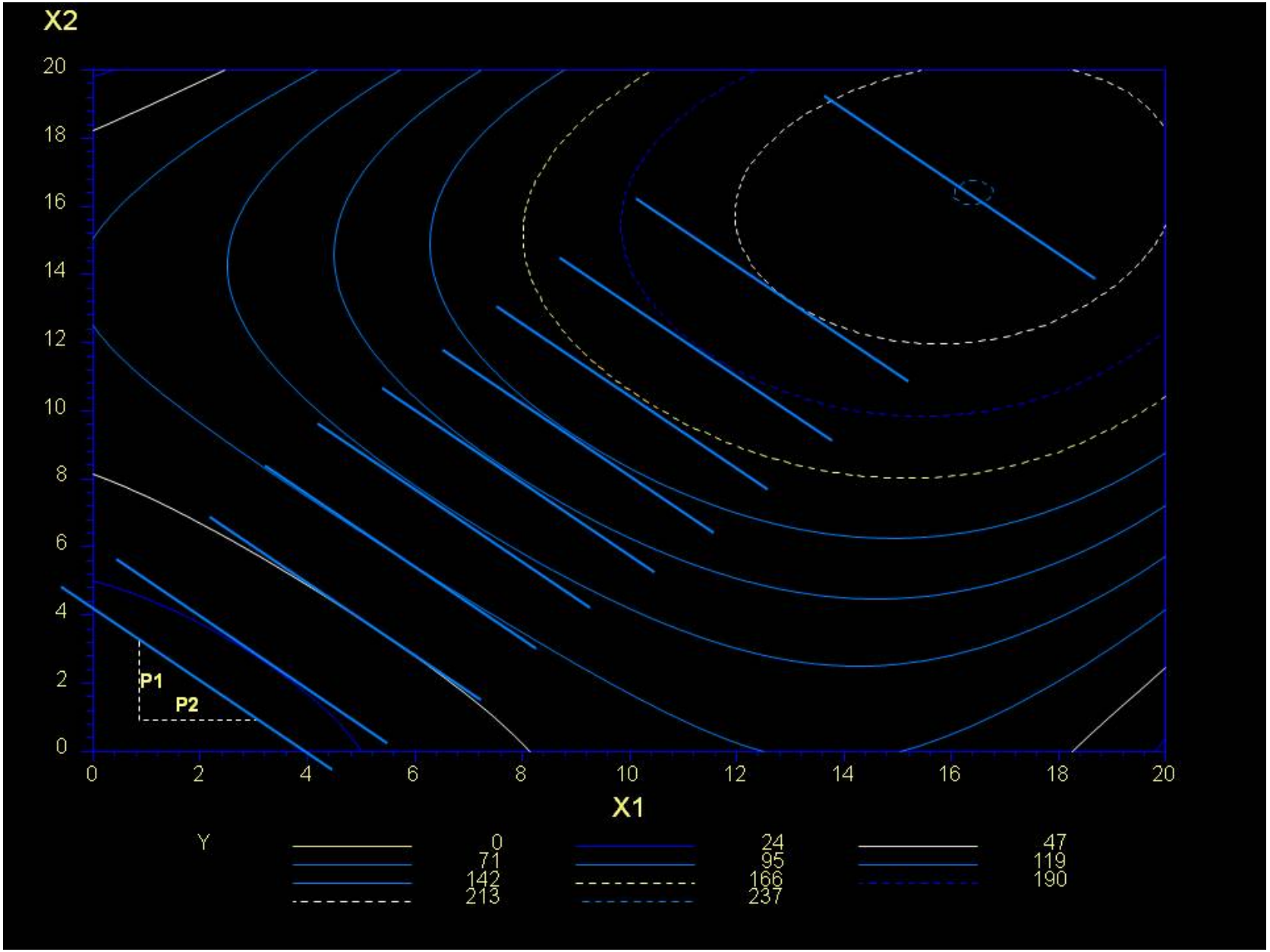
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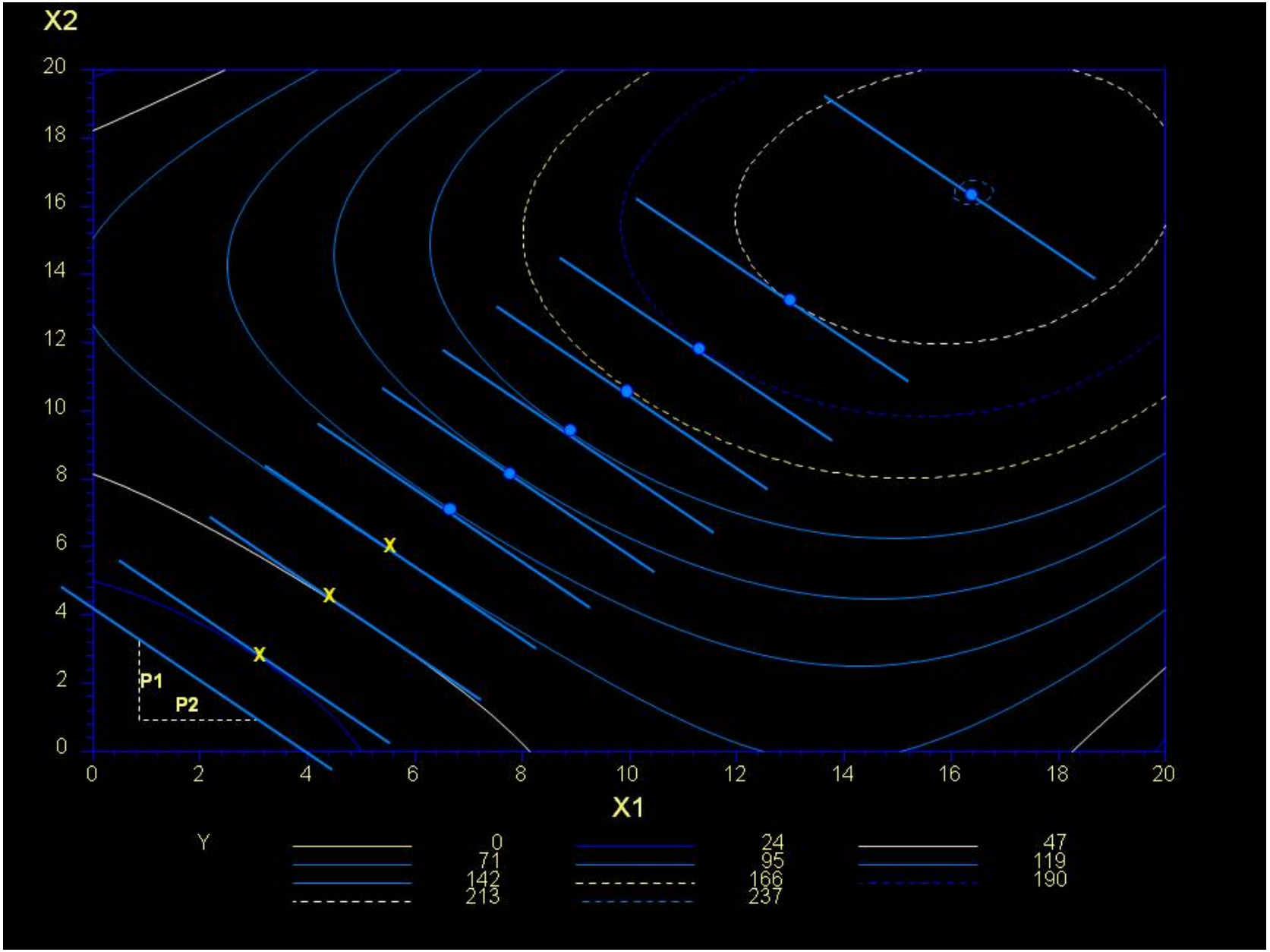
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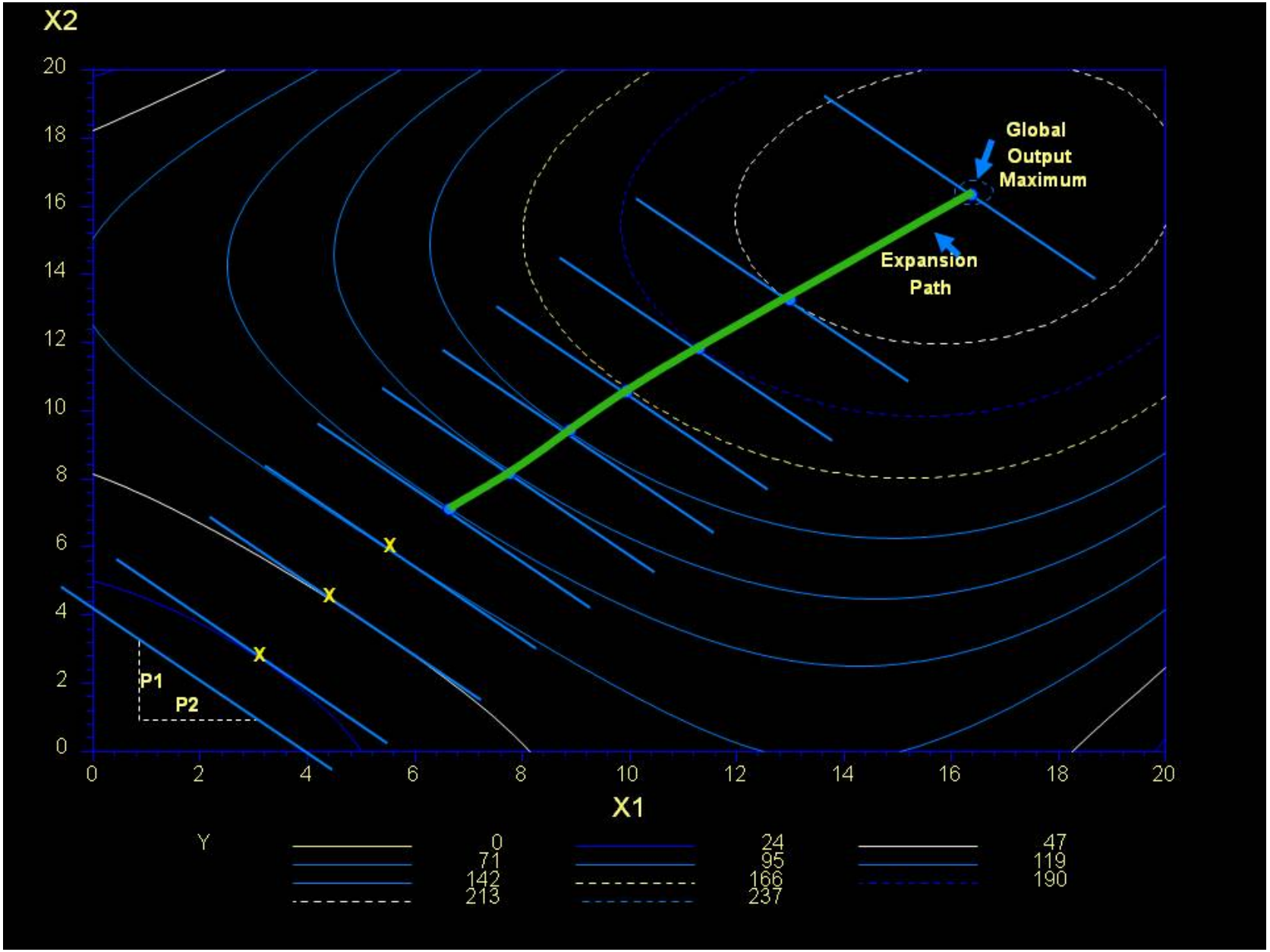
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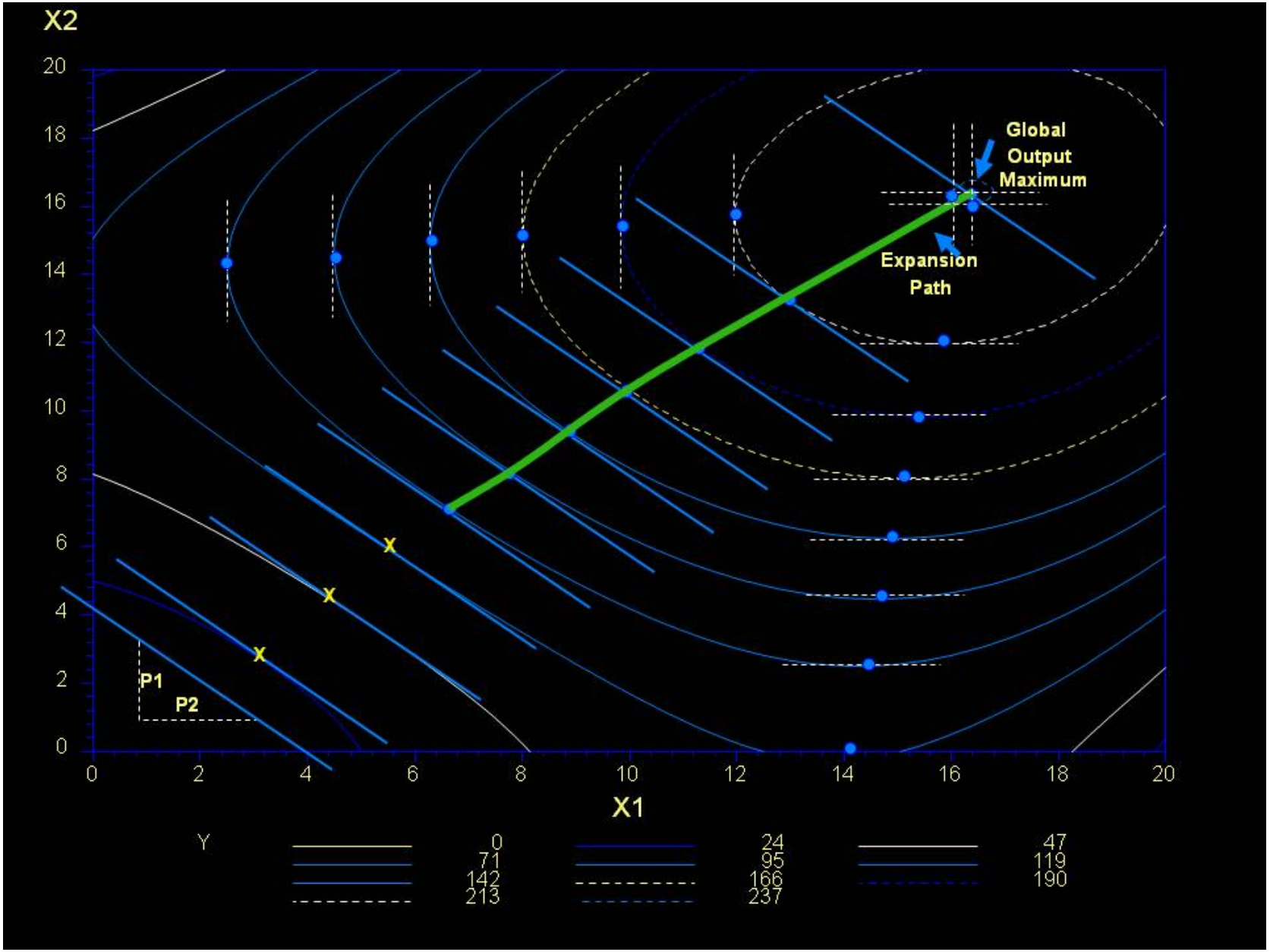
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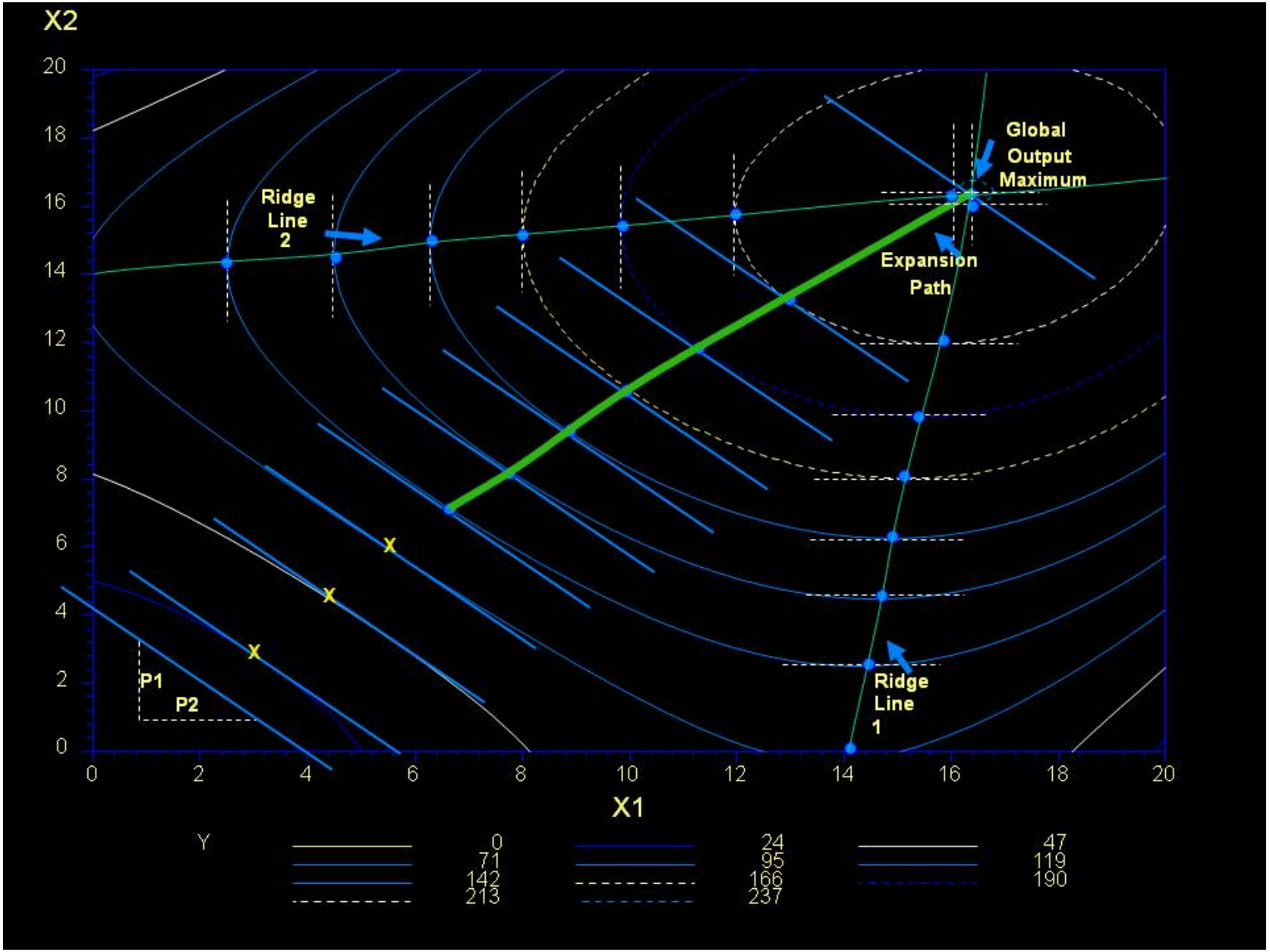
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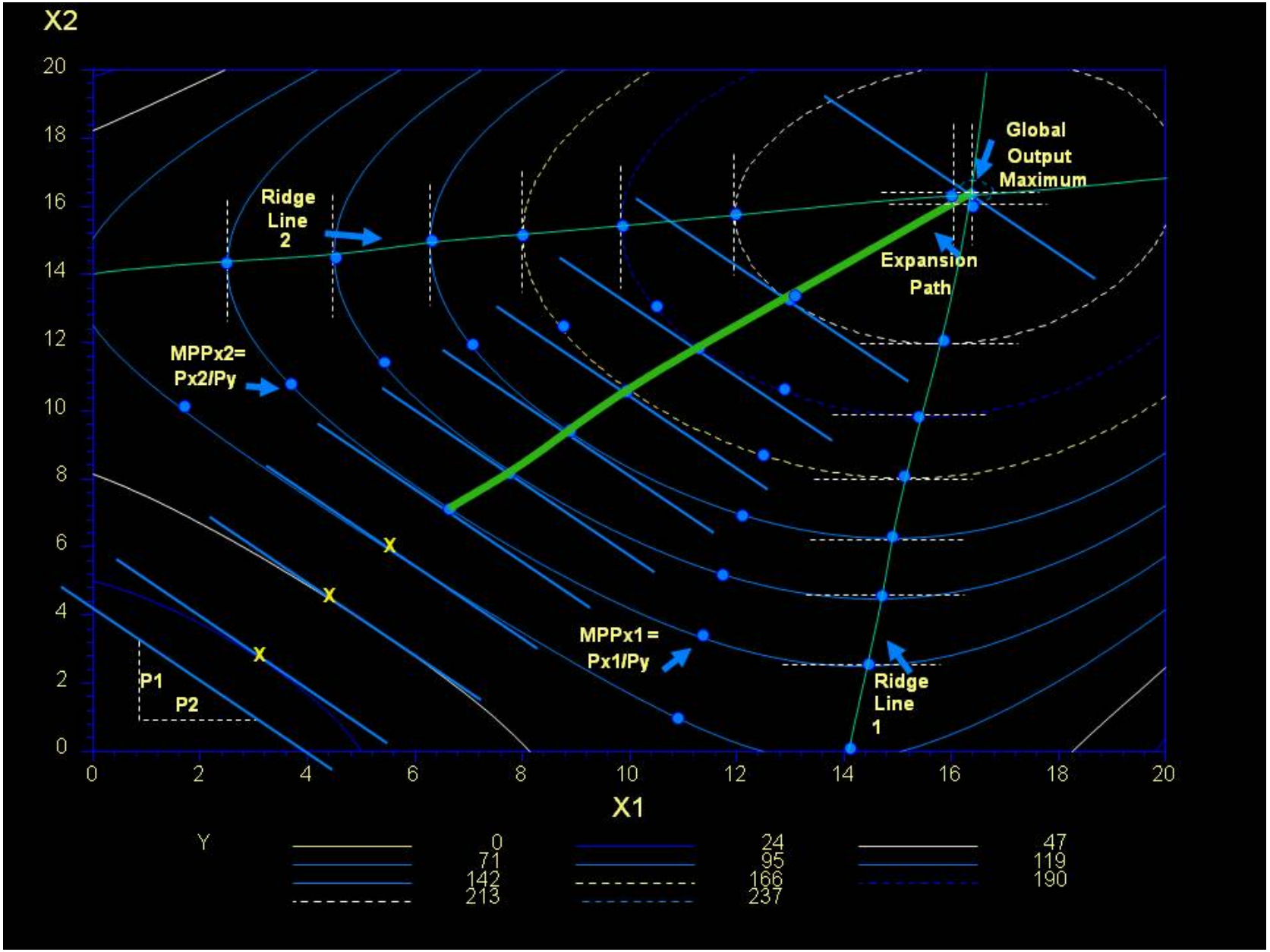
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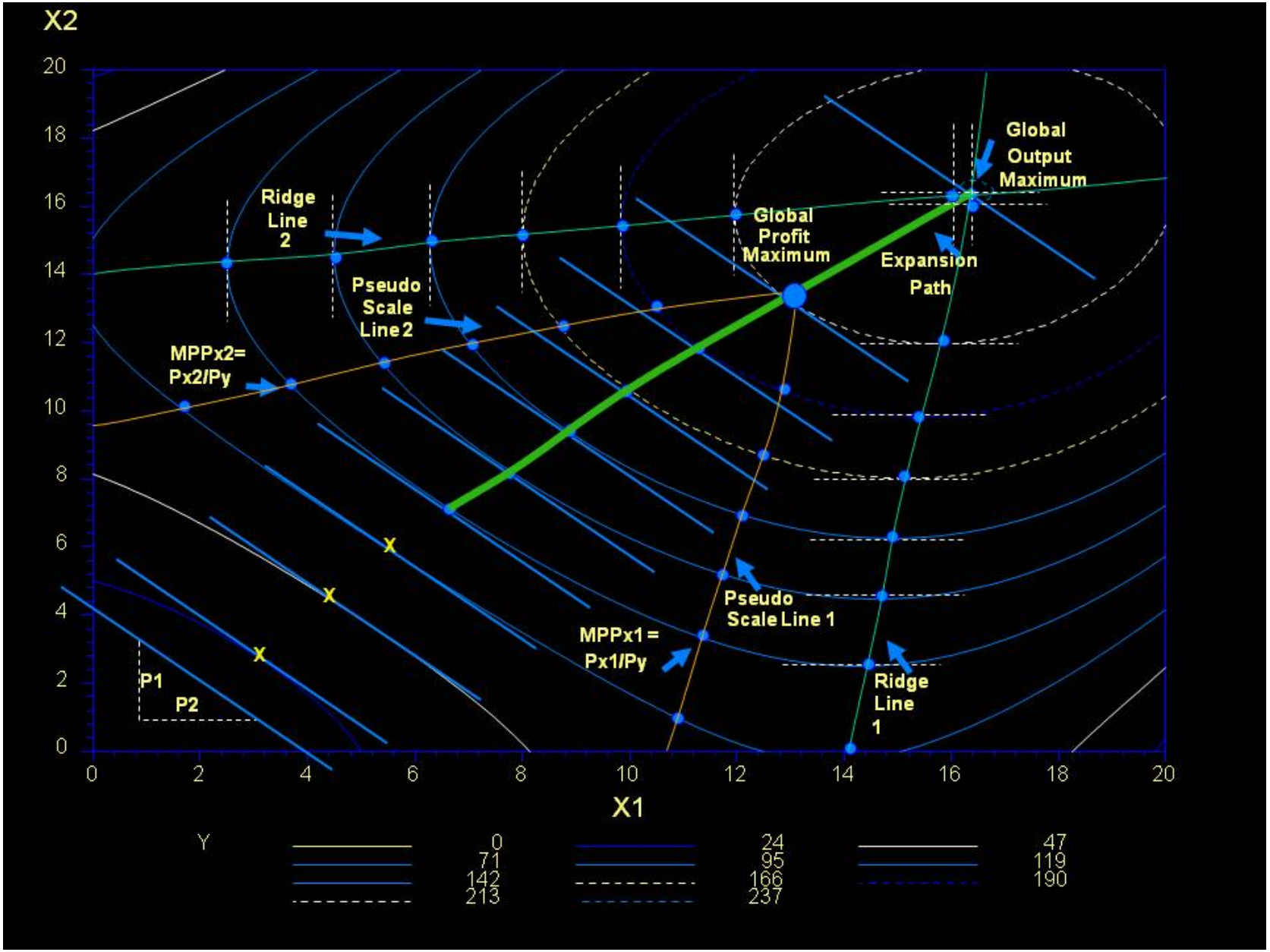
X2
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Ridge Line 2
Pseudo Scale Line 2
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Global Profit Maximum
Global Output Maximum
Expansion Path
Pseudo Scale Line 1
MPPx1 = Px1/Py
Ridge Line 1
P1
P2
Y
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119
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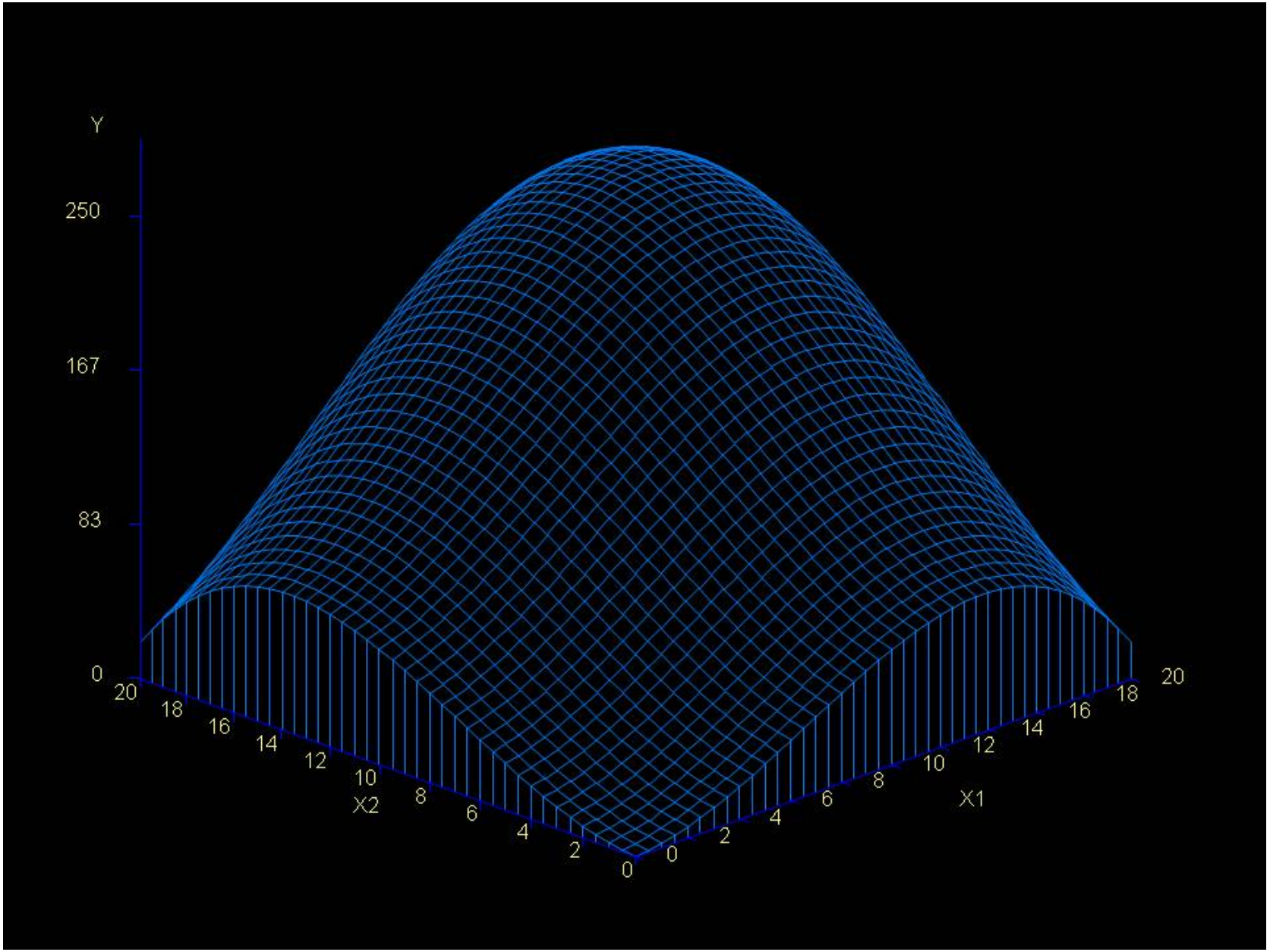
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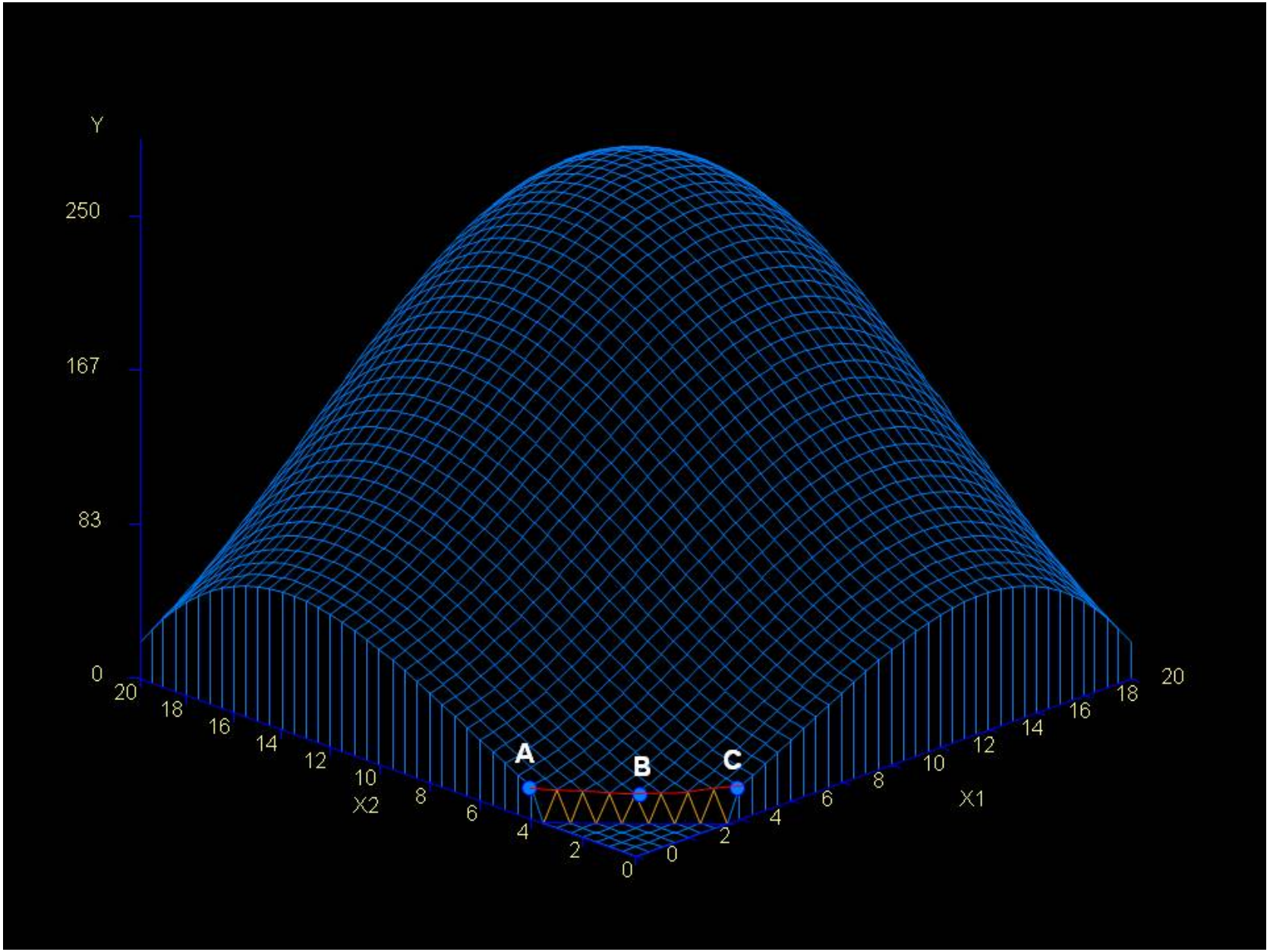
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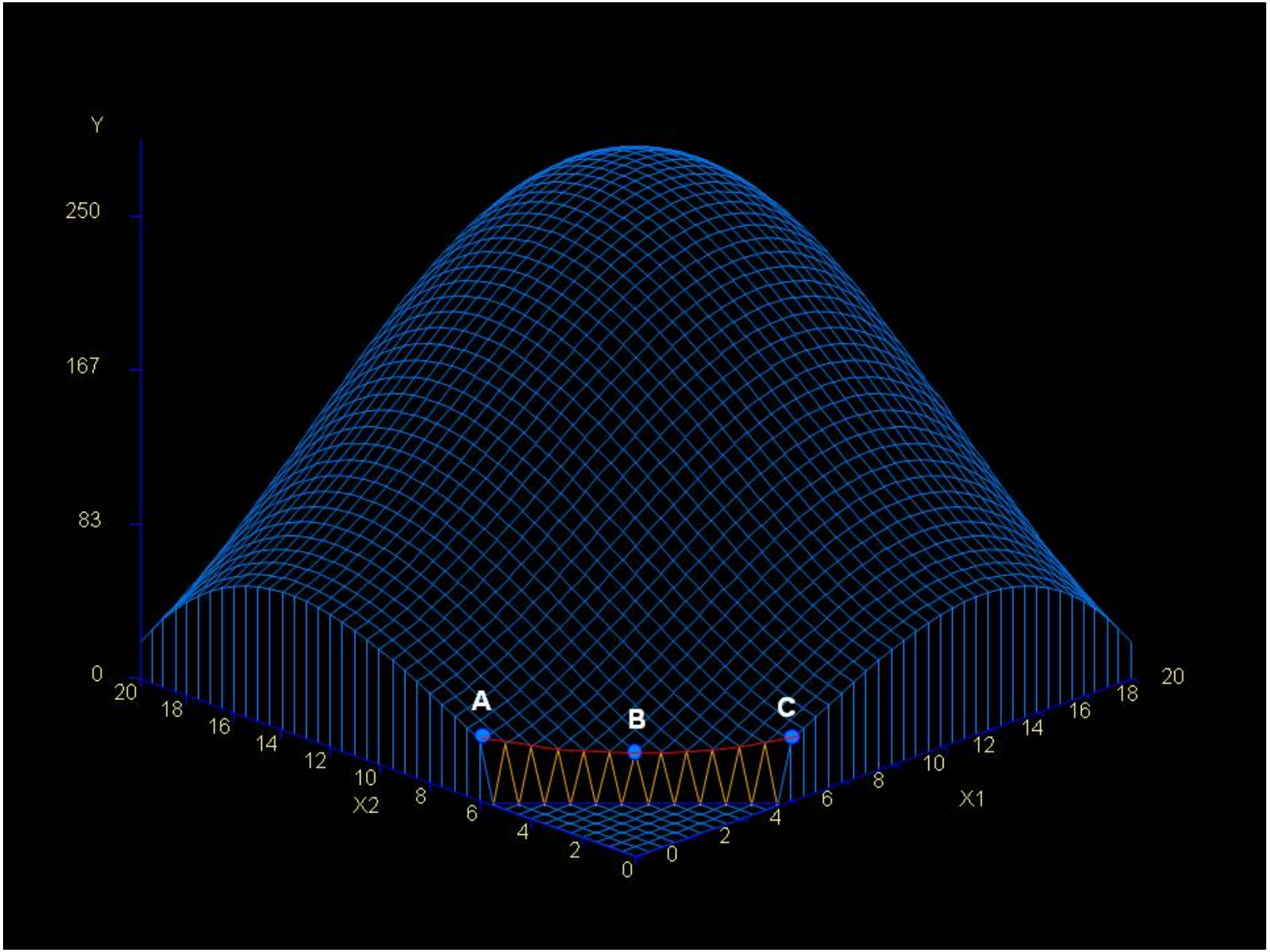
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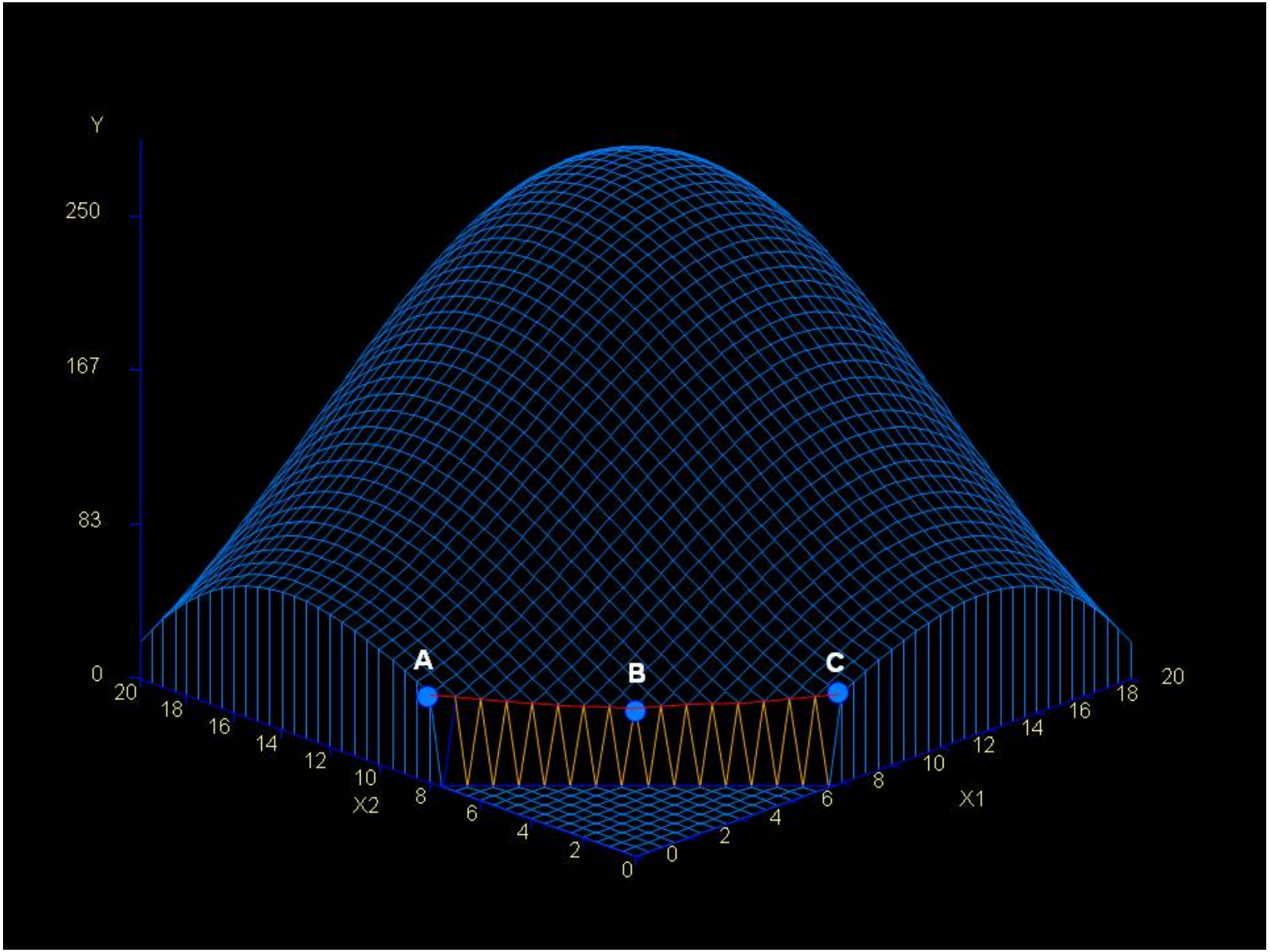
Y
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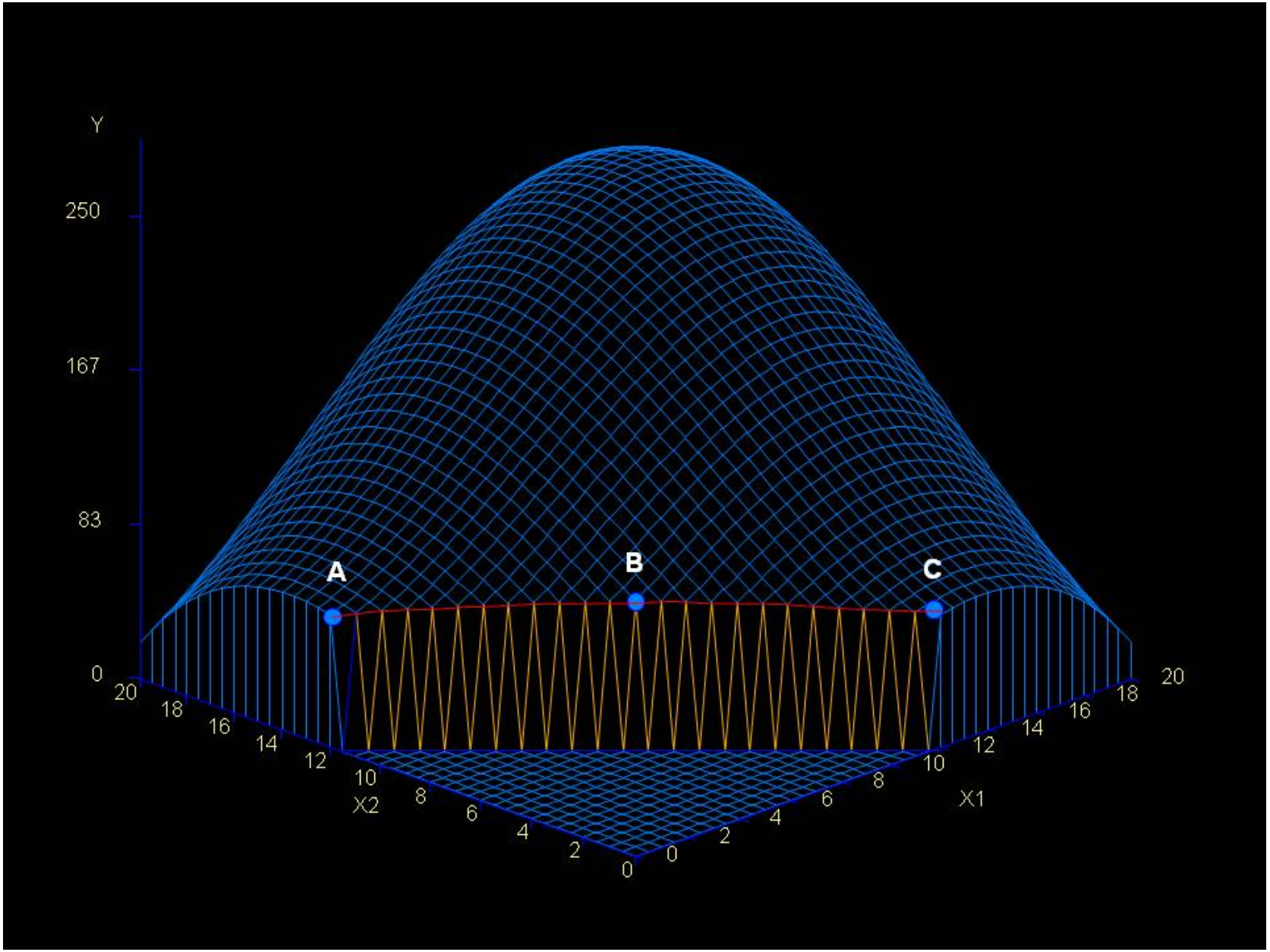
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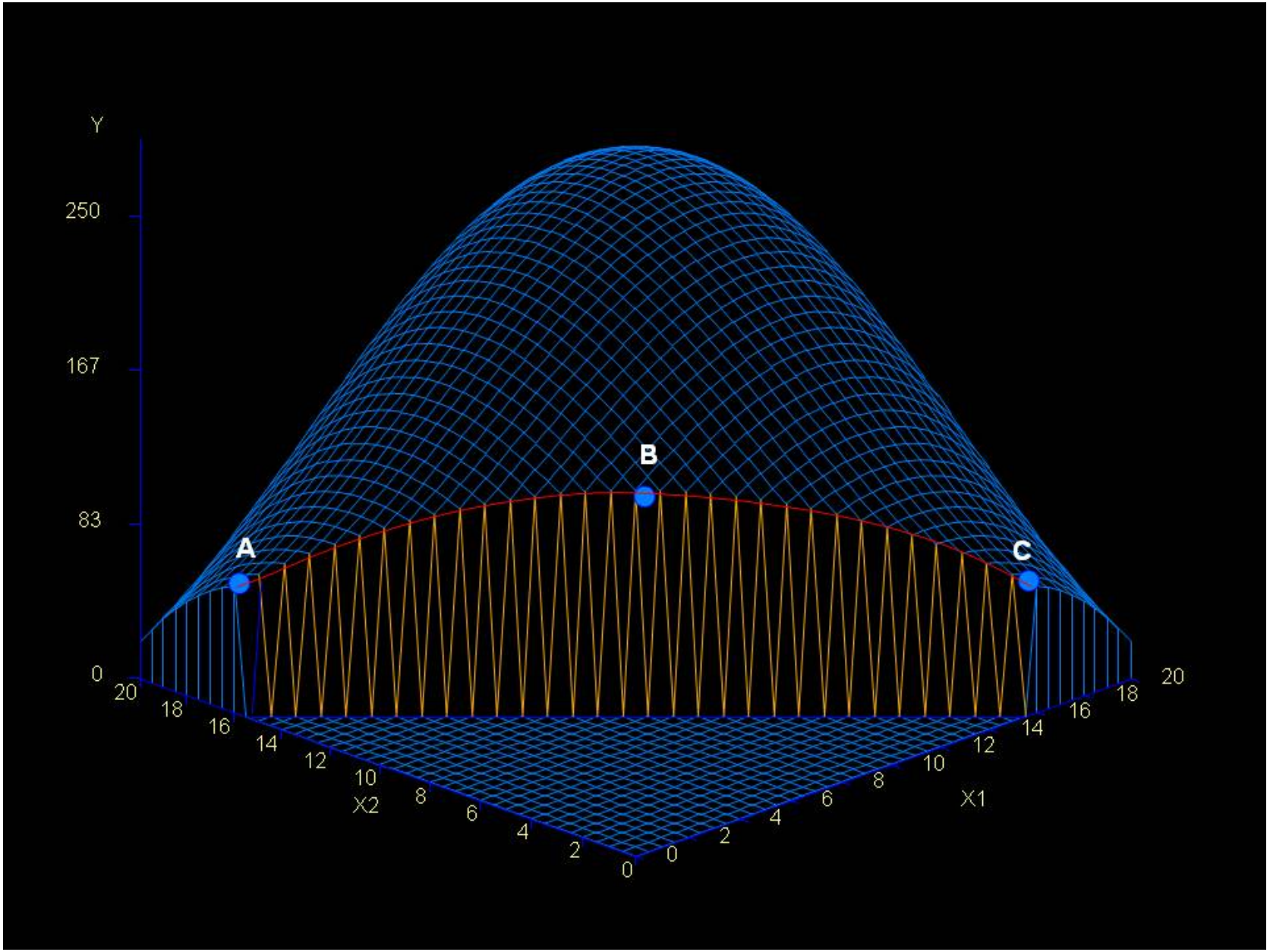
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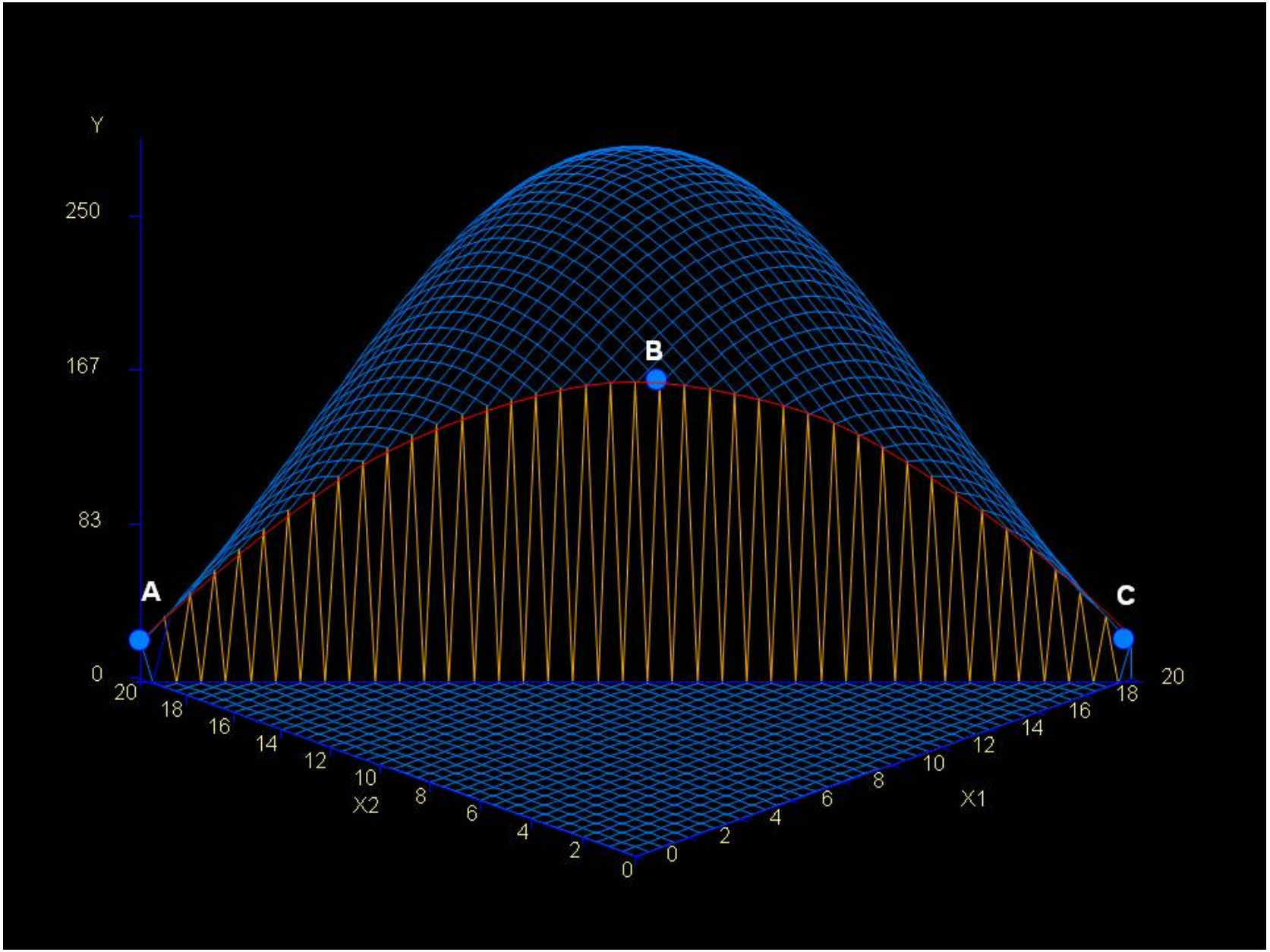
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X2
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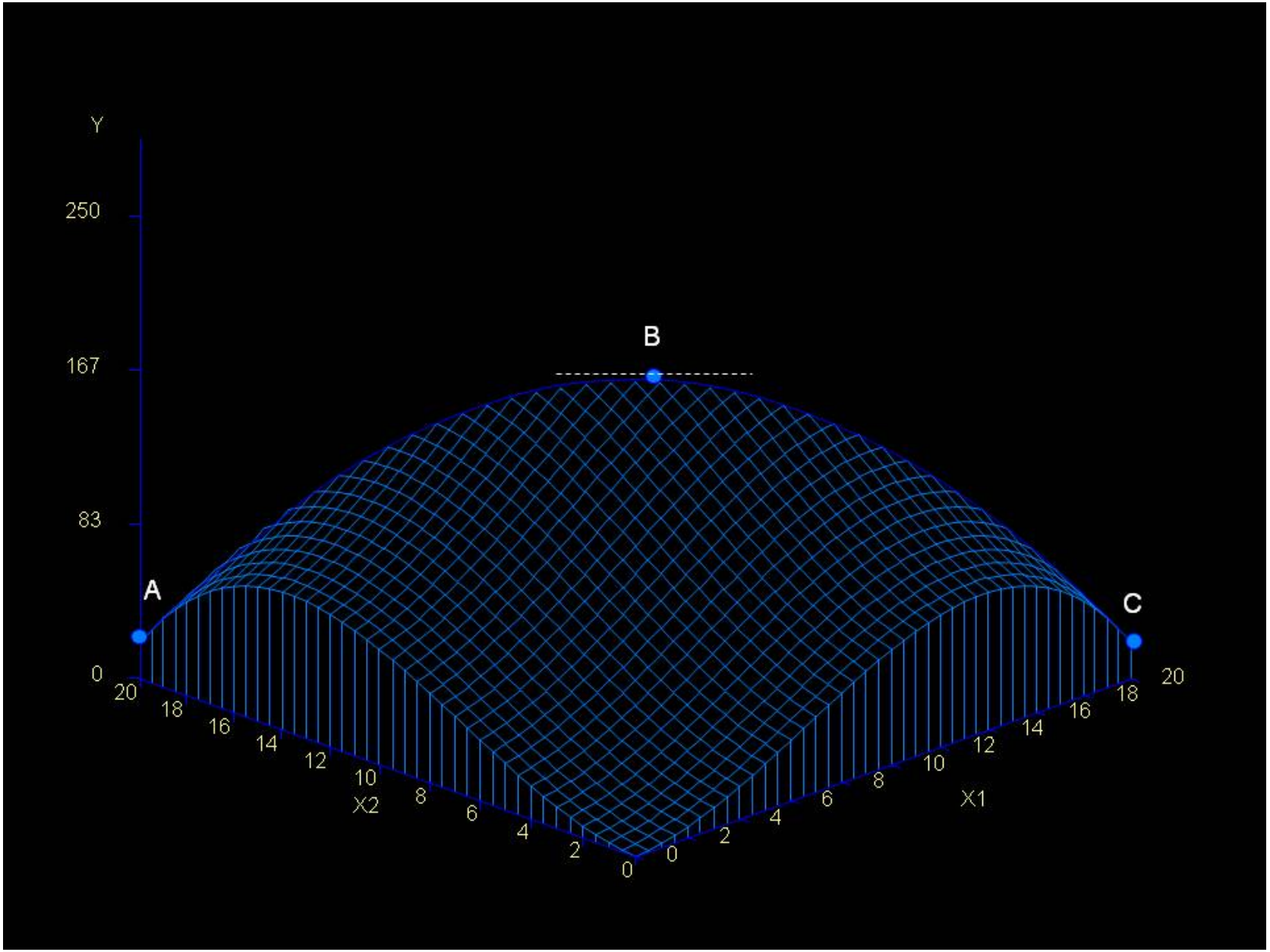
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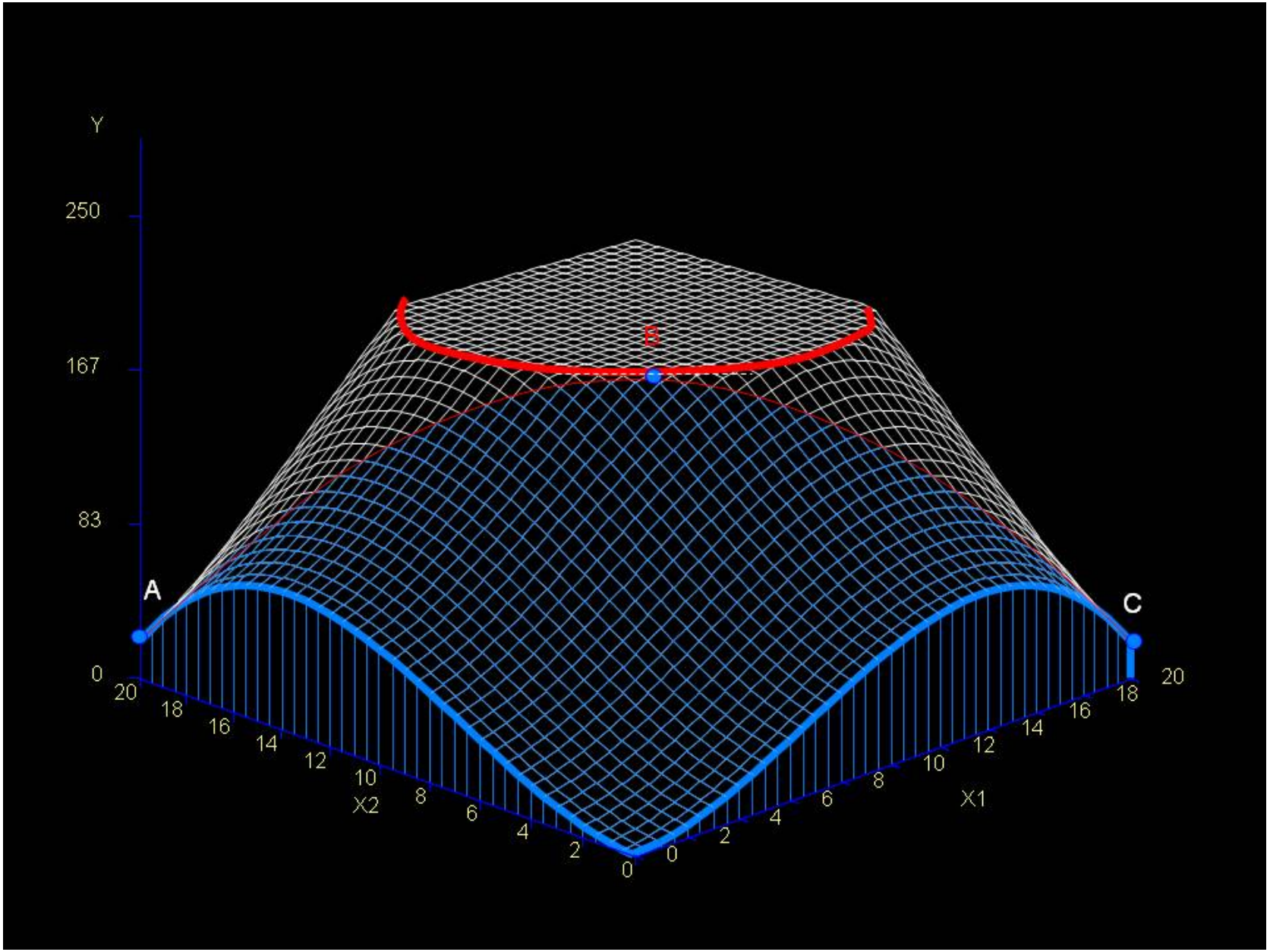
Y
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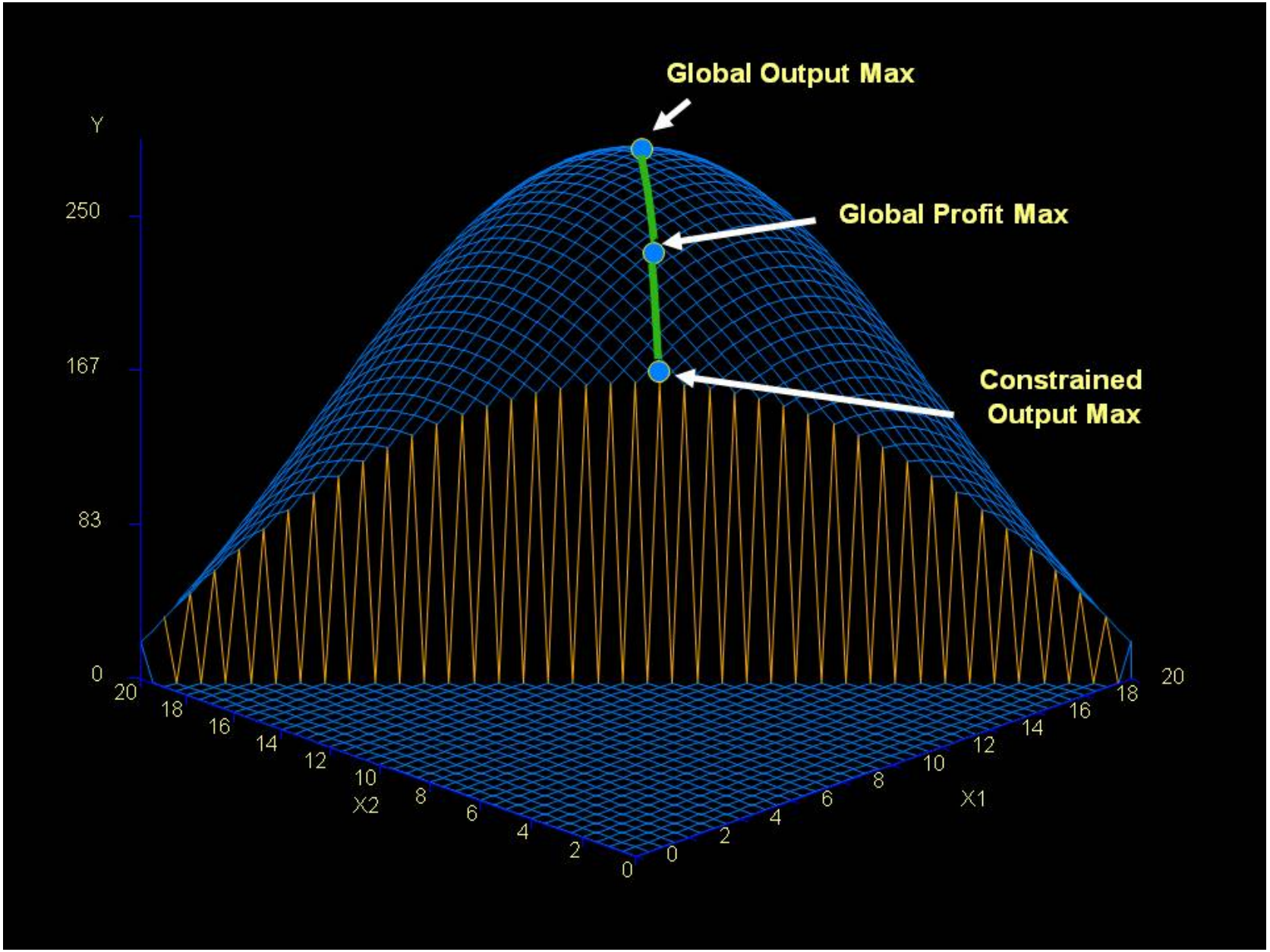
Global Output Max
Global Profit Max
Constrained
Output Max
Y
250
167
83
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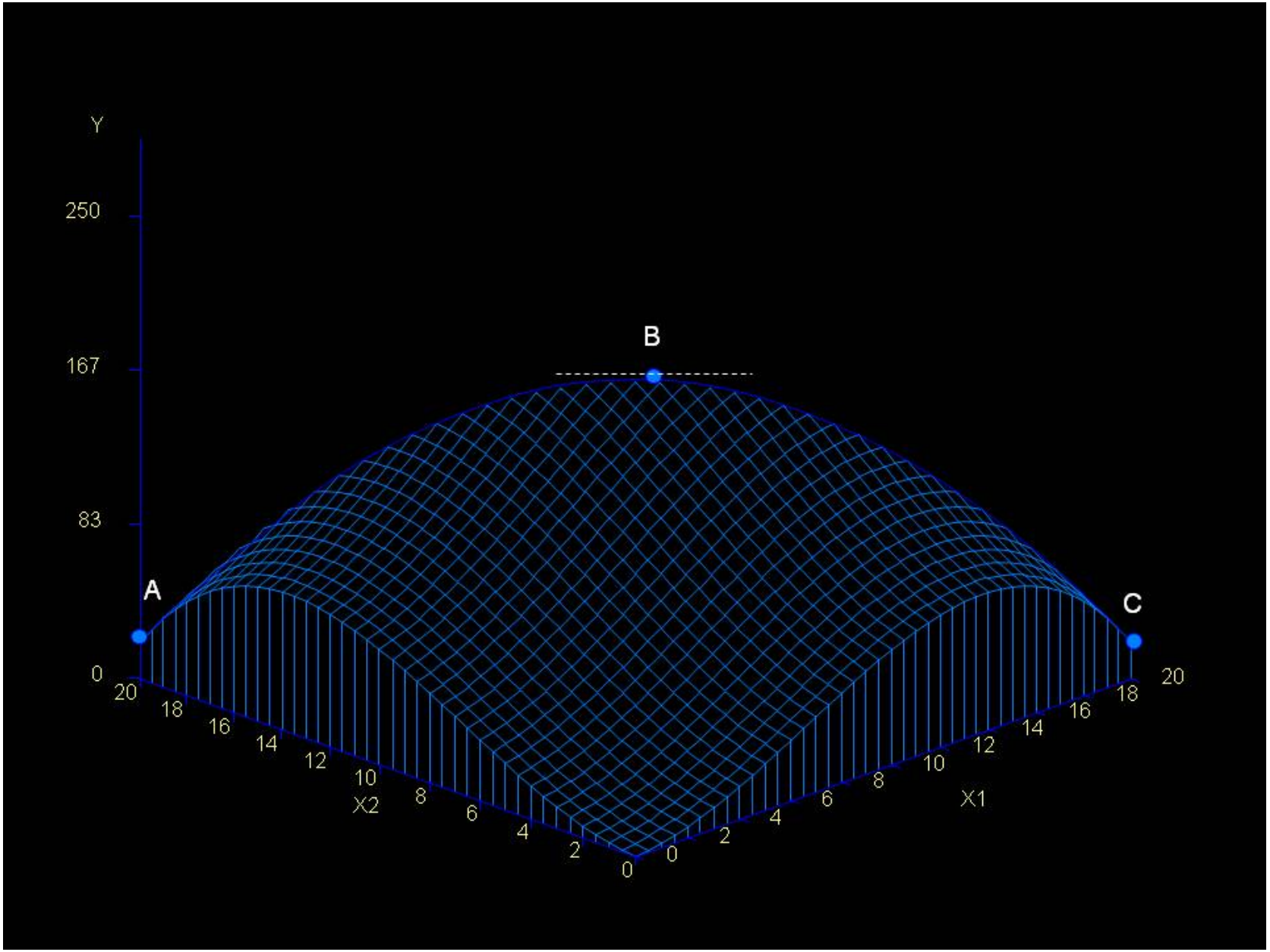
18
20
X1


Y
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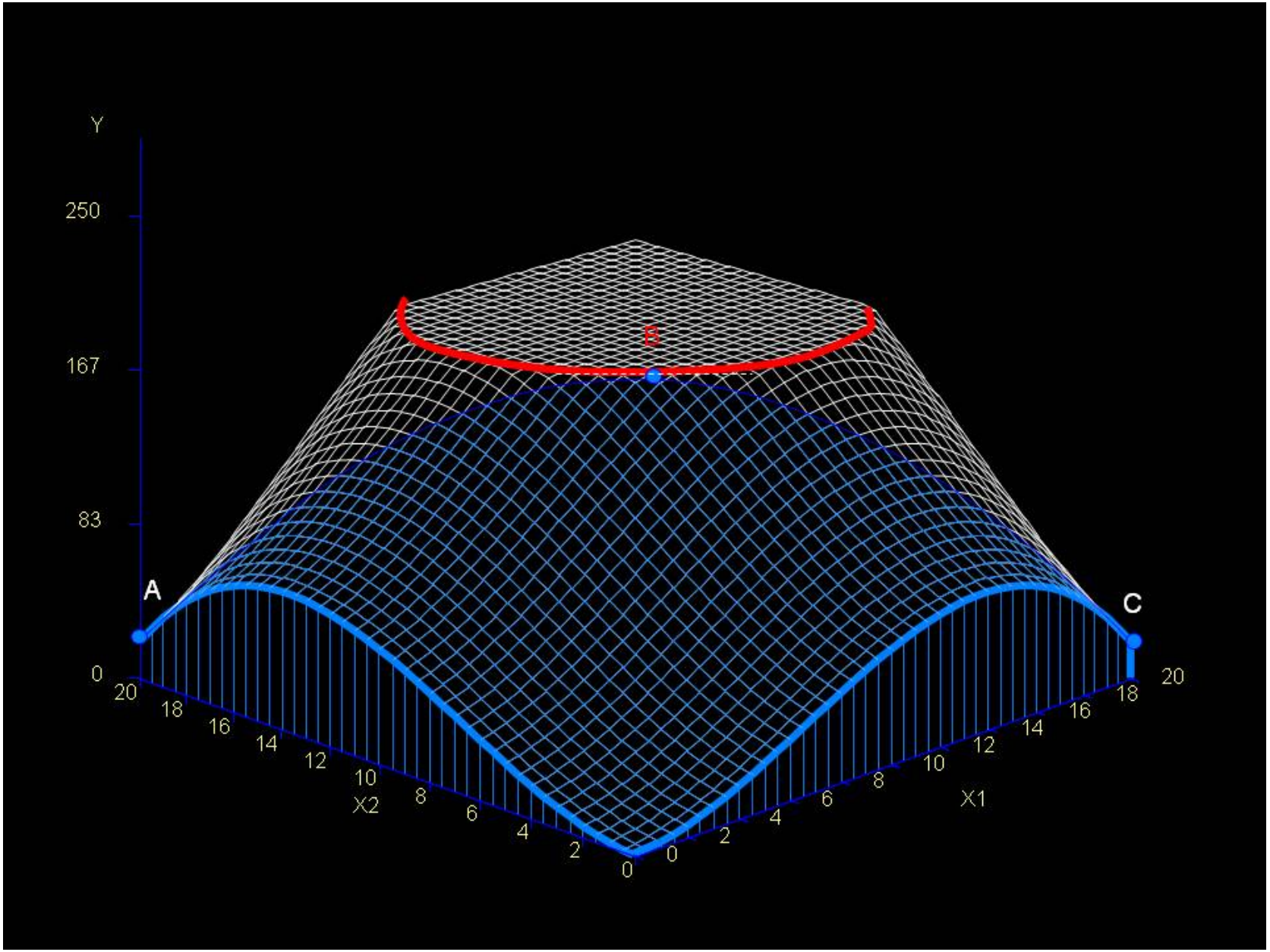
16
18
20
X1


Y
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X2
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12
14
16
18
20
X1

The following sequences illustrate three approaches for precisely locating Pseudo Scale Lines and the point of Global Profit Maximization.

Profit maximization in the single input case always occurs at the point where

$Py\ MPPx = Px$
or $MPPx = Px/Py$, where Px is the price of x.

Consider this relationship in the two-input case.

$Mppx_1 = P1/Py$
$MPPx_2 = P2/Py$, where
$P1$ = price of x_1; $P2$ = price of x_2.

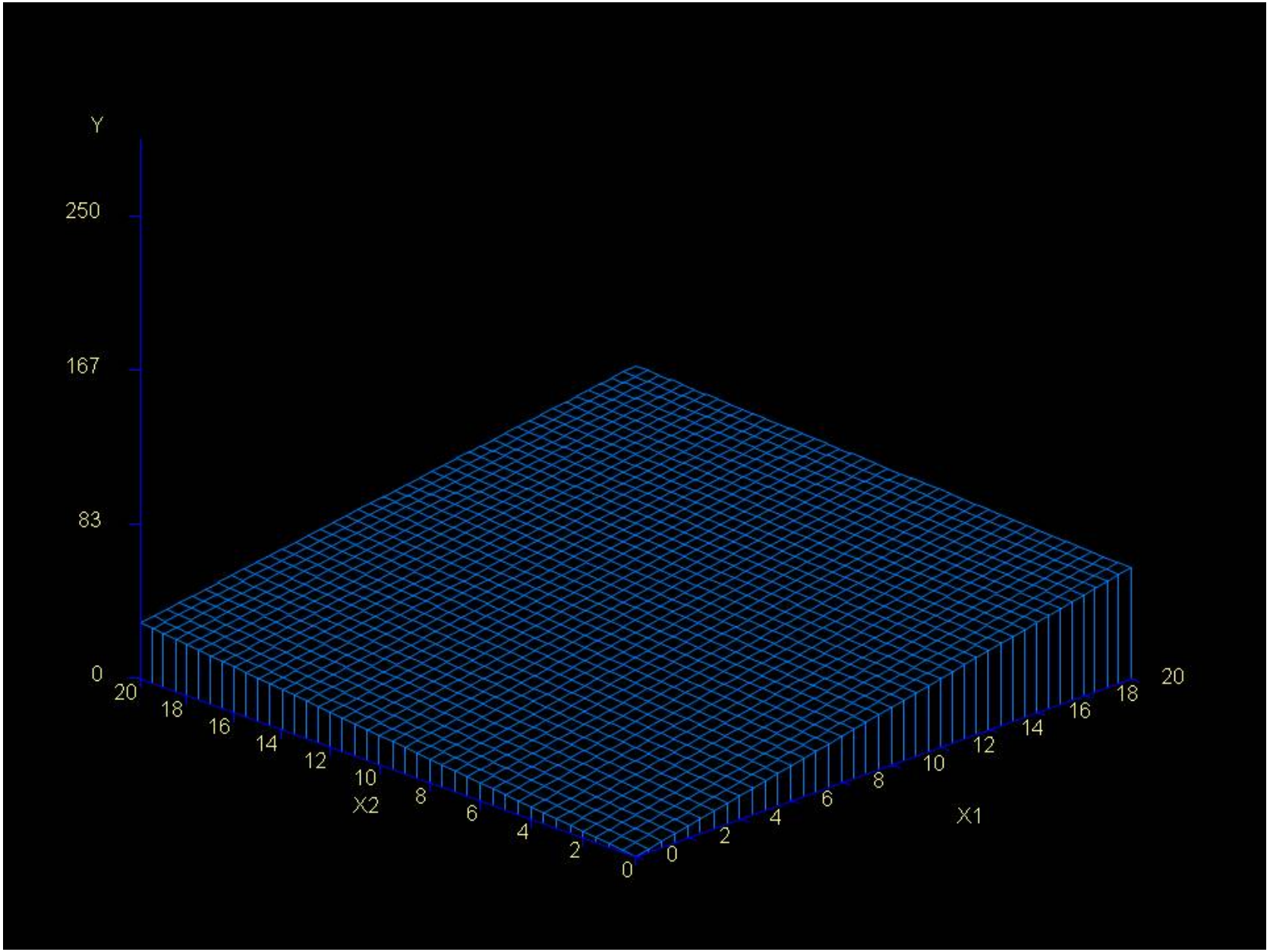
In this sequence, the budget constraint

$C = \$3.00\ x_1 + 1.5\ x_2$ is represented by a hyperplane of constant slope (since input prices are constant.

The output quantity is multiplied by the output price, assumed to be $1.00 per unit.

The hyperplane, shown in red, is raised to locate the position of the Pseudo Scale Lines and the global point of profit maximization.

$MPPx_1 = P1/Py \quad MPPx_2 = P2/Py.$


Y
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167
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X2
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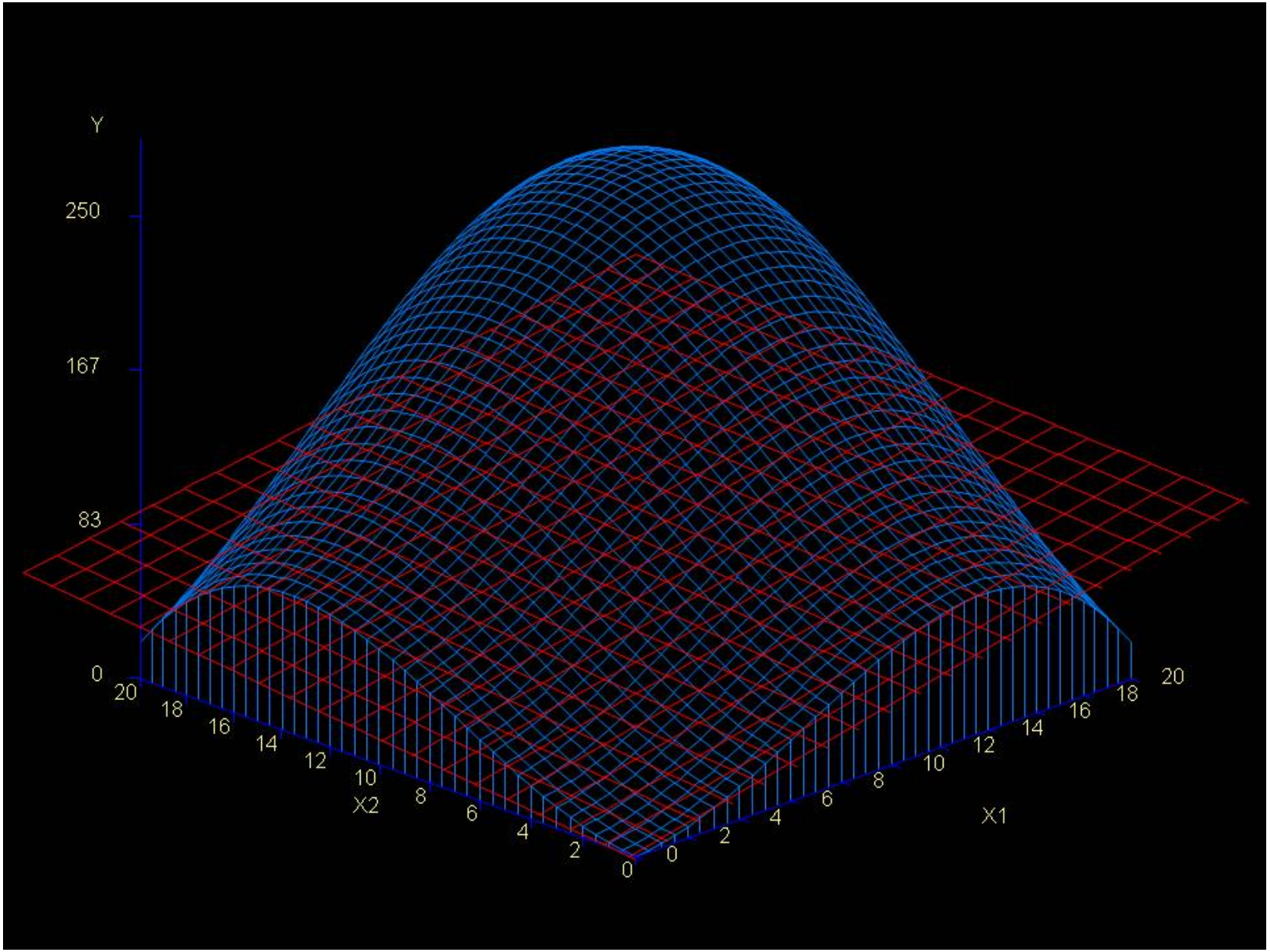
14
16
18
20
X1


Y
250
167
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X2
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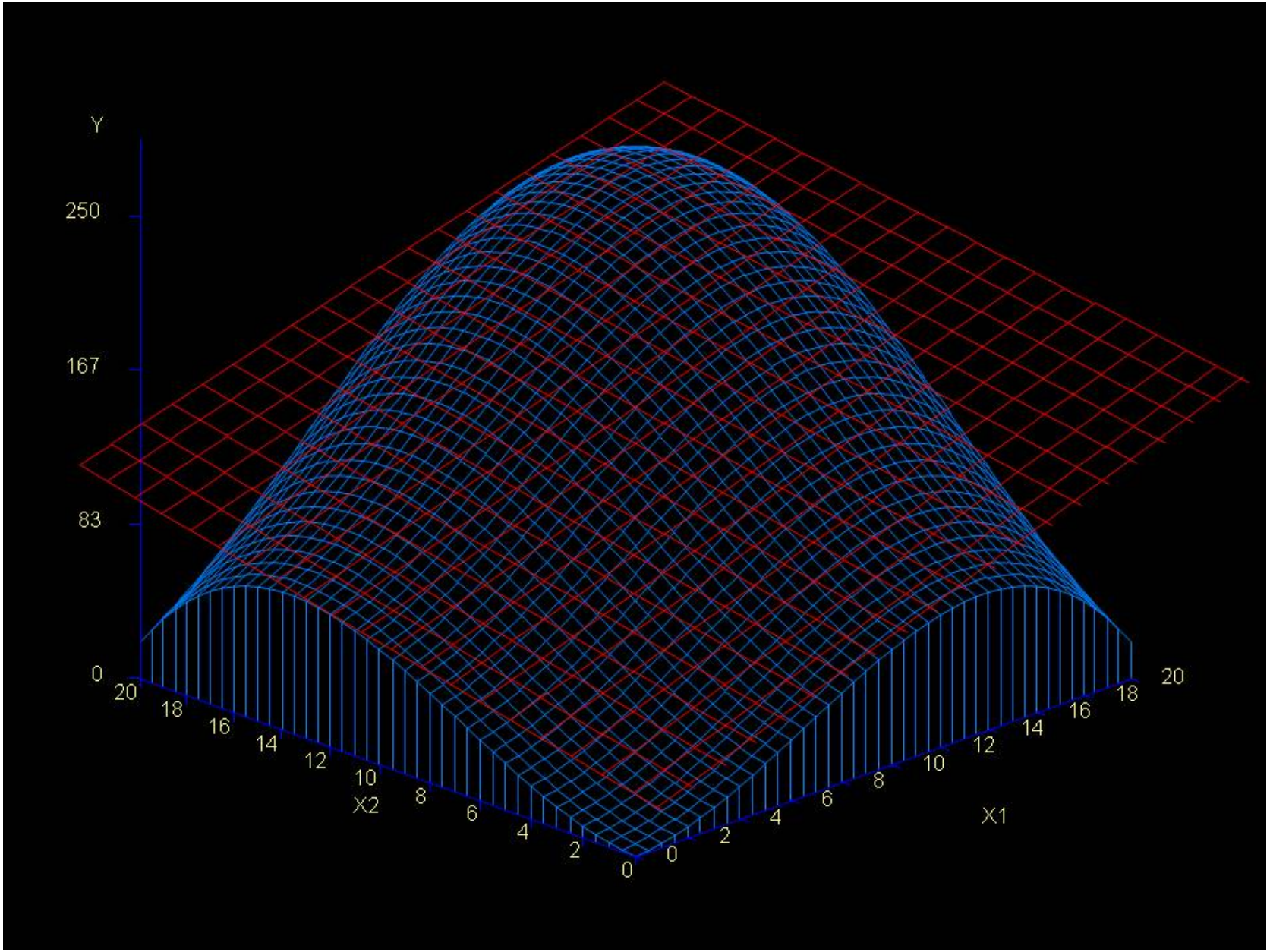
18
20
X1


Y
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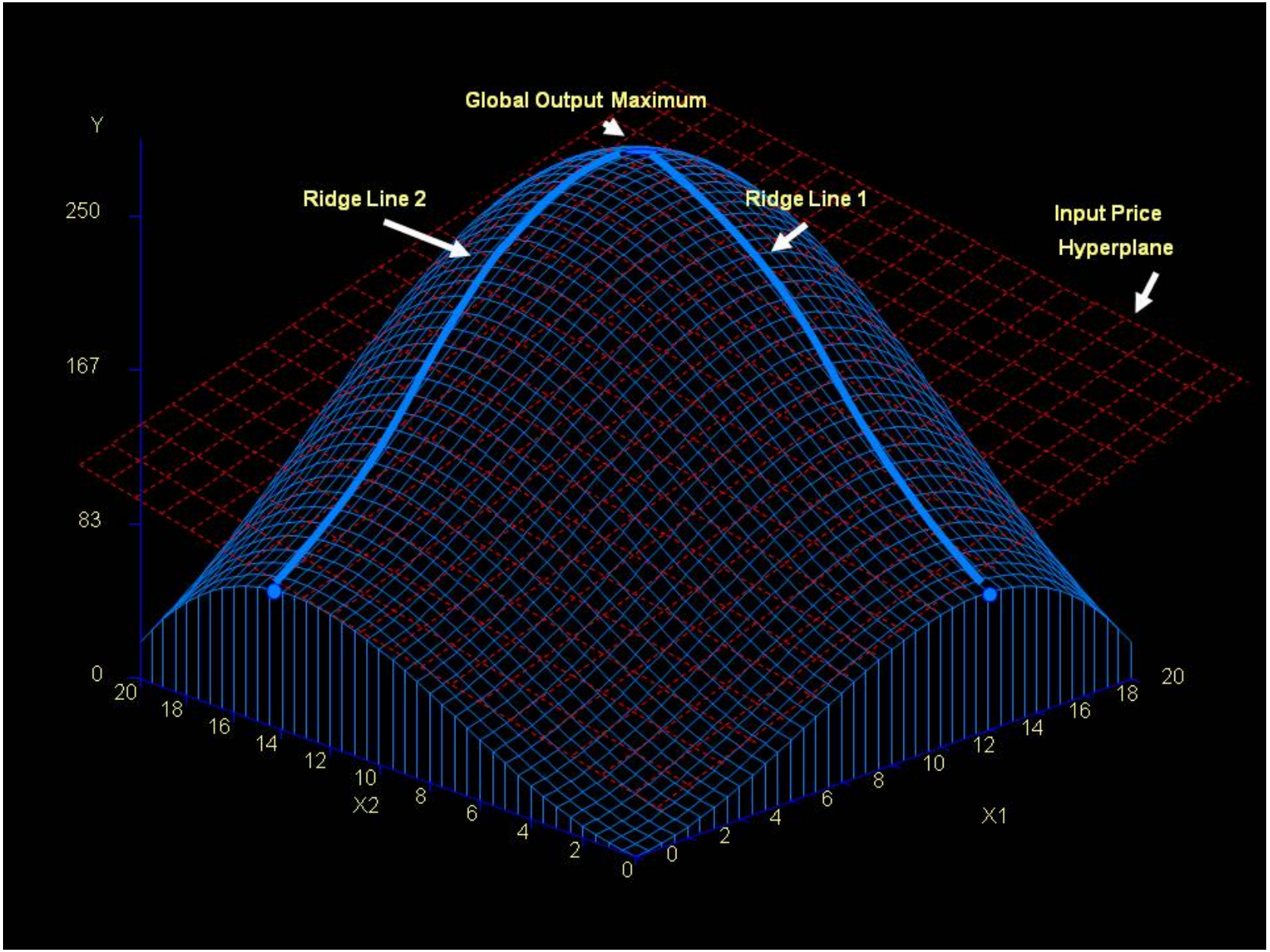
X1


Global Output Maximum
Ridge Line 2
Ridge Line 1
Input Price
Hyperplane
Y
250
167
83
0
20
18
16
14
12
10
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X2
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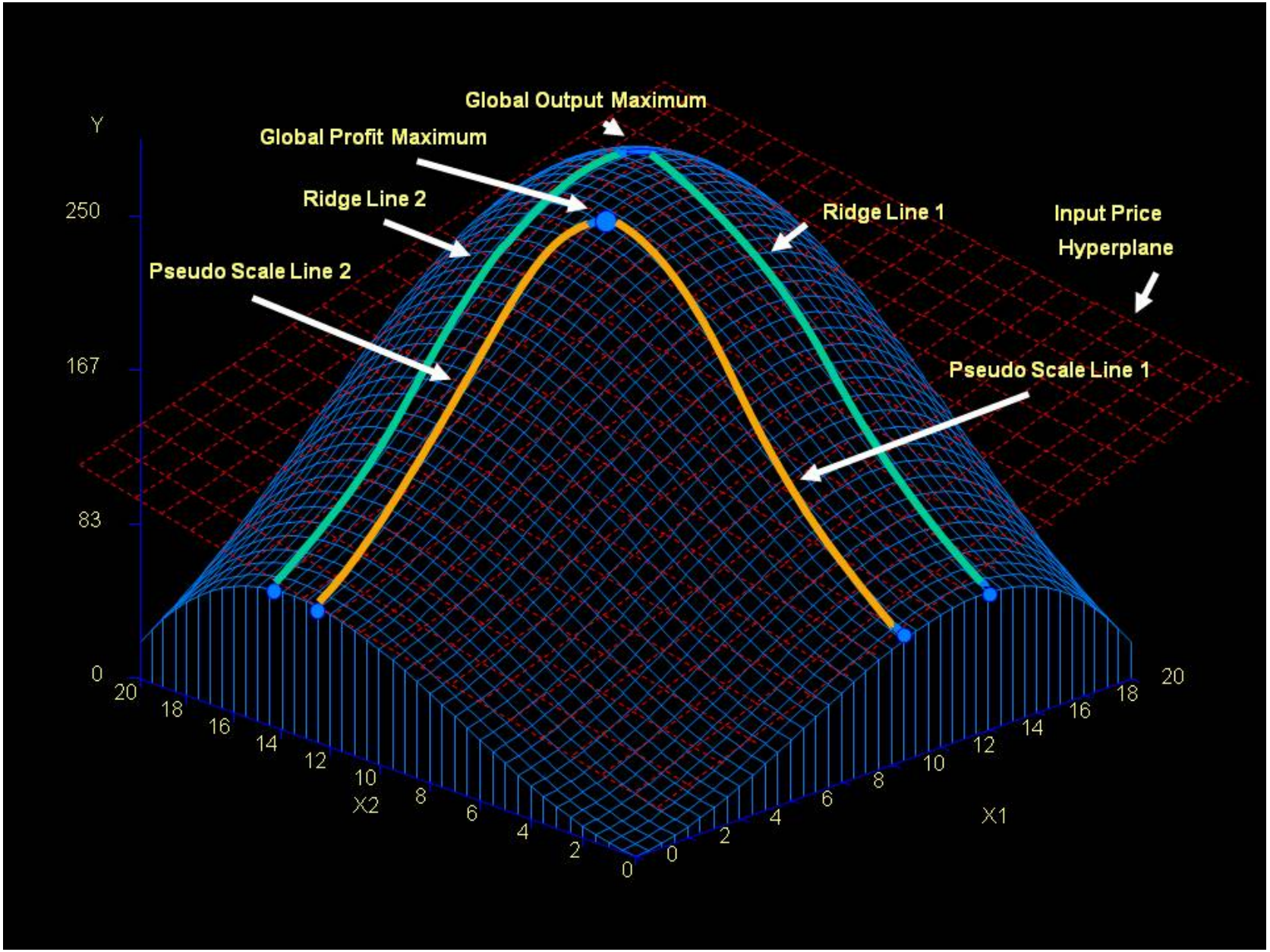
16
18
20
X1

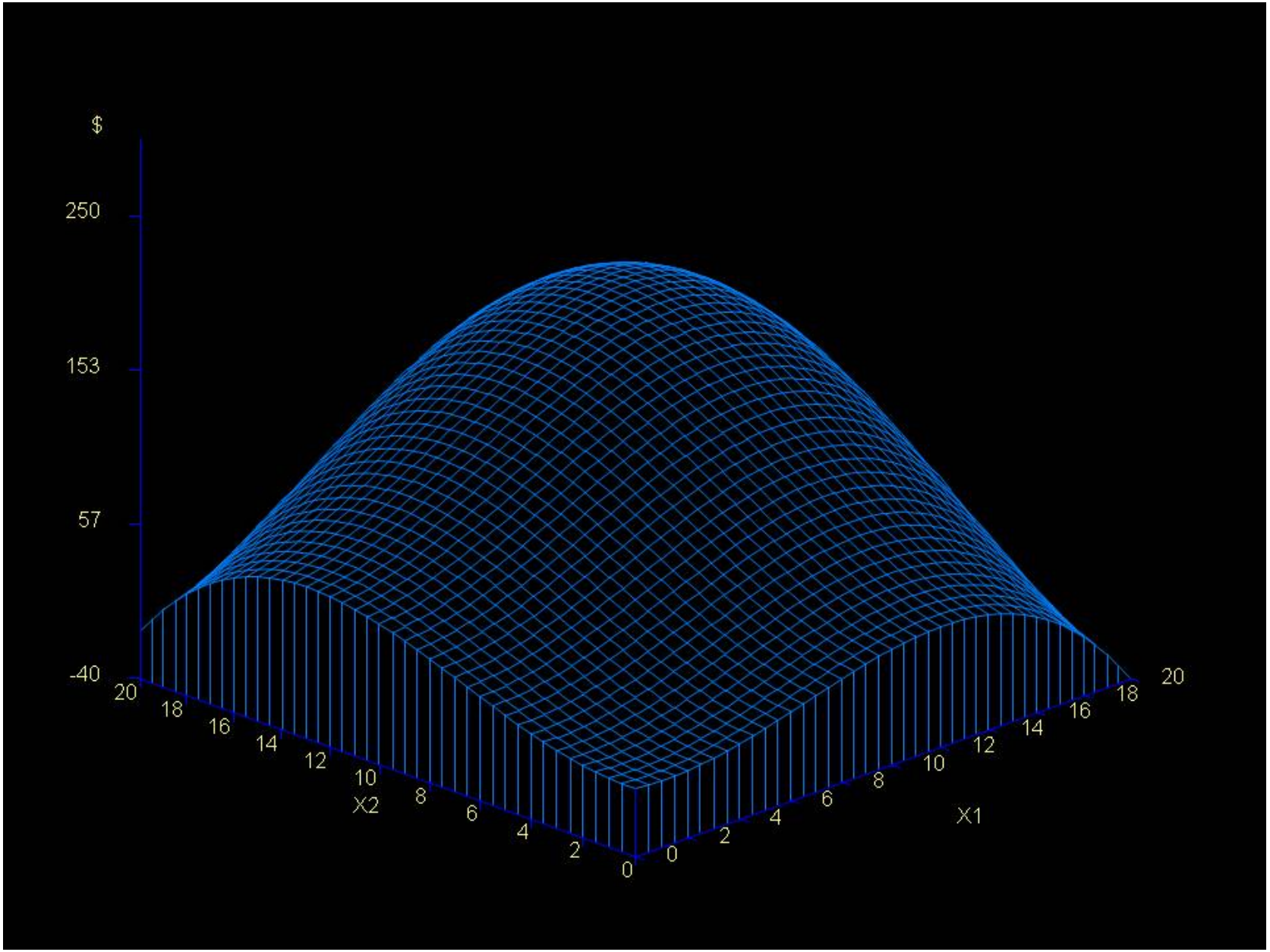

Global Output Maximum
Global Profit Maximum
Ridge Line 2
Ridge Line 1
Input Price
Hyperplane
Pseudo Scale Line 2
Pseudo Scale Line 1
Y
250
167
83
0
20
18
16
14
12
10
8
6
4
2
0
X2
0
2
4
6
8
10
12
14
16
18
20
X1

In this sequence, the gross revenue surface is superimposed on the profit surface.

The gross revenue surface appears in white, the profit surface in blue.

The global point of revenue maximization and the global point of profit maximization are both shown.

Py = $1.00; P1 = $3.00; P2 = $1.50


$
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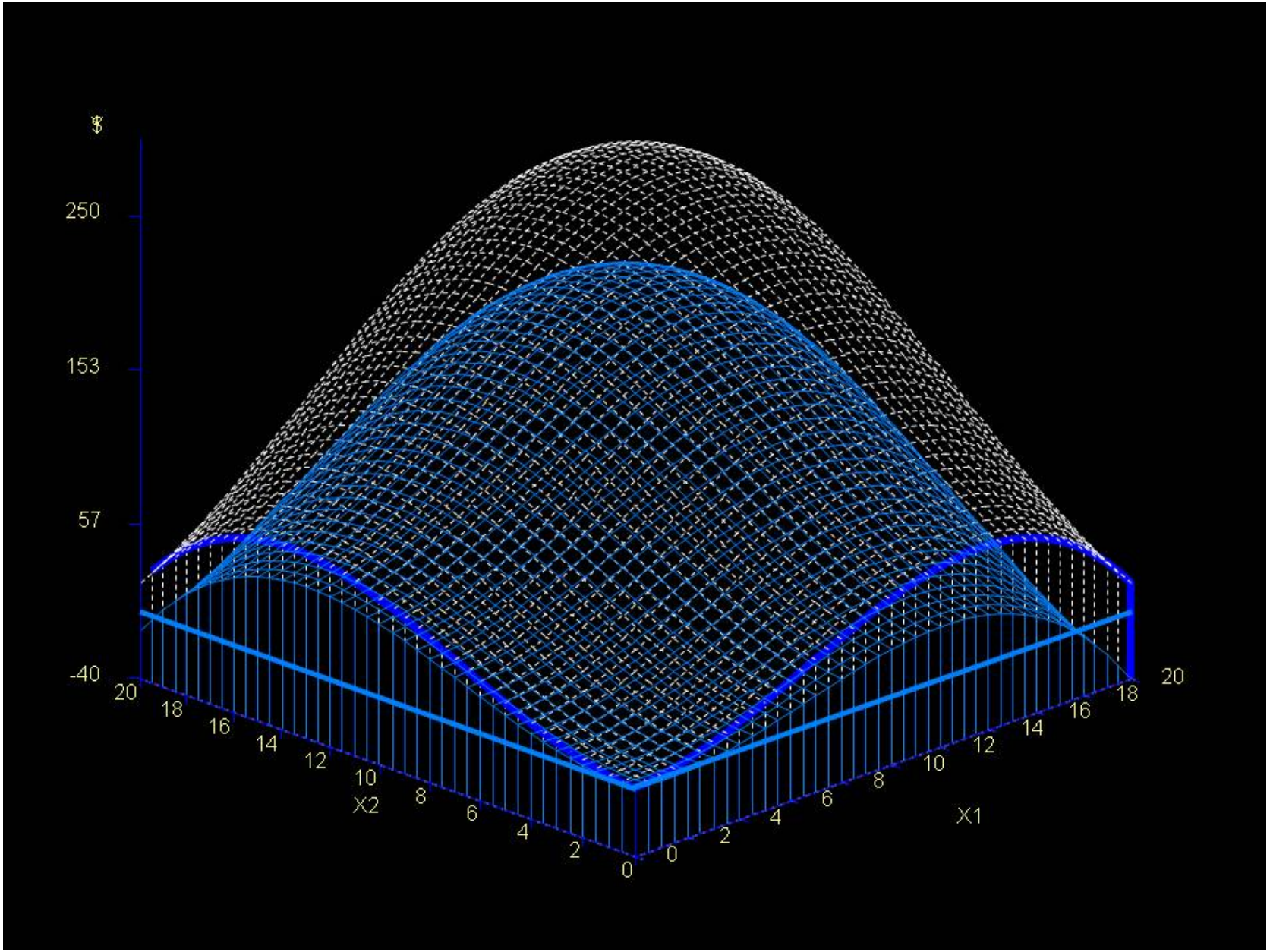
14
16
18
20
X1


$
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153
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-40
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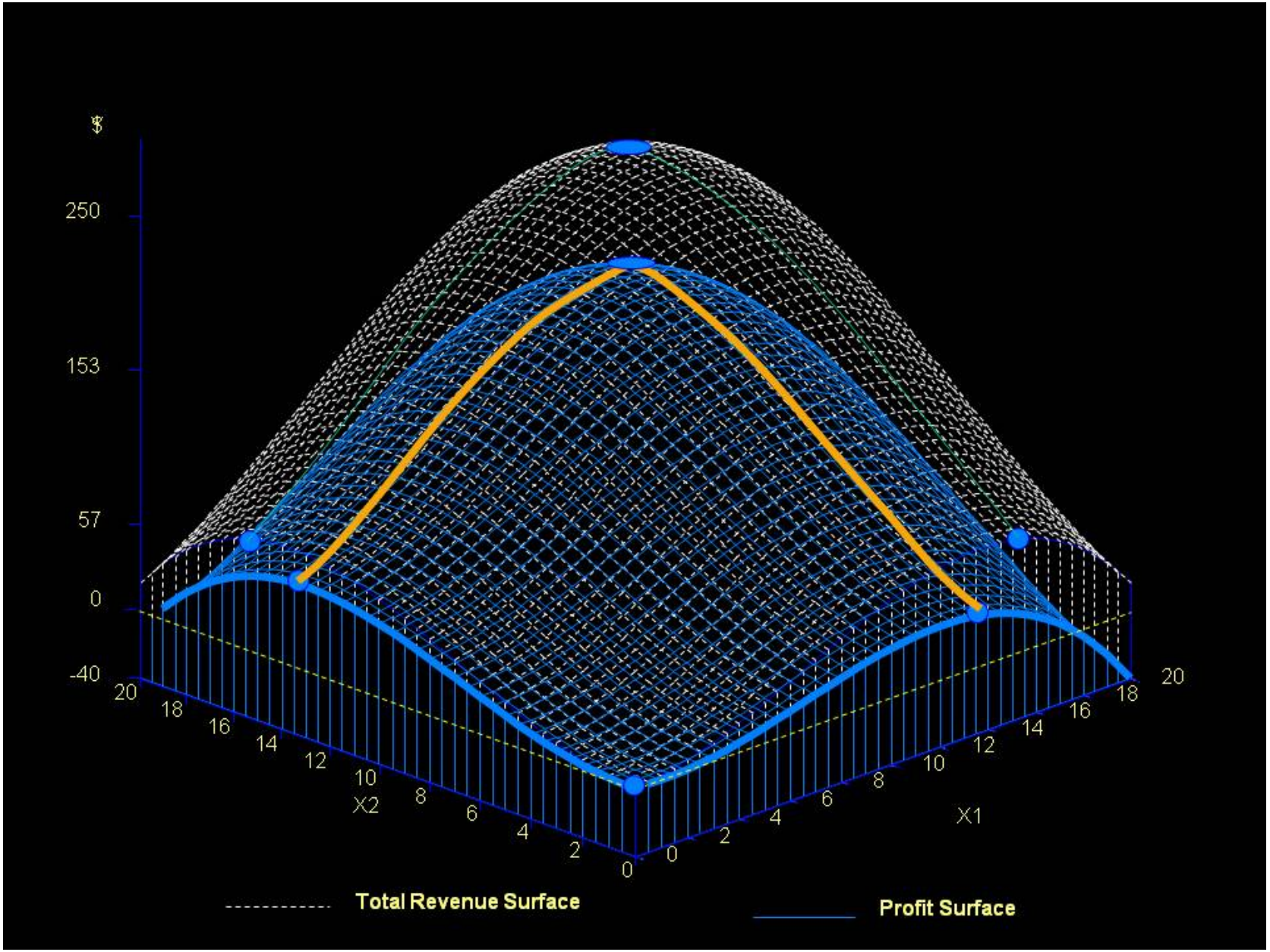
8
10
12
14
16
18
20
X1


$
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153
57
0
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20
18
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12
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8
6
4
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X2
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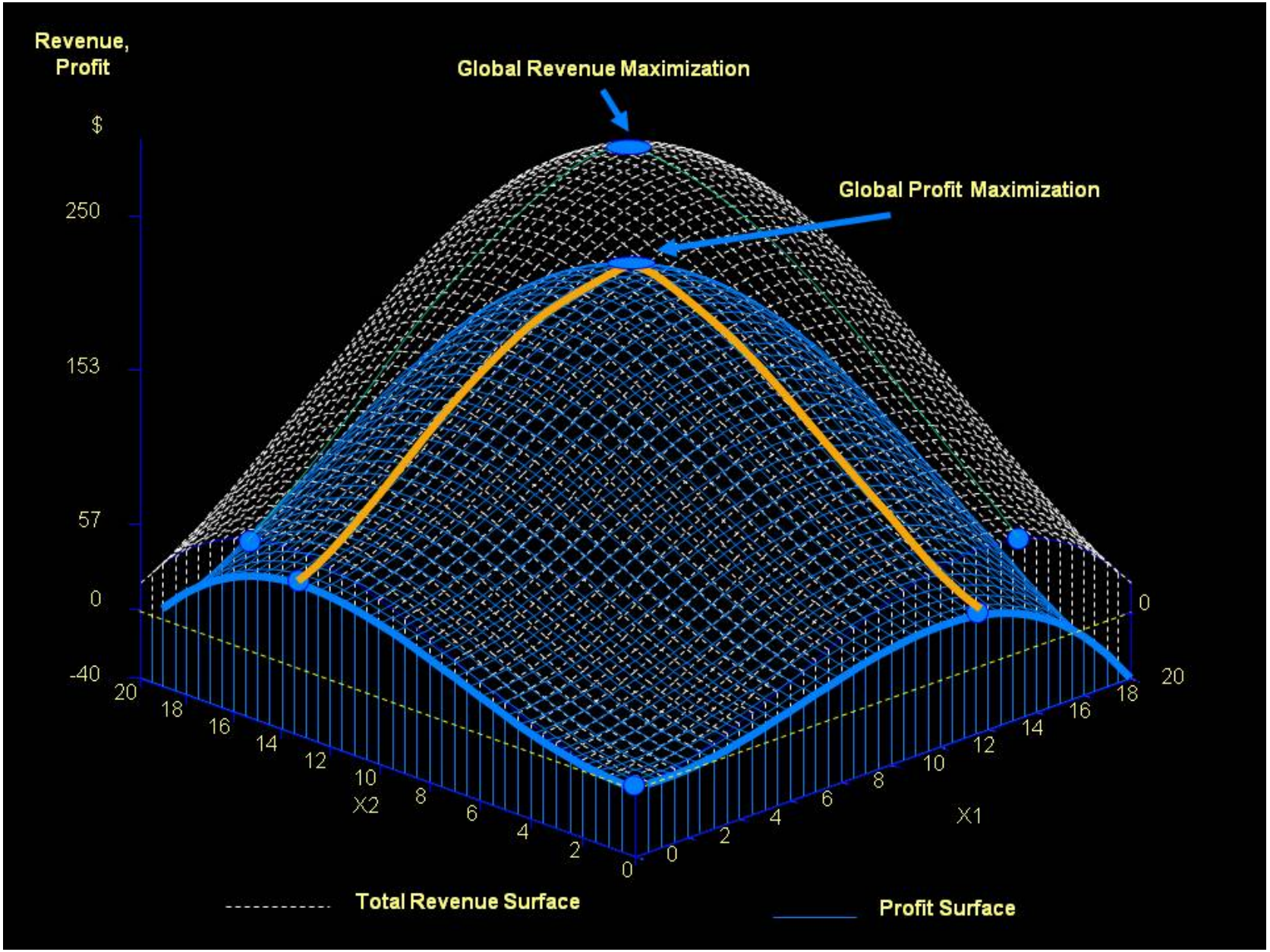
6
8
10
12
14
16
18
20
X1
Total Revenue Surface
Profit Surface


Revenue,
Profit
$
250
153
57
0
-40
Global Revenue Maximization
Global Profit Maximization
20
18
16
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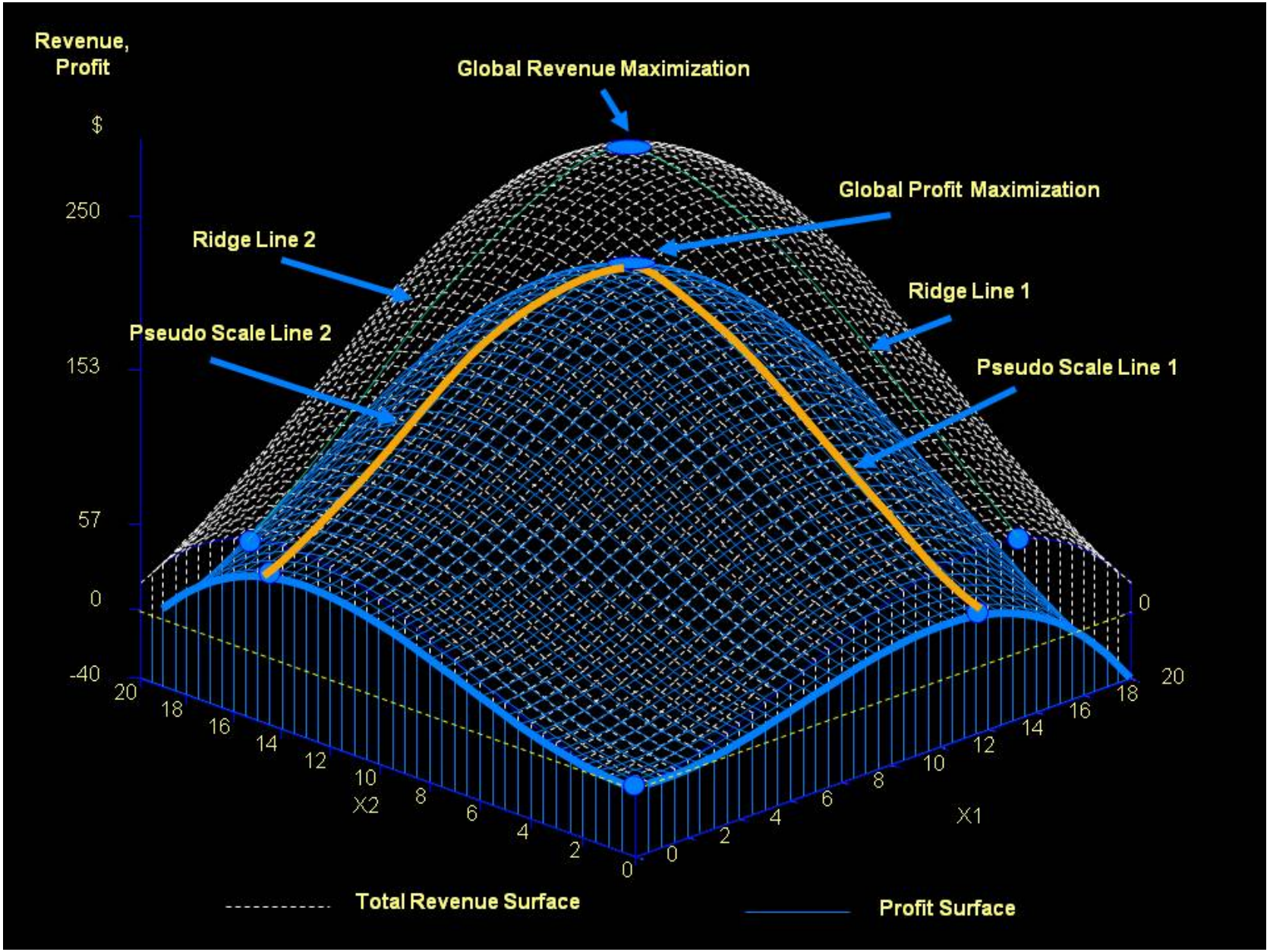
X1
Total Revenue Surface
Profit Surface


Revenue,
Profit
$
250
153
57
0
-40
Global Revenue Maximization
Global Profit Maximization
Ridge Line 2
Ridge Line 1
Pseudo Scale Line 2
Pseudo Scale Line 1
20
18
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12
10
8
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4
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X2
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2
4
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8
10
12
14
16
18
20
X1
Total Revenue Surface
Profit Surface

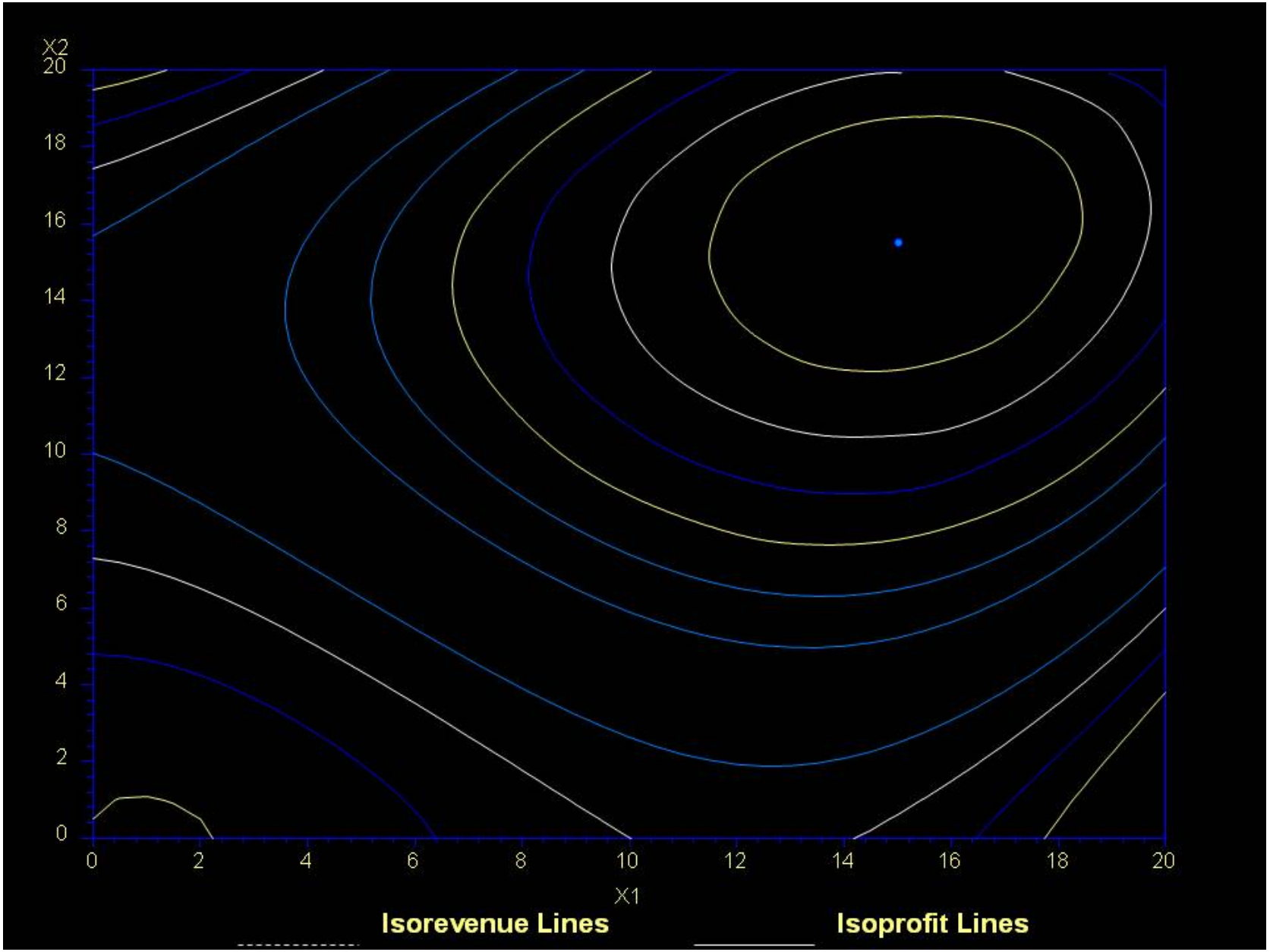
In this sequence, isorevenue contour lines represent points of constant total revenue.

They appear as dotted lines. Except for the units change, they look just like isoquants.

Isoprofit contour lines are superimposed on the same graph. They are solid lines.

Ridge lines connect points of zero or infinite slope on the isorevenue lines (isoquants). These are shown in cyan.

Pseudo Scale Lines connect points of zero or infinite slope on isoprofit lines. They are shown in orange.


X2
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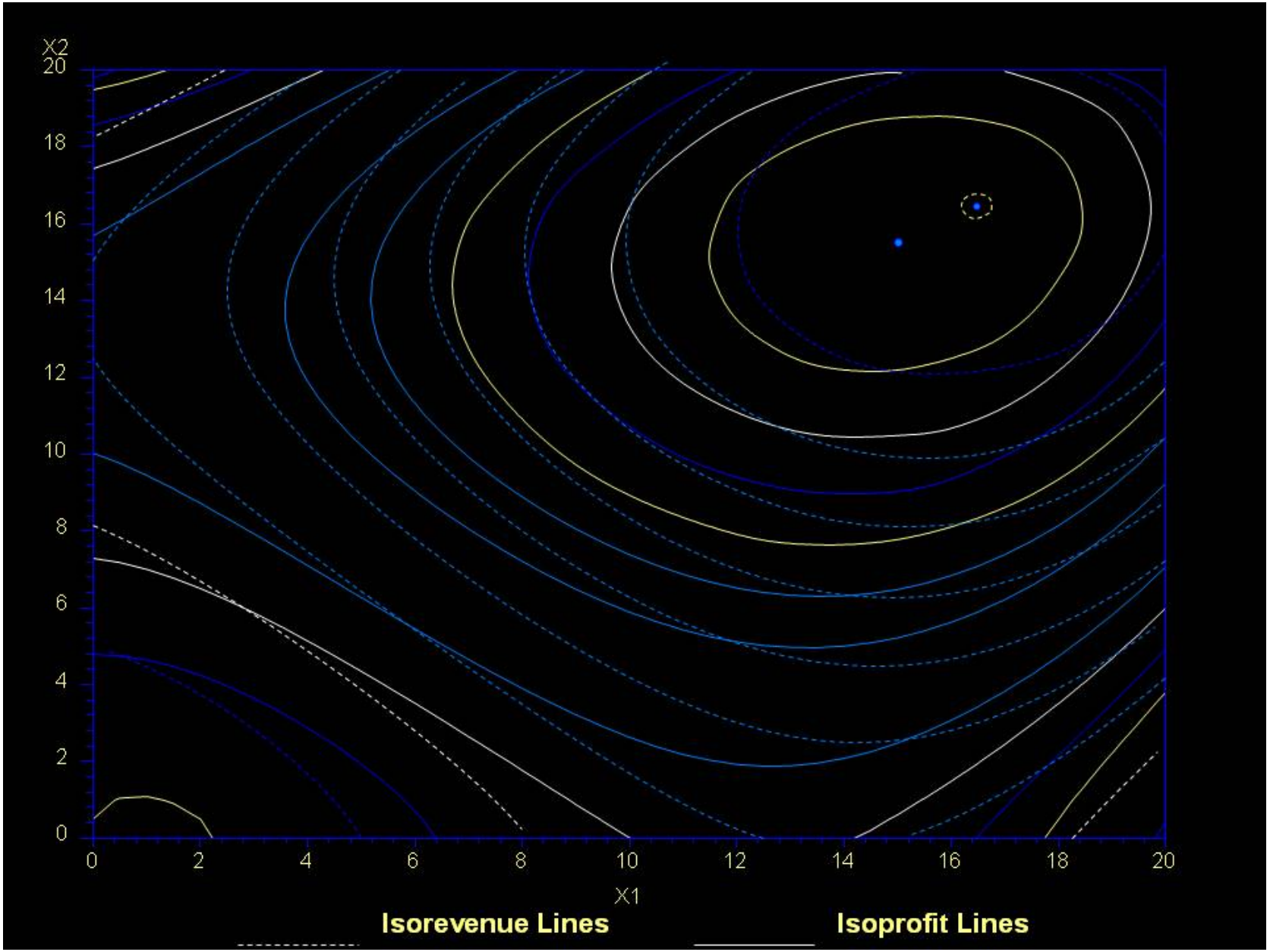
18
20
X1
Isorevenue Lines
Isoprofit Lines


X2
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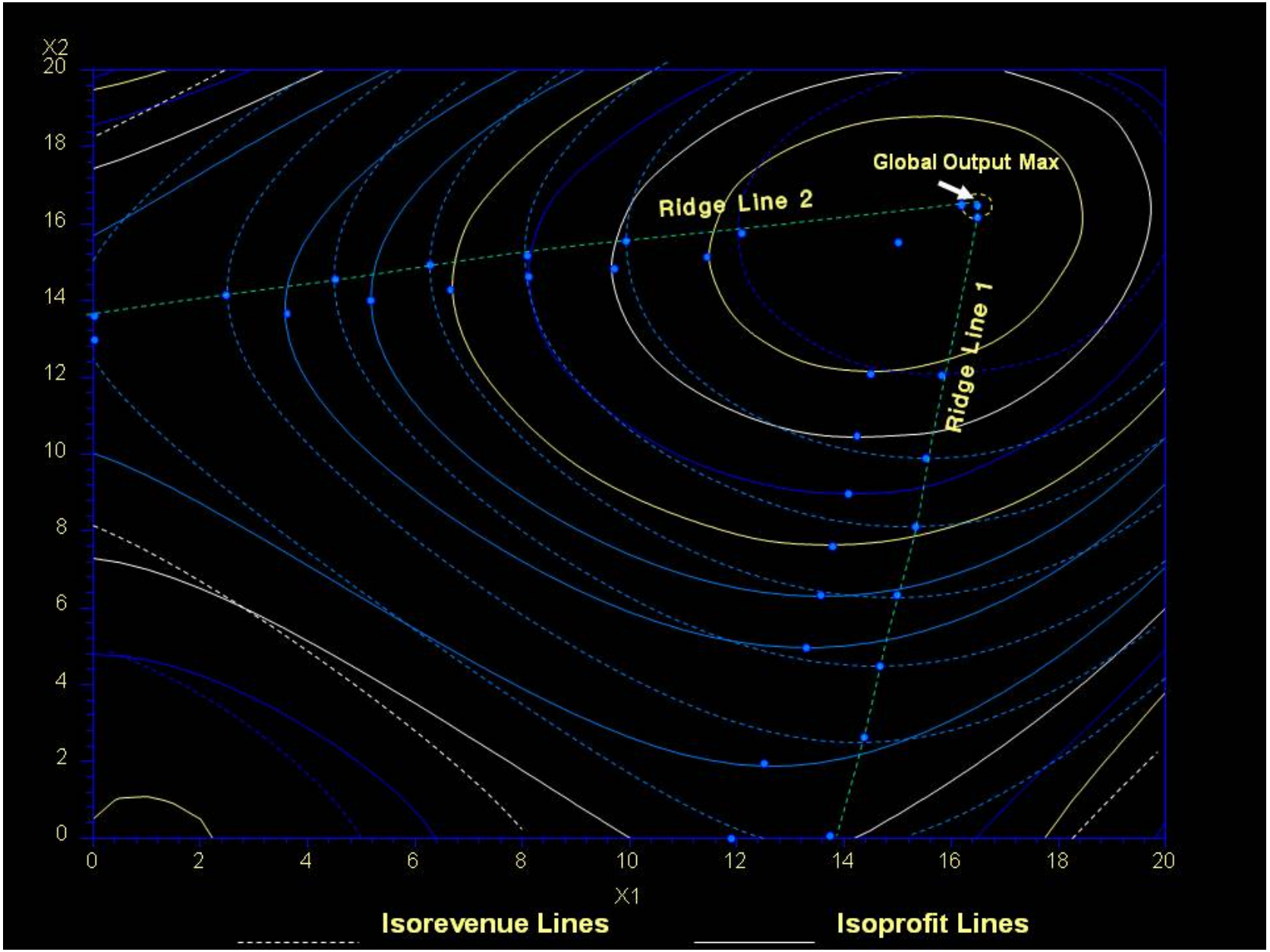
X1
Isorevenue Lines
Isoprofit Lines


X2
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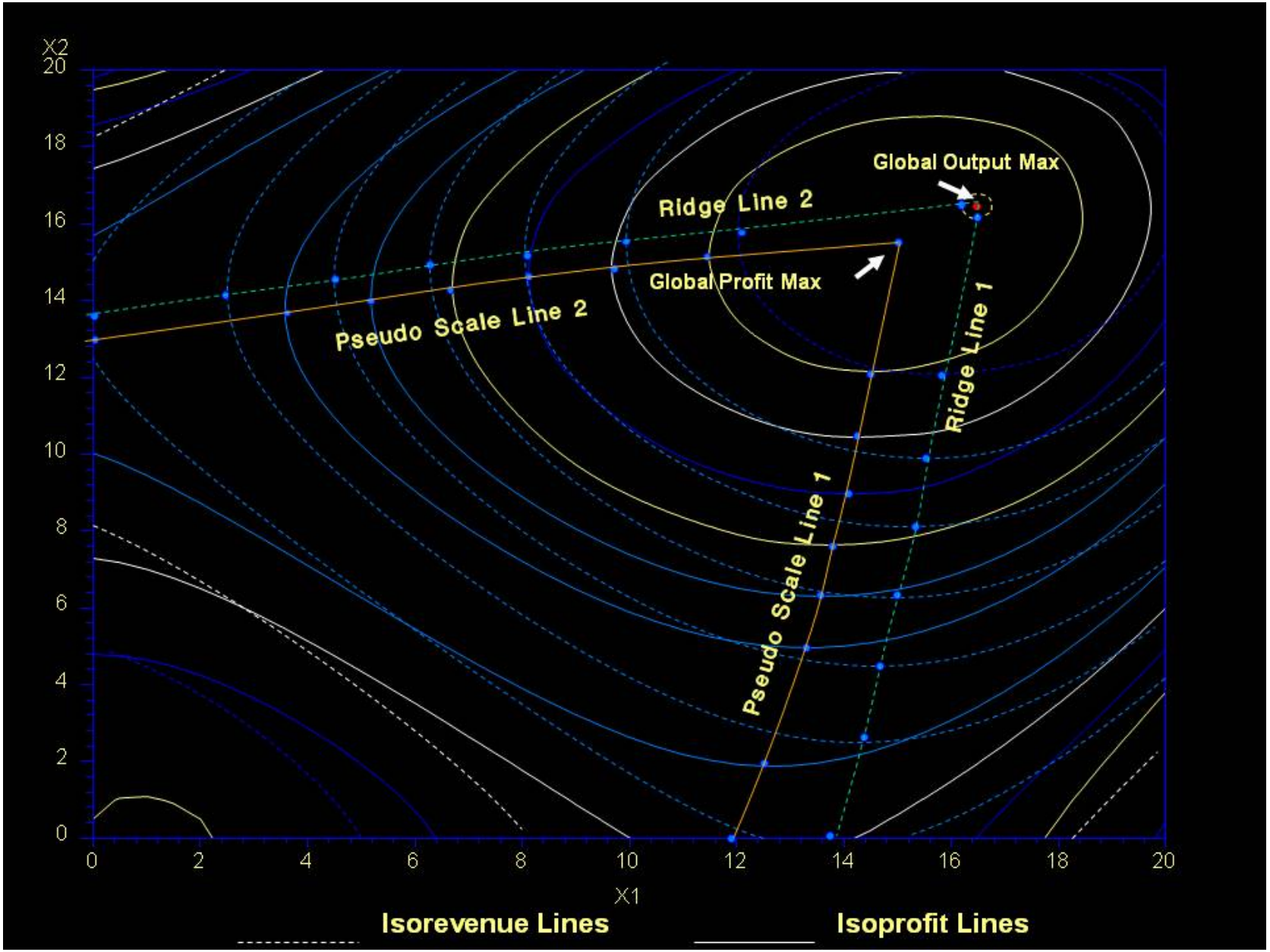
12
14
16
18
20
X1
Global Output Max
Ridge Line 2
Ridge Line 1
Isorevenue Lines
Isoprofit Lines


X2
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10
12
14
16
18
20
X1
Global Output Max
Ridge Line 2
Global Profit Max
Pseudo Scale Line 2
Ridge Line 1
Pseudo Scale Line 1
Isorevenue Lines
Isoprofit Lines