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Optimal Dynamic Irrigation Schemes

by

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SUMMARY

Optimal control methods are employed to derive irrigation management schemes accounting both for the dynamic response of the biomass yield to soil moisture and for the cost of irrigation water. Moisture dynamics depend on the irrigation rate and on the current biomass and moisture states. We find that the optimal irrigation policy has turnpike characteristics: soil moisture in the root zone should be brought to some optimal target level *as rapidly as possible* and kept at that level until some time prior to harvest, when irrigation should be ceased. The target moisture level and the stopping time vary across crops, soil types, climatic conditions and economic (price) factors, but the turnpike structure of the optimal irrigation policy persists under general circumstances. An empirical example demonstrates that the optimal scheme significantly outperforms the policy of yield maximization in actual practice.

KEY WORDS: dynamic optimization, turnpike, irrigation management.

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1. INTRODUCTION

The notion that technological progress in agricultural production is the main vehicle to avoid the threats of Malthusian starvation is well established. Part of this progress is due to the development of high-yield varieties that are resistant to pests and diseases. Equally important are efficient production technologies that save on costly inputs such as labor, land, water, nutrients and pesticides by adjusting carefully the timing and quantities in which inputs are applied to the needs of each particular crop. Among these inputs, irrigation water plays an increasingly important role, as limitations on the quality and quantity of water are hampering growth in many areas around the globe. The amount of water applied, then, is an important consideration in evaluating agricultural production procedures.

Efficient irrigation management requires the understanding of the relation between water input and crop yield. Consequently, a considerable amount of research has been devoted to elucidate this relation. The two basic approaches to this problem can be classified as either static or dynamic. The static approach considers the total amount of water available during the growing period and ignores intra-seasonal variations [1,2]. Irrigation management in this approach entails allocating the total amount of irrigation water without specifying how this amount is to be distributed during the growing period. A dynamic framework, on the other hand, accounts for intra-seasonal variations in the irrigation profile [3]. Dynamic irrigation management entails allocating irrigation water at each point of time or at various stages of the crop growth [4-6].

The present effort adopts the second (dynamic) approach. Unlike previous dynamic irrigation management models in this vein that rely on numerical algorithms applied specifically to each particular case, we apply Optimal Control methods to identify and characterize an irrigation policy that is optimal for many crops and under a wide range of circumstances. Underlying the irrigation management model is a dynamic yield production

process in which the plant converts biomass into marketable yield (when the biomass itself constitutes yield, the conversion reduces to identity). The biomass, in turn, grows from emergence to harvest at a rate that depends, inter alia, on climatic and soil conditions as well as on the current state of the plant's biomass. Soil moisture (water content) in the root zone is a key factor affecting biomass growth and is the focus of interest when irrigation management is under consideration. The harvested yield is thus an outcome of a growth process that involves two state variables—soil moisture and plant biomass—and one control—irrigation rate.

The ensuing optimal irrigation policy consists of the following basic rule: bring the soil moisture in the root zone to some optimal target level $\hat{\theta}$ (the *turnpike*) as rapidly as possible, keep it constant on the turnpike until some time prior to harvest, and then cease irrigation. The optimal policy is thus specified in terms of two parameters, namely the turnpike moisture level and the irrigation stopping time. Both can vary across crops, soil and climatic conditions, as well as across economic (price) specifications. Nonetheless, the turnpike structure persists under general conditions. Characterizing the optimal irrigation policy in each circumstance boils down to determining these two parameters; implementing the ensuing policy is then straightforward.

Similar finite-horizon turnpike policies are obtained for a wide variety of economic and management problems (see, e.g. [7] pp. 195-205 for the Vidal-Wolfe [8] advertising model and [7] pp. 295-298 or [9] for an epidemic control problem). Our model differs from these simple examples by the presence of two inter-dependent state variables (note that the biomass keeps on growing while moisture remains on the turnpike), and by the lack of end-constraints that fix these variables at some specified end-values. Instead, the final states are determined indirectly via appropriate transversality conditions. The analysis yielding this policy is, therefore, of interest on its own.

As is often the case, maximizing profit turns out to be quite different from maximizing yield. This is particularly true when one of the inputs (e.g. irrigation water in arid or semi-arid regions) contributes significantly to production expenses. In fact, for the empirical example presented in this work we find that the maximal yield policy inflicts a loss whereas the optimal policy gives rise to a positive profit.

The next section specifies the problem and characterizes the optimal irrigation policy. In Section 3 we apply the model to sunflower growth in the Arava Valley, Israel. Section 4 considers several extensions and establishes the robustness of the turnpike policy. Finally, Section 5 concludes and the Appendix contains the technical derivations.

2. OPTIMAL IRRIGATION MANAGEMENT

Let $m(t)$ represent the plant biomass at time $t \in [0,T]$, where T denotes the length of the growing period (or time from emergence to harvest). Marketable yield is derived from the biomass according to the yield function $y(m)$. If yield and biomass are the same, then $y(m) = m$. Often, however, $y(m)$ is small for m below some critical level but above this level it increases at a rate that exceeds that of the biomass. At each point of time the biomass grows at a rate that depends on the current biomass state (the accumulated growth up to this time) as well as on a host of factors including availability of water and nutrients in the root zone, sunlight intensity, day length and ambient temperature. Some of these factors (e.g., water content) can be controlled by the growers via input application and are denoted by $\theta(t)$.

The plant biomass rate of growth depends on $\theta(t)$ and $m(t)$ according to

$$\frac{dm(t)}{dt} \equiv \dot{m}(t) = g(\theta(t))h(m(t)). \tag{2.1}$$

Implicit in (2.1) is the assumption that the biomass growth rate can be factored to terms depending on θ and m separately. The functions g and h are assumed to be strictly concave in their respective arguments, and g obtains a maximum at some input value $\theta_{\max}$ (too much

moisture harms growth). The specification of these functions in any particular context is based on empirical evidence or physiological crop models.

Focusing attention on irrigation management, $\theta(t)$ in this work represents the water content in the root zone. Mass conservation implies that the change in $\theta(t)$ at each point of time must equal water input through irrigation (x) minus losses due to evapotranspiration (ET) and drainage (D). (Rainfall can also be incorporated in this framework, but to focus on irrigation management we assume no rainfall.)

Evapotranspiration rate is specified as

$$ET(\theta, m) = \beta \cdot g(\theta) f(m) \tag{2.2}$$

where the coefficient β depends only on climatic conditions and is independent of m and θ and $0 \leq f(m) \leq 1$ is a crop scale factor representing the degree of leaves exposure to solar radiation [10]. The use of the same moisture function $g(\theta)$ in (2.1) and (2.2) is based on the linear relation between biomass production and evapotranspiration, suggested by deWit [11] and established for a variety of climates and crops [12].

The rate of water drainage $D(\theta)$ is assumed to be positive, increasing and convex for the relevant soil moisture range. When all the flow rates are measured in mm·day^{-1} and θ is a dimensionless water concentration, the soil water balance can be specified as

$$Z\dot{\theta}(t) = x(t) - \beta g(\theta(t)) f(m(t)) - D(\theta(t)), \tag{2.3}$$

where Z is the root depth and $Z\theta(t)$ measures the total amount of water in the root zone (mm).

Let P and W denote the crop (output) and water (input) prices, respectively, assumed fixed throughout the growing season. The grower pays for the total amount of water input at harvest time T, at which time he also receives the revenue $Py(m(T))$. For a growing season that lasts a few months we can ignore discounting, and the return to water (not including expenses on other inputs) is

$$Py(m(T)) - W\int_0^T x(t)dt\ . \tag{2.4}$$

The irrigation management problem entails finding the irrigation policy $\{x(t), 0 \le t \le T\}$ that maximizes (2.4) subject to (2.1), (2.3), $m(0) = m_0$, $\theta(0) = \theta_0$ and $0 \le x(t) \le \bar{x}$, where m_0 and θ_0 are the initial biomass and soil moisture levels and $\bar{x}$ is an upper bound on the feasible irrigation rate, reflecting physical constraints on irrigation equipment or on soil water absorption capacity. The formulation, thus, involves one control variable (the irrigation rate x) and two state variables (the biomass m and the water content θ).

The derivation of the optimal policy is detailed in the Appendix. It uses Optimal Control techniques to characterize the optimal trajectories of the control, $x(t)$, and of the state variables $\theta(t)$ and $m(t)$ during the course of the growing period. The ensuing optimal policy itself turns out to be straightforward: It is defined in terms of two parameters: a turnpike soil water content $\hat{\theta}$ and a date $t_2 < T$, such that the optimal soil water process, $\theta^*(t)$, must be brought from its initial level θ_0 to the turnpike $\hat{\theta}$ *as rapidly as possible* and maintained at that level until t_2, at which time irrigation ceases. The termination of irrigation prior to harvest is because the gain from the contribution to yield that could have resulted from maintaining the soil water content at $\hat{\theta}$ during the remaining period is not sufficient to cover the cost of the water needed for this purpose.

The optimal soil water and biomass processes, denoted $\theta^*(t)$ and $m^*(t)$, are derived from (2.1) and (2.3), given θ_0 and m_0 and the policy parameters $\hat{\theta}$ and t_2. Noting (2.3), the corresponding optimal irrigation policy when $\theta_0 \le \hat{\theta}$ is given by

$$x^*(t) = \begin{cases} \bar{x} & \text{when } 0 \le t \le t_1 \\ \beta g(\hat{\theta}) f(m^*(t)) + D(\hat{\theta}) & \text{when } t_1 < t \le t_2 \\ 0 & \text{when } t_2 < t \le T \end{cases} \tag{2.5}$$

where t_1 is the time the soil water process reaches $\hat{\theta}$. If $\theta_0 > \hat{\theta}$, then $x^*(t)$ vanishes also during the first stage. The derivation of $\hat{\theta}$ and of t_2 is presented in the Appendix.

Equation (2.5) reveals that the optimal policy consists of three stages: in the first stage, $\theta^*(t)$ is brought to the turnpike $\hat{\theta}$ as rapidly as possible; during the second (singular) stage, $\theta^*(t)$ is maintained on the turnpike and during the third stage irrigation ceases. Normally, all three stages are implemented sequentially. It is possible, however, that $t_1 = t_2$, in which case irrigation is applied at a full capacity until t_2 and then ceases (i.e. there is no time for the singular stage). It is also possible that $t_2 = 0$, which occurs when irrigation is not profitable hence never applied. Finally, when the initial water content is high enough (i.e., $\theta_0 > \hat{\theta}$) it pays to let the soil dry up to the turnpike and then begin irrigating at the singular rate until t_2.

3. AN EMPIRICAL EXAMPLE

We illustrate the performance of the optimal policy by applying it to control the growth of Ornamental sunflower (*Helianthus annuus* var dwarf yellow) in the Arava Valley in Israel. Lack of precipitation throughout the growing period and deep groundwater (120 m below soil surface) ensure that irrigation is the only source of water. Biomass growth has been modeled specifying a quadratic function for $g(\theta)$ and a logistic function for $h(m)$ [13], with parameters estimated via a field experiment under local conditions with high frequency drip irrigation. The drainage function is estimated using the hydraulic model of Brooks and Corey [14]: $D(\theta) = K_S[(\theta - \theta_R)/(\theta_S - \theta_R)]^\eta$, with the numerical values of the saturated hydraulic conductivity K_S, the residual and saturated water contents θ_R and θ_S and the exponent η fitted to local soil properties. Finally, Based on recommended extension service practices, the crop factor function is specified as $f(m) = m(1 - m/785.6)/196.4$.

Inserting the numerical parameter estimates, the equations of motion (2.1) and (2.3) are specified as

$$\dot{m} = (1.21\Theta - 1.71\Theta^2)m(1 - m/491) \tag{3.1}$$

and

$$\dot{\theta} = [x - 0.19(1.21\Theta - 1.71\Theta^2)m(1 - m/785.6) - 3600\Theta_d^{5.73}]/600 \tag{3.2}$$

where $\Theta = (\theta - 0.09)/0.31$ and $\Theta_d = (\theta - 0.04)/0.36$.

In this experiment, marketable yield was obtained only at biomass levels above 350 g·m^{-2}. At the maximal biomass (m = 491 g·m^{-2}), the yield comprises 80 percent of the biomass. Assuming a linear increase, this implies the following yield function:

$$y(m) = \begin{cases} 0 & \text{if } m < 350\,\text{g}\cdot\text{m}^{-2} \\ -976.5 + 2.79m & \text{if } m \geq 350\,\text{g}\cdot\text{m}^{-2} \end{cases} \tag{3.3}$$

The initial soil water and biomass levels were taken at $\theta_0 = 0.1$ (just above water content at the wilting point $\theta = 0.09$ where Θ and the growth rate vanish) and m_0 = 10 g·m^{-2} (about 2% of the maximal obtainable biomass). The maximal feasible irrigation rate is $\bar{x}$ = 41.8 mm·day^{-1}. Water prices in the Arava vary around \$0.2-0.4·m^{-3} and the price of sunflower seeds received by the farmers after the growing period of T = 45 days is about \$1·kg^{-1}. We therefore consider the optimal policy using the cost ratio $w \equiv W/P$ = 0.3 kg·m^{-3}.

Results and discussion

A numerical implementation of the optimal policy based on the above specifications gave rise to the following optimal parameters:

$\hat{\theta}$ = 0.148 (about 74% of $\theta_{\max}$ = 0.2),

t_2 = 42.2 days.

It is of interest to compare the results of the optimal irrigation policy with the outcome of an ad hoc policy that aims at maximum yield (by raising the water content to $\theta_{\max}$ as rapidly as possible and

maintaining this maximal growth moisture level until harvest). Trajectories for $\theta(t)$, $m(t)$, $x(t)$, $ET(t)$ and $D(t)$ under the optimal and maximal yield policies are presented in Figures 1-5. With maximal irrigation rate of $\bar{x} = 41.8$ mm·day^{-1} (Figure 3), it takes about 0.7 day to bring the soil water content from its initial level $\theta_0 = 0.1$ to the turnpike level $\hat{\theta}$=0.148 (Figure 1), and about two days to elevate the water content to θ_{max}. As soon as the soil moisture under the optimal policy reaches the turnpike (at $t_1 = 0.7$ day), irrigation is tuned so as to maintain the soil water content at $\hat{\theta}$ until t_2. The singular stage on the turnpike extends, therefore, over the major part of the growing period of $T = 45$ days. During the last 2.8 days irrigation is avoided because the gain in yield due to continued irrigation is not sufficient to cover the cost of the water needed to maintain the high soil water content. Thus, at harvest time, the water content is decreased back to about 60 percent of θ_{max}.

Biomass growth and the corresponding yield are depicted in Figure 2 for both the optimal and the maximal yield policies. It is evident from Figure 2 that the main effect of reducing irrigation rate and thus water content in the root zone is in slowing down growth rate, causing the plant to produce seeds and add biomass during a longer period. The corresponding harvested yield is 350.6 g·m^{-2} for the optimal policy—about 10% below the maximal attainable yield. The irrigation level required to maintain water content in the root zone for maximal yield is about four times higher than the irrigation needed for optimal policy. Most of the added irrigation is wasted as a result of the higher drainage rate (see Figure 5) rather than higher evapotranspiration (Figure 4).

With irrigation costs of \$1020 ha^{-1} (about half of which is due to drainage), the net income (excluding labor and other inputs) from the optimal policy amounts to \$2480 ha^{-1}. Under the ad hoc maximal yield policy, irrigation cost is significantly higher and amounts to \$5380 ha^{-1}, over 80% of which is due to drainage, entailing a net loss of \$1500 ha^{-1}. The optimal policy is thus seen to represent an advantageous compromise of the tradeoffs between high yield and saving on

the water bill. Indeed, when the relation describing drainage losses involves a high exponent, as in the case of the sandy loam soil considered here, adjusting the water content to the proper level is of prime importance.

4. EXTENSIONS

The analysis above is based on some restrictive assumptions such as given harvest time, unchanging climatic conditions and no constraint on the available amount of water. It turns out that the turnpike nature of the optimal policy is preserved also when these assumptions are relaxed. We discuss below the modifications associated with each of these three extensions.

Endogenous harvest time: In some cases it may be desirable to adjust the harvest time to changing market conditions, such as a favorable crop price for early marketing. In such situations, T is treated as a choice variable, while the revenue is redefined as $R(m(T),T) = y(m(T))P(T)$, where y is the yield function and P is the output price that depends on the harvest time. Early marketing is advantageous when $\partial R/\partial T < 0$. Inspecting the derivation in the Appendix, we see that none of the conclusions is affected, except that the fixed harvest date is replaced by an additional transversality condition $H^* + \partial R(m(T),T)/\partial T = 0$. Thus, the optimal θ–process preserves its turnpike structure, although the values of $\hat{\theta}$ and of the associated transition dates must be modified to account for the new transversality condition.

Limited water quota: When water resources are scarce, the irrigation policy may be limited not only by the price of water but also by restrictions imposed on water use by some regulatory agency in order to account for the limited availability. Suppose that the total amount of water available to the grower until harvest time is $\overline{Q}$. This situation can be modeled by defining the remaining quota $Q(t) = \overline{Q} - \int_0^t x(s)ds$ as an additional state variable and adding the constraints $\dot{Q} = -x$, $Q(0) = \overline{Q}$ and $Q(T) \geq 0$. Denoting the corresponding costate variable by κ, we see that the Hamiltonian (see A4 in the Appendix) should be supplemented by the additional term $-\kappa x$. Since the Hamiltonian is independent of Q, κ is

constant, and it must be non- negative to meet the slackness conditions associated with the constraint on $Q(T) \geq 0$. We see that the imposed water quota implies the effective price $w+\kappa$, which accounts also for the opportunity (or scarcity) cost of water κ. Of course, if the water price w is such that it does not pay to consume all the quota (i.e. it is optimal to leave $Q(T)$ strictly positive), then κ vanishes and we are back with the original problem. However, if the entire quota is to be used, the opportunity cost obtains a positive value and the analysis must account for the full effective price of water. In such a case, it is easy to verify that the irrigation problem is equivalent (in the sense of yielding the same optimal policy) to a problem without water quantity constraint but with a higher water price that equals the minimal price under which the water quota would be unbinding, (i.e., under which it pays to use exactly the water allotment). The rest of the analysis, and the classification derived thereof, are not affected.

Time-dependent climatic conditions: Another extension of the model allows the climatic conditions (represented here by the evapotranspiration coefficient β) to change in time. For example, if $\beta(t)$ increases (due to rising ambient temperature, or to a change in the wind regime), equation (A4) in the Appendix implies that the Hamiltonian decreases with time, although (A9) ensures that it is positive at all times. It follows that the turnpike $\hat{\theta}$, defined by (A11), turns into a decreasing process $\hat{\theta}(t)$ to be followed by the optimal θ-process during the singular stage. Indeed, the decreasing turnpike reflects the reduced profitability associated with each moisture level because of the enhanced rate of water loss. This feature is similar to the Non-Standard Most Rapid Approach policy derived by Tsur and Zemel [15,16] in a different context. Apart from this change, the reasoning behind the turnpike characterization under a constant β remains unaltered.

5. CONCLUDING COMMENTS

In an increasingly large portion of the Globe the pressure on available water resources renders efficient irrigation essential for viable agriculture. This observation applies not only to physical irrigation technologies and to water quality differentiation, but also to the intra-seasonal distribution of water applied during the growing period. In fact, irrigating at a rate that exceeds the immediate needs of the plant implies increased drainage losses. Given the relative prices of water and yield, the tradeoff between biomass growth and the need to save on water losses entails an optimal (turnpike) moisture level in the root zone. The optimal policy established in this work is to drive the water content towards the turnpike as rapidly as possible, and then to irrigate at the (variable) rate required to maintain the soil water content at that level. Towards the harvest date, however, keeping this moisture level is no longer advantageous because the additional growth during this last period cannot compensate for the irrigation cost. Therefore, after a certain date the optimal policy requires to cease irrigation and let the plant grow on the remaining moisture in the soil until the harvest. Evidently, this policy does not provide the maximum possible yield. However, by carefully accounting for the irrigation cost, it gives rise to a positive profit also under price specifications in which the policy that maximizes yield would entail a significant loss to the farmers.

The optimal policy has been derived under conditions that are quite rigid: The climatic conditions (represented by the evapotranspiration coefficient β) are assumed to be constant during the growing period and the length of the growing period (from emergence to harvest) is assumed exogenously given. We further assume that the yield price is fixed, and the farmers' revenues depend only on the final yield. In fact, none of these assumptions is essential for the derivation. As demonstrated in Section 4, treating the harvest time T as an additional decision variable, to be determined mainly according to time variation of the yield price, the same optimal policy is obtained (although the numerical values of the turnpike water content and of the irrigation stopping date may vary). Similarly, considering time

dependent climatic conditions merely transforms the constant turnpike into a time-dependent process, to be followed by the optimal water content process until the irrigation stopping date. Otherwise, the characteristics of the optimal policy remain unaltered. The main features of the turnpike policy, therefore, are robust under a wide variety of agricultural, climatic and economic conditions.

APPENDIX: DERIVATION OF THE OPTIMAL POLICY

The irrigation management problem (2.4) is formulated in terms of the relative cost of water $w = W/P$ as

$$V(m_0, \theta_0) = P \cdot Max_{\{x(t)\}} \left\{ \int_0^T -wx(t)dt + y(m(T)) \right\} \tag{A1}$$

subject to

$$dm(t)/dt \equiv \dot{m}(t) = g(\theta(t))h(m(t)), \tag{A2}$$

$$\dot{\theta}(t) = [x(t) - \beta g(\theta(t))f(m(t)) - D(\theta(t))]/Z, \tag{A3}$$

$0 \le x(t) \le \bar{x}$ and m_0 and θ_0 given (c.f. 2.1 and 2.3).

It is assumed that θ_0 lies in the interval $(\theta_{\min}, \theta_{\max})$ where $g(\theta_{\min}) = 0$, $g'(\theta_{\max}) = 0$, $g'(\theta) > 0$ in $(\theta_{\min}, \theta_{\max})$ and $g''(\theta) < 0$ for all θ. We further assume that $D(\theta)$, $D'(\theta)$ and $D''(\theta)$ are all positive in $(\theta_{\min}, \theta_{\max})$ and that the upper bound $\bar{x}$ exceeds the water loss terms of (A3) throughout the relevant ranges of m and θ, so θ is increasing when $x = \bar{x}$. The yield function $y(m)$ is assumed to increase with m.

Preliminaries: Let λ and ρ represent the co-state variables of m and θ, respectively and $\mu = \rho/Z$. Ignoring the constant term P in front of the objective of (A1), we obtain the Hamiltonian

$$H(m, \theta, x, \lambda, \mu) = [\mu - w]x - g(\theta)[\beta\mu f(m) - \lambda h(m)] - \mu D(\theta) \tag{A4}$$

(To simplify notation, the time argument t is suppressed from m, θ, x, λ and μ when no confusion arises. All quantities below refer to optimal processes.) Note that the Hamiltonian does not

depend explicitly on time hence its value along the optimal policy is constant [17, p. 190]; we denote this constant by H^*. The necessary conditions for optimum include: $x(t)$ maximizes H, implying

$$x(t) = \begin{cases} \bar{x} & if\ \mu(t) > w \\ 0 & if\ \mu(t) < w \end{cases}, \tag{A5}$$

(the singular $x(t)$ process when $\mu(t) = w$, is derived below),

$\dot{\lambda} = -\partial H / \partial m$ and $Z\dot{\mu} = -\partial H / \partial \theta$, giving

$$\dot{\lambda} = g(\theta)[\beta\mu f'(m) - \lambda h'(m)] \tag{A6}$$

and

$$Z\dot{\mu} = g'(\theta)[\beta\mu f(m) - \lambda h(m)] + \mu D'(\theta), \tag{A7}$$

and the transversality conditions

$$\lambda(T) = y'(m(T)) \quad \text{and} \quad \mu(T) = 0. \tag{A8}$$

Using (A8), (A5) and (A4), the Hamiltonian at the harvest time T is evaluated as

$$H^* = y'(m(T))g(\theta(T))h(m(T)) \geq 0. \tag{A9}$$

It is expedient to introduce the function

$$\xi(\theta) = g(\theta)D'(\theta) / g'(\theta) - D(\theta). \tag{A10}$$

The derivative $\xi'(\theta) = [g(\theta)D'(\theta) / g'(\theta)][D''(\theta) / D'(\theta) - g'(\theta) / g''(\theta)]$ is positive over $(\theta_{\min}, \theta_{\max})$. Moreover, $\xi(\theta_{\min}) < 0$ while $\xi(\theta_{\max})$ diverges, hence the equation $\xi(\theta) = c$ has a unique solution in $(\theta_{\min}, \theta_{\max})$ for any non-negative constant c.

The singular path: According to (A5), the optimal θ-process is classified as *increasing* ($x(t) = \bar{x}$), *decreasing* ($x(t) = 0$) or *singular*, depending on whether μ exceeds, falls short or equals w, respectively. The singular path occurs when $\mu(t) = w$ holds during a time interval (the specification of $x(t)$ when $\mu(t) = w$ for an isolated point of time is inconsequential.) During this

interval $\mu(t)$ is constant, and (A7) implies that $\lambda h(m) - \beta w f(m) = wD'(\theta) / g'(\theta)$. Using (A4) and (A10) we find

$$H^* = g(\theta)[\lambda h(m) - \beta w f(m)] - wD(\theta) = w\xi(\theta), \tag{A11}$$

which possesses a unique root $\hat{\theta}$ in $(\theta_{\min}, \theta_{\max})$. It follows that along the singular path $\theta(t) = \hat{\theta}$ which is the (constant) solution of $\xi(\theta) = H^*/w$. We refer to $\hat{\theta}$ as the *turnpike*. According to (A3), the corresponding irrigation rate is

$$x(t) = \beta g(\hat{\theta}) f(m(t)) + D(\hat{\theta}), \tag{A12}$$

where the time dependence of $x(t)$ on the turnpike is due to the growth of the biomass $m(t)$.

We see from (A5) and (A12) that at each point of time the optimal process $\theta(t)$ must either increase as rapidly as possible ($x(t) = \bar{x}$), decrease as rapidly as possible ($x(t) = 0$) or remain fixed on the turnpike $\hat{\theta}$ ($x(t)$ given in A12). This appears to permit many possible optimal processes that switch back and forth among these decreasing, increasing and singular stages. It turns out, as verified in Claims 1-2 below, that $\mu(t)$ cannot cross w more than once. This, in turn, implies that the irrigation policy can have at most three stages: (i) a rapid approach to the turnpike level $\hat{\theta}$; (ii) a singular stage during which $\theta(t)$ is maintained at $\hat{\theta}$; and (iii) a final stage during which irrigation ceases. Of these three stages, only the third (no irrigation) must always be implemented (see A8); whether or not the other two are implemented depends on the parameters of the problem, particularly the initial soil moisture θ_0, the length of the growing period T and the relative water price w. When irrigation is not profitable, only the third stage is implemented. If irrigation is profitable, the system normally begins with the first stage of rapid approach to $\hat{\theta}$, which may be increasing (when $\theta_0 < \hat{\theta}$) or decreasing (when $\theta_0 > \hat{\theta}$). The second (singular) stage is implemented only if there is enough time to reach $\hat{\theta}$ before the third stage is entered. If the initial soil moisture level equals the turnpike ($\theta_0 = \hat{\theta}$), the first stage is skipped and the system immediately enters the second stage.

Accounting for all possible cases, the optimal policy must assume one of the following four types:

Type 1: $x(t) = 0$ for all t (i.e., irrigation is not profitable and only stage (iii) is implemented).

Type 2: Initially $x(t) = 0$ and $\theta(t)$ decreases until some time t_1 when $\theta(t_1) = \hat{\theta}$; $x(t)$ is then tuned so as to maintain $\theta(t) = \hat{\theta}$ during a singular time interval of duration τ, (with $t_2 = t_1 + \tau < T$), following which irrigation ceases (all three stages are implemented).

Type 3: The same as Type 2 except that $x(t) = \bar{x}$ and $\theta(t)$ increases during the initial period $t \in [0, t_1]$ (all three stages are implemented).

Type 4: Initially $x(t) = \bar{x}$ and $\theta(t)$ increases until some time $t_2 < T$, at which time irrigation ceases (only stages (i) and (iii) are implemented).

Remark: Processes of Types 2 or 3 allow also for vanishing initial periods (i.e. $t_1 = 0$) so that the process is initially singular (when $\theta_0 = \hat{\theta}$) and the first stage is skipped.

The above classification is a result of the property that the optimal μ-process cannot attain (or cross) w more than once, as established in Claims 1-2 below.

Claim 1: If at some date t', $\theta(t') > \hat{\theta}$ and $\mu(t') > w$, then from time t' on the process $\mu(t)$ cannot decrease back and approach w from above.

Proof: Since $x(t) = \bar{x}$ while $\mu(t) > w$, $\theta(t)$ remains above $\hat{\theta}$. Suppose that $\dot{\mu} < 0$ so that (A7) implies

$$g'(\theta)[\lambda h(m) - \beta\mu f(m)] > \mu D'(\theta) > 0. \tag{A13}$$

Since the optimal Hamiltonian $H^* = [\mu - w]\bar{x} - g(\theta)[\beta\mu f(m) - \lambda h(m)] - \mu D(\theta)$ is non-negative and the first term of H^* falls short of the third term when μ is close enough to w, it must be that $\lambda h(m) - \beta\mu f(m) > 0$. It follows from (A13) that $g'(\theta) > 0$ and $\lambda h(m) - \beta\mu f(m) > \mu D'(\theta) / g'(\theta)$. Thus,

$$\mu\xi(\hat{\theta}) > w\xi(\hat{\theta}) = H^* > g(\theta)[\lambda h(m) - \beta\mu f(m)] - \mu D(\theta) > \mu\xi(\theta),$$

contradicting the assumption that $\theta > \hat{\theta}$. □

The claim immediately implies

corollary 1: $\theta(t)$ must decrease while $\theta > \hat{\theta}$.

Proof: $\theta \neq \hat{\theta}$ cannot support a singular policy. If $\theta(t)$ increases, it must be that $\mu(t) > w$ and the process $\mu(t)$, according to claim 1, cannot fall short of w at a later date, violating the transversality condition $\mu(T) = 0$ (cf. A8). □

For $\mu(t)$ below w we have

Claim 2: If at some date t', $\theta(t') < \hat{\theta}$ and $\mu(t') < w$, then from time t' on the process $\mu(t)$ cannot increase to approach w from below.

Proof: Since $x(t) = 0$ while $\mu(t) < w$, $\theta(t)$ must remain below $\hat{\theta}$. If $\dot{\mu}(t) > 0$ then (A7) implies

$$g'(\theta)[\lambda h(m) - \beta\mu f(m)] < \mu D'(\theta). \tag{A14}$$

When $\theta < \hat{\theta}$, $g'(\theta) > 0$ and (A14) implies $\lambda h(m) - \beta\mu f(m) < \mu D'(\theta)/g'(\theta)$. With vanishing $x(t)$, $\mu\xi(\hat{\theta}) < w\xi(\hat{\theta}) = H^* = g(\theta)[\lambda h(m) - \beta\mu f(m)] - \mu D(\theta) < \mu\xi(\theta)$, contradicting our assumption that $\theta < \hat{\theta}$. □

It is now straightforward to verify that the optimal policy must assume one of the four Types listed above. Consider the value of H^* as defined in (A9) and the corresponding state $\hat{\theta}$ obtained using (A11). Then,

corollary 2: When initiated at $\theta_0 > \hat{\theta}$, the θ–process is either of Type 1 or Type 2.

Proof: Above $\hat{\theta}$, the θ–process must decrease. Arriving at $\hat{\theta}$, it can go on decreasing (with $\mu(t) < w$), which implies, according to Claim 2, that it must decrease throughout, yielding a Type 1 process. Alternatively, the process might switch to a singular stage. To satisfy (A8), $\mu(t) = w$ cannot be maintained until T, and the singular stage must end prior to T. Corollary 1 forbids an increasing stage, and Claim 2 ensures that once the decreasing stage starts, the process must continue decreasing until T, yielding a Type 2 process. □

Remark: Initiated at $\theta_0 = \hat{\theta}$, the θ–process behaves according to Corollary 2, except that the initial decreasing stage leading to $\hat{\theta}$ is omitted.

Corollary 3: Initiated at $\theta_0 < \hat{\theta}$, the θ–process must be of Type 1, Type 3 or Type 4.

Proof: If the θ–process is initially decreasing, then, according to Claim 2 it must continue decreasing until *T*, yielding a process of Type 1. If the process is initially increasing, it must stop increasing at or prior to arriving at $\hat{\theta}$. Suppose that the optimal θ-process arrives at the turnpike $\hat{\theta}$. According to Corollary 1, the process cannot increase any further. It can, however, stay constant at that level for a while and then decrease, yielding a Type 3 process. Alternatively, it can decrease promptly upon arrival at $\hat{\theta}$, yielding a Type 4 process. Suppose now that the optimal process ceases to increase below $\hat{\theta}$. In this case, a singular stage cannot be supported and the process must decrease promptly until the harvest time; a Type 4 process is again obtained. □

It is noted that the value H^* of the Hamiltonian (or equivalently, the turnpike $\hat{\theta}$), is not a-priori given. In actual implementations, this parameter, together with the duration of the various stages, must be adjusted so as to satisfy the transversality conditions (A8). This task is readily carried out via a numerical integration scheme for any specification of the model functions.

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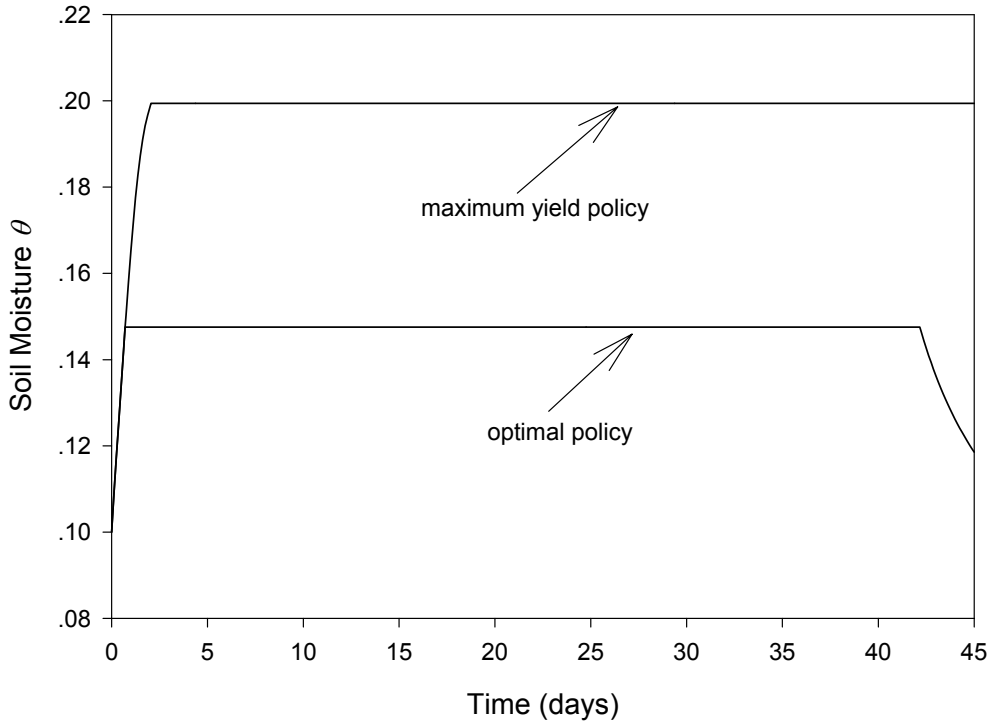

Figure 1: Soil moisture $\theta(t)$ as a function of time (days from planting) under the optimal and the maximal yield policies.

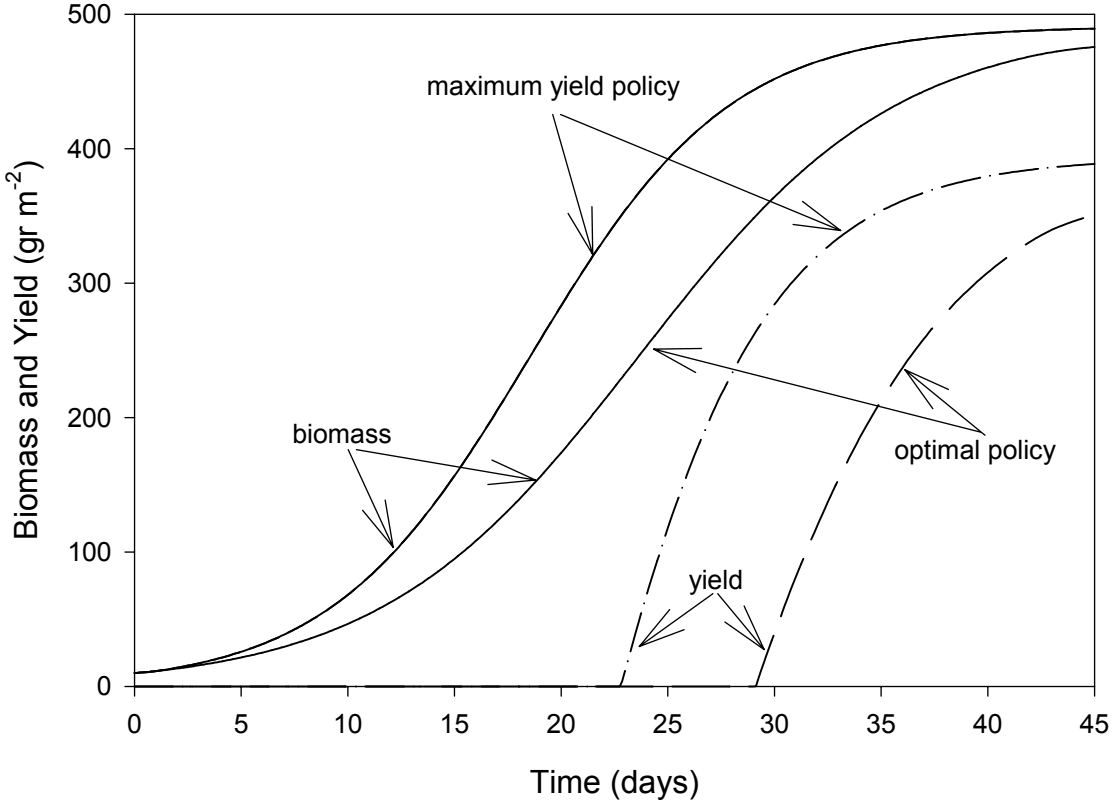

Figure 2: Biomass $m(t)$ (solid lines) and marketable yield (dashed lines) as functions of time (days from planting) under the optimal and the maximal yield policies.

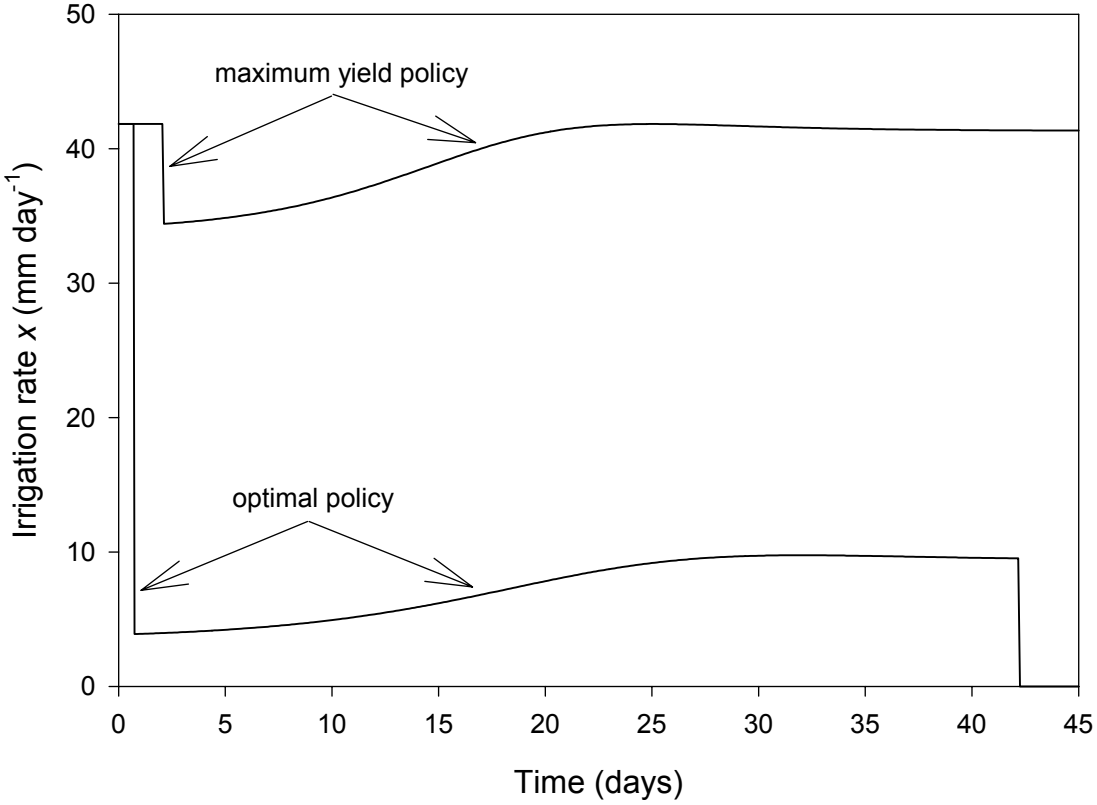

Figure 3: Irrigation rate $x(t)$ as a function of time (days from planting) under the optimal and the maximal yield policies.

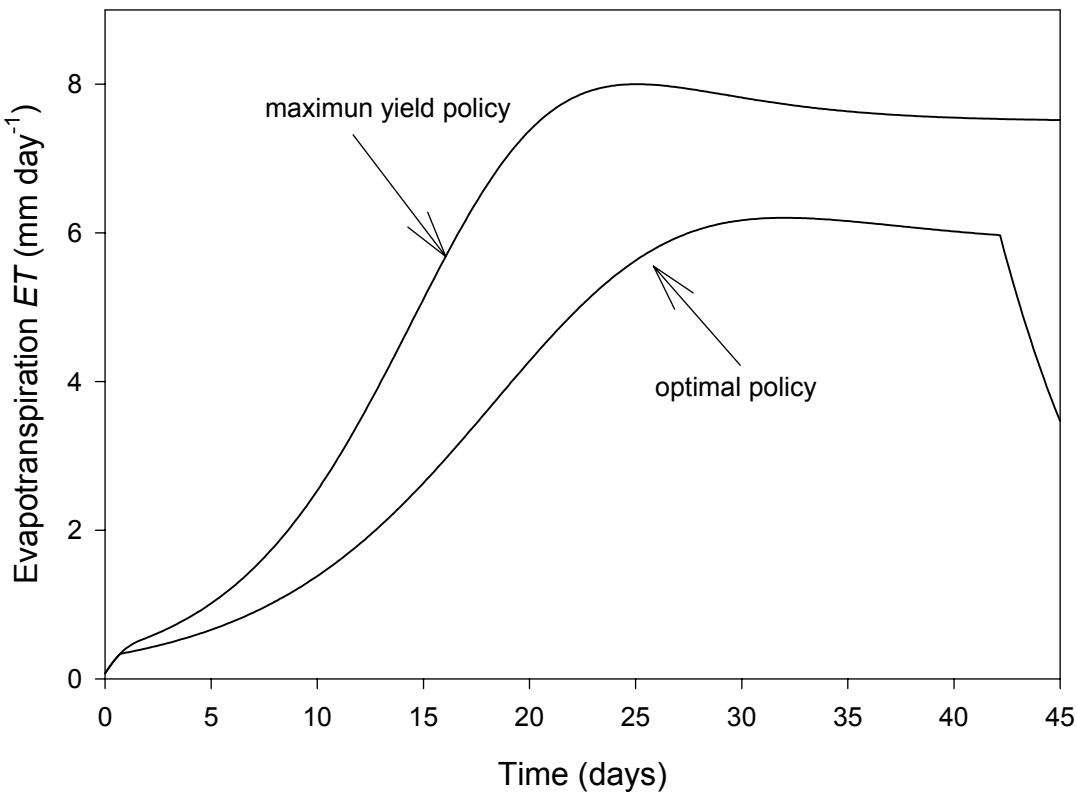

Figure 4: Evapotranspiration rate $ET(t)$ as a function of time (days from planting) under the optimal and the maximal yield policies.

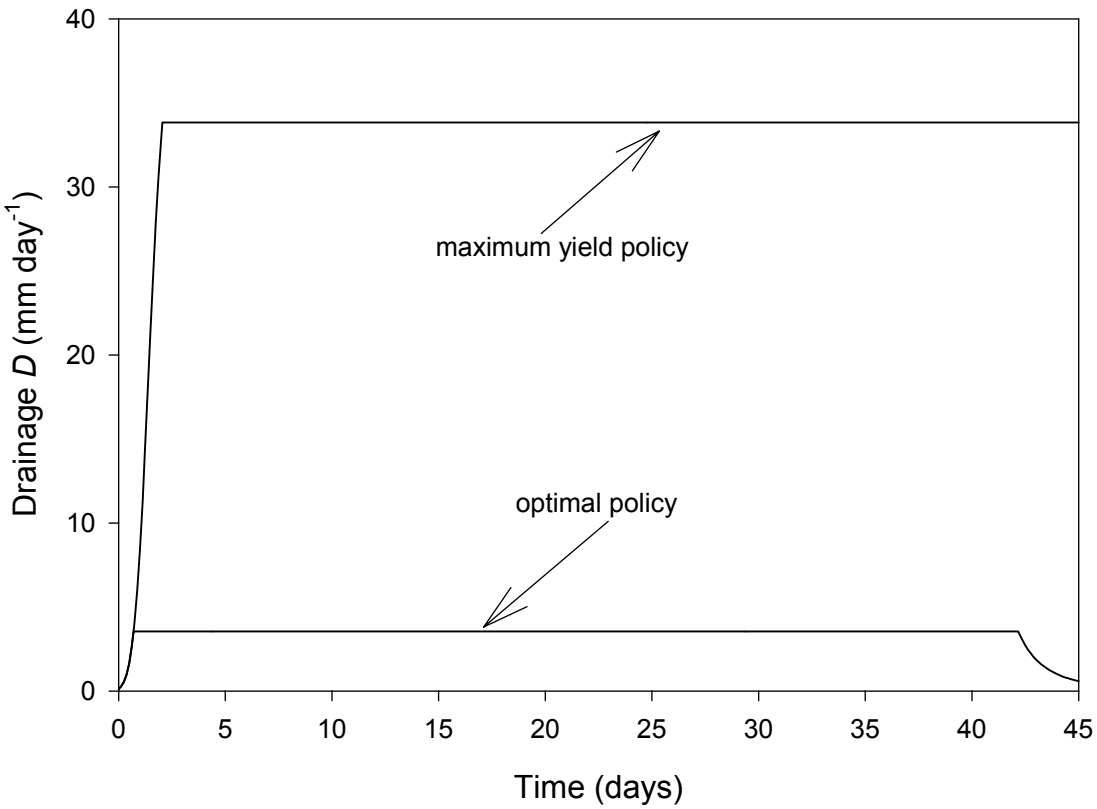

Figure 5: Drainage rate $D(t)$ as a function of time (days from planting) under the optimal and the maximal yield policies.