Dynamic Programming and the Economics of Optimal Grain Storage

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Abstract

Understanding the dynamic programming algorithms used in the optimal grain storage literature is a prerequisite to understanding the findings of this literature This article introduces these dynamic programming algorithms by examining several in terms of their underlying economic behavior These are Gustafson's original algorithms and algorithms developed by Gardner and Ippolito that include rational producer response in addition to optimal grain storage, making them the most advanced in the optimal grain storage literature

Keywords

Dynamic programming, optimal storage, price stabilization, rational expectations

Grain prices have fluctuated widely since the early seventies One method of dampening price fluctuations is to store grain in bumper crop years for use in lean years This possibility has revitalized interest in applying dynamic programming techniques to the analysis of grain carryover Dynamic programming algorithms determine grain carryover rules that are optimal under specified market assumptions, thus, a common title for this literature is optimal grain storage This article provides an introduction to the dynamic programming algorithms in the optimal grain storage literature This literature is growing rapidly because both the objective to be maximized and the market assumptions can be changed to reflect a multitude of types of market situations

The dynamic programming method determines carryover from one harvest to the next by maximizing a specific objective function, such as the value of grain consumption The method also accounts for the expected impact of one year's carryover on carryover levels in future years This consideration makes the carryover determination optimal

We examine several dynamic programming algorithms from the optimal grain storage literature in terms of the economic behavior of storers (speculators) and producers

We concentrate on dynamic programming algorithms developed by Gardner (*3*) and by Ippolito (*7*) [1] Their work contains improvements in the incorporation of economic behavior, particularly producer supply response, into the dynamic programming method We also examine Gustafson's (*5*) original dynamic programming analysis of grain storage—the foundation for the optimal grain storage literature

The basic dynamic programming algorithm maximizes an objective function, subject to the influence of a random variable The algorithm accomplishes this by finding a sequence of decisions concerning the levels of a control variable One type of grain storage problem fits particularly well into this algorithm the problem of maximizing the value of consumption over a long time period [2] The control variable is the size of grain carryover from

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[1] Italicized numbers in parentheses refer to items in the References at the end of this article

[2] The objective of maximizing total expected revenue can be used instead of total expected value of consumption Maximizing total revenue corresponds to a grain marketing board that acts as a monopolist for the benefit of grain producers Maximizing total value of consumption corresponds to a competitive grain market

this year to next year The decision each year is how much grain to carry into the next marketing year The random variable in this problem is production Its variability is largely attributable to the unpredictability of weather

Gustafson developed a dynamic programming algorithm to find optimal grain carryover levels with random production Grain carryover is determined given the total grain supply after harvest, which equals current production plus carryover from the previous year The carryover decisions are made optimally in the sense that the sum of the current value and expected future value of consumption less storage cost is maximized

Demand variability and forward-looking producer reponse are additional elements needed in a comprehensive dynamic programming algorithm for a grain market Both these elements influence the optimal grain carryover level for a given level of total grain supply Including them places optimal grain storage firmly in an economic context, but complicates the dynamic programming computations. Demand variability includes fluctuations in national income If demand includes export demand, then its variability is also attributable to grain supply variability in other countries

Forward-looking producer response adds rational expectations to the model This addition allows producers to react to differing levels of grain carryover and allows speculators to react to differing levels of expected production

Table 1 shows the objectives maximized and the demand and supply assumptions for the algorithms we examine The table provides a starting point for understanding the similarities and differences among the algorithms One maximizes the objective functions of the algorithms by finding the optimal carryover rules and, if production is given

Table 1—Specifications for the optimal carryover algorithms

Algorithm description	Objective maximized	Current year demand	Current year production
Gustafson			
1A Random production and a stable demand curve	Value of consumption	Linear function of current year's price	Random variable
1B Random production and a stable demand curve	Returns from carryover	Linear function of current year's price	Random variable
2A Random production and a fluctuating demand curve	Value of consumption	Linear function of current year's price plus a random term	Random variable
2B Random production and a fluctuating demand curve	Returns from carryover	Linear function of current year's price plus a random term	Random variable
Gardner			
1A Rational production and a stable demand curve	Economic surplus	Linear function of current year's price	Linear function of current year's expected price plus a random term[1]
1B Rational production and a stable demand curve	Returns from carryover and returns from production	Linear function of current year's price	Linear function of current year's expected price plus a random term[1]
Ippolito			
1 Rational production and a fluctuating demand curve	Returns from carryover and returns from production	Linear function of current year's price plus a random term	Linear function of current year's expected price plus a random term[1]

[1] The production response for a particular year is dependent on the expected price for that year in both Gardner's algorithms and Ippolito's algorithm However, there is a difference in the order of solving for the production response Gardner's algorithms solve for the current year's carryover and for next year's production Ippolito's algorithm solves for the current year's carryover and for the current year's production

as a function of expected price, by also finding the rational production response Under specific market assumptions, maximizing the value of consumption is equivalent to maximizing the returns from carryover in a competitive market [3] Furthermore, maximizing economic surplus is equivalent to maximizing returns from carryover and returns from production in a competitive market under these market assumptions These objectives result in identical carryover rules and identical production responses To make the explanation easier, we show Gardner's algorithms with a stable demand curve Gardner included a random demand component using the method pioneered by Gustafson In this article, we explain Gustafson's method of including a random demand component

First, we examine a dynamic programming algorithm to find optimal carryover levels under random production and stable demand, using the value of consumption as the objective function We then show that the first-order condition for maximizing the value of consumption suggests an alternative algorithm based on the objective of maximizing the returns to storers in a competitive market Next we show how a stochastic demand component is added to these two dynamic programming algorithms Finally, we examine algorithms that include rational producer response and explain the relationship between rational production response and optimal grain carryover

Optimal Grain Carryover Under Random Production

One measure of consumer welfare is the area under the demand curve This measure is a convenient way to put a value on consumption In this section we examine the dynamic programming algorithm for finding optimal carryover levels using the area under the demand curve as the objective function In addition, we review the first-order conditions for maximizing the value of consumption These first-order conditions draw attention to the economic behavior of speculators in grain storage Profit-maximizing behavior by speculators in a market satisfying specific economic assumptions will bring about competitive equilibrium levels of producer and consumer behavior We review an alternative algorithm for determining the optimal grain carryover using the grain prices suggested by the first-order conditions These algorithms were developed by Gustafson, who also extended them to include demand variability

The first algorithm finds the optimal grain carryover given the level of total supply by a trial-and-error search Discrete specifications of carryover and total supply are required because the search for the optimal carryover level must be restricted to a finite number of possibilities and the search procedure can only be used a finite number of times In addition, a discrete specification is required for production as total supply equals carryover from the previous year plus production The parameters of the first two methods include storage cost, discount rate, variability and mean level of production, and the level (intercept) and slope of the demand equation

Estimating Optimal Carryovers with Value of Consumption as the Objective Function

The value of consumption algorithm solves for optimal carryovers under random production by the dynamic programming method known as value iteration [4] It finds the optimal carryover, $C_{k,t}$, for each possible level of the total supply after harvest, $S_{i,t}$, where t is the current year, and i and k denote discrete intervals over the range of values of the variables The subscript j replaces i to indicate levels of total supply for the next year, $S_{j,t+1}$

The optimal carryover decisions maximize the value of consumption in year t, $R_{i,t}$, plus the discounted expectation of the value of future consumption, $r \sum_{j=1}^{I} [Pr(S_{j,t+1} \mid C_{k,t})] f_{j,t+1}$, minus the current year's storage cost, $SC(C_{k,t})$ The constant r is the discount rate, which equals $1/(1+\rho)$, where ρ is an interest rate

Equations (1) and (2) show the computations involved in finding the optimal carryover levels

[3] See Gustafson (5, p 49) for a succinct review of these assumptions

[4] The approach used to solve equations (1) and (2) is known as value iteration in the dynamic programming literature Dynamic programming problems can also be solved by Howard's policy iteration approach (6) The value iteration approach is used in all the algorithms of this article

$$f_{i,T} = \max_{k=1} [R_{i,T} | S_{i,T} - C_{k,T}] \quad i = 1,2, \quad , I \qquad (1)$$

$$f_{i,t} = \max_{k} (R_{i,t} | S_{i,t} - C_{k,t}) - SC(C_{k,t}) + r \sum_{j=1}^{I} [Pr(S_{j,t+1} | C_{k,t})] f_{j,t+1} \qquad (2)$$

$i,j = 1,2, \quad , I$
$k = 1,2, \quad , K$

where

$t = T, T-1, T-2,$	indicates year starting with the most distant year considered (the horizon), year T,
$i,j = 1,2, \quad , I$	are intervals indicating discrete levels of total supply after harvest in the current year (year t) and in the following year (year t+1), respectively (for example, level $i = 2$ might represent 10 million bushels),
$k = 1,2, \quad , K$	indicates discrete carryover levels in the current year (year t),
$Pr(S_{j,t+1} \mid C_{k,t})$	is the probability that total supply next year will be at level j when carryover from the current year is at level k

The first equation is used in the first (most distant) year considered, year T Carryout from year T is assumed to equal zero (k = 1) for all possible levels of total supply, implying that we need not consider storage cost and expected value of future consumption In prior years these terms must be included, so equation (2) is used

In the optimal grain storage literature, demand is often specified as a linear function of price Let $D_{i,t}$ represent the following demand curve $Q^D_{i,t} = \alpha - \beta P_{i,t}$ A linear demand function gives a value of consumption, $R_{i,t}$, in the current year equal to

$$R_{i,t} = (\alpha/\beta)(S_{i,t} - C_{k,t}) - \frac{(S_{i,t} - C_{k,t})^2}{2\beta} \qquad (3)$$

Equation (3) is the integral of the inverse demand function (price as a function of quantity) from zero to q, $\int_0^q (\alpha/\beta - \frac{1}{\beta} q)dq$, where q equals total supply minus the chosen level of carryover This is the area under a linear demand curve from zero to the level of total supply minus carryover, $S_{i,t} - C_{k,t}$

The algorithm operates backwards in time from the last year considered, year T The logic is that the optimal carryover decisions for future total supply levels and the probabilities that these total supply levels will occur must be known prior to making the optimal carryover decision for the current year

The algorithm starts by using equation (1) to calculate the maximum value of consumption, $f_{i,t}$, for each level of total supply, $S_{i,t}$, in year T, the last year considered Each of these maximum values has a zero carryover, level 1 of subscript k represents zero carryover The carryover in the last year considered is usually specified to be zero with the value iteration approach This restriction does not influence the final results of the value iteration approach as the influence of zero carryover for the last year is gradually dissipated as we move backward in time

Next, using equation (2), one finds the maximum expected value of consumption in the preceding year, t = T-1, by searching among the K possible carryover levels for each level of total supply, $S_{i,t}$ The values $f_{j,t+1}$, $j = 1, \quad , I$ are the maximums found in equation (1) The term $Pr(S_{j,t+1} | C_{k,t})$ is the probability that next year's total supply will be at level j when carryover from the current year is at level k The probabilities of the various levels of total supply occurring around its mean are determined by the probability distribution of production outcomes

The mean of this probability distribution equals carryover plus the mean level of production and is determined by the carryover decision

$$E(Pr(S_{j,t+1} \mid C_{k,t})) = C_{k,t} + E(PROD_{t+1} \mid C_{k,t}) \quad (4)$$

The expected level of production is the unconditional mean, $E(PROD_{t+1})$, when random production is assumed

Next, equation (2) is used to find the optimum carryover for the preceding year, this time year t = T-2 The procedure is as previously described for equation (2) except that the maximum values, $f_{j,t+1}$, are those found for year T-1

Assuming that the parameters in equations (1) and (2) are stable over time, additional years are included until the optimal carryovers (levels of the control variable) converge to a particular level for each level of total supply At this point, the influence of the zero carryover restriction for year T is completely dissipated The correspondence between these optimal carryovers and total supply represents the optimal storage behavior (optimum carryover decisions) in a perfectly competitive market It is optimal because these levels of carryover maximize the current plus the expected future value of consumption minus the storage cost over an indefinite time span In this situation, this optimal storage behavior (set of carryover rules) remains in effect for each new total supply, that is, for each new harvest until a change occurs in one or more of the parameters

Gustafson (*5*) developed the preceding algorithm specifically for estimating optimal grain carryover Howard (*6*) independently developed a similar algorithm to estimate optimal decisions We recommend Howard's book for further background reading on dynamic programming

The search for the optimum carryover level (given the level of total supply) is not a brute force search over all possible carryovers If the optimum carryover level is greater than zero, then the value of equation (2) increases, reaches a maximum, and then decreases as carryover increases (as the welfare measure is quadratic) In this situation, the algorithm evaluates equation (2), using successively larger carryover levels until the maximum is passed The optimal carryover level is the next to last value used If the value of equation (2) decreases as carryover increases, the optimum carryover level is zero This occurs when total supply is low A value of carryover less than zero might maximize an unconstrained version of equation (2) in this situation, but only zero or positive values for carryover are physically possible Negative carryover values imply that grain can be borrowed from next year's harvest for use in the current year

The determination of the optimal current-year carryover by use of equation (2) is an application of Bellman's principle of optimality (*1*, p 83) This principle states that a necessary condition for the current decision to be optimal is that future decisions must constitute optimal behavior with regard to the effect of the current decision This rather elusive concept implies that the trial-and-error search for the optimal grain carryover in the current year, given the level of total supply, requires knowledge of how future optimal decisions are affected by the current grain carryover decision This knowledge is contained in the maximums ($f_{j,t+1}$, j = 1, ,I) found in the previous solutions of equation (2)

We can see a profit-maximizing motive for storage by examining the first-order condition for maximizing equation (2) We derived the standard expression for this first-order condition, expression (5 3), by taking the partial derivative of the maximum found in equation (2) with respect to the current year's carryover

$$\frac{\partial f_{i,t}}{\partial C_{k,t}} = \frac{\partial(R_{i,t} \mid S_{i,t} - C_{k,t})}{\partial C_{k,t}} - \frac{\partial SC(C_{k,t})}{\partial C_{k,t}} + r \sum_{j=1}^{I} (Pr(S_{j,t+1} \mid C_{k,t})) \frac{\partial f_{j,t+1}}{\partial S_{j,t+1}} \leq 0 \quad (5)$$

$$= -[\alpha/\beta - \frac{1}{\beta}(S_{i,t} - C_{k,t})] - MSC + r \sum_{j=1}^{I} (Pr(S_{j,t+1} \mid C_{k,t}))P_{j,t+1} \leq 0 \quad (5\ 1)$$

$$= -P_{i,t} - MSC + rE(P_{t+1} \mid C_{k,t}) \leq 0 \quad (5\ 2)$$

so

$$P_{i,t} \geq rE(P_{t+1} \mid C_{k,t}) - MSC \quad (5\ 3)$$

where E is the expectations operator and MSC is marginal storage cost

Expression (5 3) is an equality when grain is stored at the optimal level In this situation, grain storage is increased and the current year's price, $P_{i,t}$, is bid up by speculators until it equals next year's expected price, $E(P_{t+1}| C_{k,t})$, times the discount factor, r, minus the marginal (per bushel) storage cost, MSC An increase in grain storage also reduces next year's expected price The marginal storage cost may either increase or remain constant with larger carryover levels, although it is convenient to assume a constant marginal cost

The cost of holding each unit equals the purchase price, $P_{i,t}$ plus the marginal storage cost The expected per unit return from storage equals expected price minus the holding cost That is, the expected return is $[E(P_{t+1})-(P_{i,t}+MSC)]$ or $[E(P_{t+1})-rE(P_{t+1})]$ Of course, the return for any single year will most likely differ from the average or expected rate

The first-order condition in (5 3) is an inequality when the optimal decision is to store no grain In this situation, the current year's price is greater than the discounted value of next year's expected price minus the marginal storage cost This situation occurs when the grain harvest in the current year falls below a critical level

Because the link between the standard expression for the first-order condition (5 3) and equation (2) is not fully explained in the optimal grain storage literature, we have included the intermediate steps (5 1) and (5 2) The derivative of the area under the demand curve, $R_{i,t}$, with respect to quantity (in this case carryover, $C_{k,t}$), is the inverse demand curve (first term in equation (5 1)) A minus sign precedes this term as increases in carryover reduce current consumption and, hence, the area under the demand curve Evaluating the inverse demand function (the first term in (5 1)) at $S_{i,t}-C_{k,t}$ produces the negative of the current year's price, the first term of expression (5 2)

The second term in equation (2) is total storage cost, given the level of carryover The partial derivative of total storage cost with respect to carryover is the marginal storage cost shown as MSC preceded by a minus sign in (5 1) and (5 2) Marginal storage cost is usually specified as a constant value in the optimal grain storage literature regardless of the level of carryover However, the total cost of storage function can be specified so that the marginal storage cost increases as carryover increases

In the third term of equation (2) the derivative of each of the maximums, $f_{j,t+1}$, with respect to next year's total supply, $S_{j,t+1}$, is

$$\frac{\partial f_{j,t+1}}{\partial S_{j,t+1}} = \frac{\partial R_{j,t+1}}{\partial S_{j,t+1}} - \frac{\partial SC(C_{k,t+1})}{\partial S_{j,t+1}} + r\sum_{\ell=1}^{I}(P_{\ell,t}C_{k,t+1})\frac{\partial f_{\ell,t+2}}{\partial S_{j,t+1}} \qquad (6)$$

which equals $(\alpha/\beta-1/\beta)(S_{j,t+1}-C_{k,t+1})$ - 0 + 0 or $P_{j,t+1}$ [5] Multiplying each possible price next year, $P_{j,t+1}$, by its probability of occurring and summing over j, as shown in the last term in (5 1), produces next year's expected price for the current year's level of carryover This expected price is shown in the last term of (5 2) The maximum $f_{j,t+1}$ equals the value of consumption next year if supply turns out to be $S_{j,t+1}$, minus the storage cost associated with this level of supply, plus the discounted expected value of the maximums for year t+2 The index $\ell = 1,2, \quad ,I$ represents the possible levels of total supply 2 years hence and $(Pr(S_{\ell,t+2} | C_{k,t+1}))$ is the probability that total supply 2 years hence will be at level ℓ if supply next year is at level $S_{j,t+1}$ The partial derivative of $f_{j,t+1}$, shown in (5), is taken with respect to next year's total supply rather than with respect to the current year's carryover This substitution can be made because a given increase in carryover from the current year increases next year's total supply by the same amount

Nonlinear specifications of demand require integration and differentiation techniques that differ from those shown when a linear specification of demand is used However, the standard expression of the first-order condition shown in (5 3) is not altered by using a nonlinear specification of demand

Price Method for Estimating Optimal Carryovers

One can find optimal carryovers by using the first-order condition in (5) The first step is to

[5] Burt's article (2) helped us understand how to take the derivative of $f_{j,t+1}$ with respect to $S_{j,t+1}$

calculate the price, $P_{i,t}$, for each level of total supply in year T, the last year, by use of the demand equation As with the value of consumption method, there are no carryovers in the last year The second step is to calculate the optimal carryover for each level of total supply in the next to last year, year T-1, using expression (5 1) The procedure is to search over the possible carryover levels (given the level of total supply) to determine the carryover level that makes (5 1) equal zero This may be impossible for low levels of total supply If equality is impossible, the optimal carryover is zero As mentioned earlier, an increase in carryover increases the current year's price and decreases next year's expected price We can calculate the current year's price, $P_{i,t}$ in (5 1) by evaluating the demand function at the given level of total supply minus the chosen level of carryover

One uses the prices found for year T, the last year, in the last term in (5 1) when searching for the carryover level for year T-1, the next to last year The probability of each of these prices occurring as well as next year's expected price shown in the last term in (5 2), is determined by the carryover level

As with the total value of consumption method in equations (1) and (2), additional years are added until the carryover converges to a particular value for each level of total supply The optimal carryovers are the same as those that one finds using equations (1) and (2)

In this example with random production, there is no computational advantage in calculating optimal carryovers by the price method, that is, by the first-order condition However, when a random demand component is also included, the price method does offer computational savings

Optimal Grain Storage When a Random Demand Component is Included

Equations (7) and (8) include a random demand component in addition to random production

$$f_{i,h,T} = \max_{k=1}[R_{i,h,T} \mid S_{i,T} - C_{k,T}, D_{h,T}]$$

$$i = 1,2, \quad , I$$
$$h = 1,2, \quad , H \qquad (7)$$

$$f_{i,h,t} = \max_{k} \Big\{ (R_{i,h,t} \mid S_{i,h,t} - C_{k,t}, D_{h,t}) - SC(C_{k,t}) - r \sum_{j=1}^{I} (Pr(S_j \mid C_{k,t})) \sum_{m=1}^{H} Pr(D_{m,t+1}) f_{j,m,t+1} \Big\}$$

$$k = 1,2, \quad , K$$
$$i = 1,2, \quad , I$$
$$h = 1,2, \quad , H \qquad (8)$$

The indexes t, i, j, and k are the same as defined for equations (1) and (2) In addition to these indexes, equations (7) and (8) contain the two indexes h,m = 1,2, ,H, which represent alternative discrete levels of the demand curve in the current year and for next year, respectively The probability of each demand level's occurring next year is represented by $Pr(D_{m,t+1})$ The value of consumption in the current year depends on both total supply minus carryover, $S_{i,t} - C_{k,t}$, and on the level of the current year's demand curve, $D_{h,t}$, which is known after harvest The total expected value of consumption for next year depends on the carryover level and the probabilities associated with next year's supply levels and demand curve levels The carryover decision influences the level of supply next year, but does not influence the level of the demand curve next year

The dynamic programming procedure for finding the optimal carryovers is slightly changed With a stochastic demand component included, the optimal carryover must be found for each combination of current year demand and supply levels The computations for calculating the expected future value of consumption involve an additional summation For each level of total supply next year, we must first find the expected value of future consumption over all levels of demand next year

If 40 levels of supply and 40 levels of demand are used in the value of consumption algorithm, then optimal carryovers must be found each year for all 1,600 combinations of supply and demand This tremendous increase in computational requirements caused by making the optimal grain storage problem more comprehensive is an example of the "curse of dimensionality" in dynamic programming However, a different formulation of a dynamic programming problem often results in substantial computational savings

The algorithm shown in equations (7) and (8) has an interesting result that can be used to reduce the number of computations, namely, the level of supply minus the level of demand determines price and carryover If i to i + 1 and h to h + 1 represent equal increases in the current year's levels of supply and of the demand curve, (for example, 10 million bushels), then price and carryover are not affected The difference between the supply and demand curves is not changed by these equal increases However, consumption is increased by an amount equal to the increase in supply (or demand)

This result is useful because it implies that one can achieve computational savings by using the price method to solve for optimal carryovers, that is, by using the first-order conditions One can achieve these computational savings by using differences between the levels of supply and demand in conjunction with the first-order conditions for maximizing profits from storage

Let us now return to the problem with 40 levels of each supply and demand If the increments between the successive levels of supply and demand are equal, then there are still 1,600 combinations of supply and demand levels but only 79 different values for supply level minus demand level This means that only 79, rather than 1,600, optimal carryover levels need be calculated for each year The probability of occurrence of each of the differences in supply and demand levels can easily be calculated from the probabilities in equation (8) The probability of a particular difference occurring equals the summation of $[Pr(S_{j,t+1})$ times $Pr(D_{m,t+1})]$ over the number of ways that the difference can occur

When we use the first-order condition given in the previous section, the dynamic programming algorithm performs equally well in solving for optimal carryovers using differences in the levels of supply and demand Now the current year's price is calculated by use of the supply level minus the demand level and the chosen level of carryover Also, next year's expected price is calculated by use of the probability of the various supply levels minus demand levels occurring in the following year, given the carryover decision

For this problem, the i index in equation (5) represents the alternative supply minus demand levels The probability distribution in equation (5) for this problem represents the probabilities of the alternative supply minus demand levels occurring in the following year, given the carryover decision An increase in carryover increases the current year's price It also increases the expected level of total supply next year, thereby increasing the probabilities of large levels of supply minus demand This situation, of course, reduces the expected price next year

Gardner (*3*, p 133) included a random demand component in his study by using the price method rather than the value of consumption method Gustafson (*5*, p 51) originally showed how to include a random demand component along with random production

Optimal Grain Storage When a Rational Production Component is Included

Gardner and Ippolito independently developed dynamic programming algorithms for including rational producer response with optimal grain storage Gardner's algorithm emphasizes the optimal carryover decision after harvest, assuming the rational production response next year Ippolito's algorithm emphasizes the rational production decision at planting, given optimal carryovers for each possible level of total supply after harvest Both algorithms represent major advances in using dynamic programming to analyze grain storage

A major difference between the two algorithms is in the way that the rational production response is found Ippolito's algorithm uses an iterative procedure that converges to the rational production response Gardner's algorithm uses a trial-and-error search procedure which is the typical approach in dynamic programming algorithms Ippolito's iterative approach diminishes the curse of dimensionality problem that is encountered when a second decision variable is introduced

Gardner's Algorithm

Gardiner's algorithm uses equations (9) and (10)

$$f_{i,T} = \max_{k=1}[R_{i,T} - C_{k,T}] \quad i = 1,2, \quad , I \tag{9}$$

$$f_{i,t} = \max_{k} \Big\{ [R_{i,t} \mid S_{i,t} - C_{k,t}] - SC(C_{k,t}) +$$

$$r \sum_{j=1}^{I} [Pr(S_{j,t+1} \mid C_{k,t}, E(PROD_g))] f_{j,t+1} - rVC_g \Big\}$$

$$k = 1,2, \quad , K$$
$$g = 1,2, \quad , G$$
$$i = 1,2, \quad , I \qquad (10)$$

The indexes t, i, j, and k are the same as defined in equations (1) and (2)

In addition to these indexes, equation (10) contains the index g = 1,2, ,G, which represents the possible levels of production next year The variable $E(PROD_g)$ in equation (10) represents next year's expected production [6] This equation includes an additional term representing variable production cost, VC_g, which, like storage cost, is subtracted Including variable production cost means that the value of consumption minus variable production cost is the objective function This means that the objective of maximizing economic surplus replaces the objective of maximizing the value of consumption

The computations for this algorithm include straightforward additions to those previously explained for random production and stable demand (equations (1) and (2)) However, there is a subtle difference in the optimal storage behavior This difference will be discussed later

The additional computations involve the probability distribution of production outcomes The rational production decision determines the mean level of next year's possible production outcomes This is reflected in the probability term in equation (10) With rational production included, there are two decision variables Both the carryover and production decisions influence next year's total supply Also, each decision influences the other That is, carryover and expected production are interdependent endogenous variables, consequently, their levels must be made consistent In this algorithm, optimal carryovers are found by a two-dimensional search among the carryover and production levels

[6] The random production specification can be considered as a special case with only one level of production response, g

The objective of the search is to find the combination that maximizes equation (10)

This algorithm, like that for random production, is started in year T, the last year considered Additional years are considered until carryover converges to a particular level for each level of total supply and until the rational production level for next year converges to a particular level for each level of carryover from the current year

An examination of the first-order conditions for maximizing equation (10) reveals the economic behavior involved Expression (11) and equation (12) show the first-order conditions for storers and producers, respectively

$$\frac{\partial f_{i,t}}{\partial C_{k,t}} = -P_{i,t} - MSC$$

$$+ rE(P_{t+1} \mid C_{k,t}, E(PROD_{g,t+1})) < 0 \qquad (11)$$

$$P_{i,t} \geq rE(P_{t+1} \mid C_{k,t}, E(PROD_{g,t+1})) - MSC \qquad (11\ 1)$$

$$\frac{\partial f_{i,t}}{\partial PROD_{g,t+1}} = rE(P_{t+1} \mid C_{k,t}, E(PROD_{g,t+1}))$$

$$- r \frac{\partial VC_g}{\partial PROD_{g,t+1}} = 0 \qquad (12)$$

$$E(P_{t+1} \mid C_{k,t}, E(PROD_{g,t+1})) = MC \qquad (12\ 1)$$

where VC is the variable production cost and MC is the marginal cost of production

In this situation, both first-order conditions must be simultaneously fulfilled The first-order condition for maximizing profits from storage includes the effect of storage on next year's expected production The first-order condition for maximizing profits from production includes the effect of carryover on production

The partial derivative with respect to carryover for the first three terms in equation (10) is iden-

tical to those in equation (2) with random production The partial derivative of the last term in equation (10) with respect to carryover is zero as production is held constant when this partial is taken The partial derivatives of the first two terms in equation (10) with respect to production next year are both zero Carryover is assumed to be held constant when these partials are taken Therefore, neither current year's consumption nor cost of storage is affected The partial derivatives of the last two terms in equation (10) with respect to production are next year's expected price and the marginal cost of production The first-order condition in equation (12) shows that producers maximize expected profits by equating expected marginal returns (price) with marginal costs Notice that the derivatives of the third term, with respect to both carryover and next year's production, are identical Identical changes in carryover and expected production will change total supply next year by identical amounts and, hence, the expected value of consumption next year by identical amounts

For each level of carryover, storers know the expected production response It is the expected production level, given carryover, that equates expected producer price and marginal production cost in the following year Thus, carryover and next year's expected production come in "ordered" pairs Storers choose that pair which also fulfills the first-order condition in expression (11) As under random production, this choice maximizes their expected profits from storage

A random demand component can be included with optimal storage and rational production just as with optimal storage and random production As under random production, computational savings are gained by the addition of the random demand component to the first-order conditions (expression (11) and equation (12)) rather than to the value of consumption equations (equations (9) and (10))

Ippolito's Algorithm

Gardner's algorithm solves for the current year's carryover and next year's expected production Ippolito's algorithm solves for the current year's expected production and the current year's carryover Because carryover and production decisions follow one another ad infinitum, the choice of which half of the cycle to put first is arbitrary

Ippolito's algorithm finds the rational production level for each level of carryin in the current year, and it finds the optimal carryover level, given the level of carryin, for each possible combination of production outcomes and demand levels in the current year As in Gardner's algorithm, the production and carryover decisions are interdependent

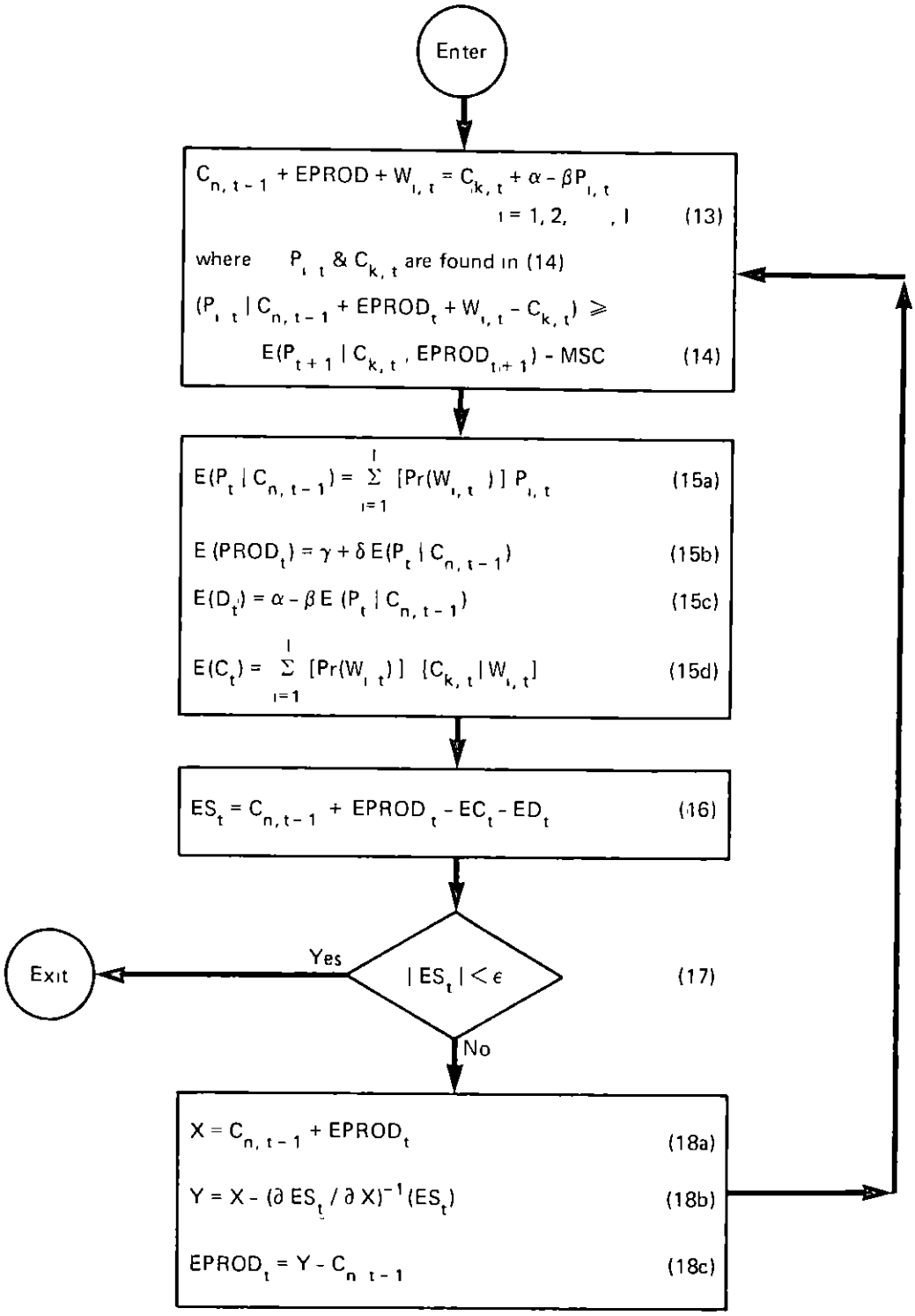
The current year's production and demand in Ippolito's algorithm are specified as $PROD_t = \gamma + \delta E(P_t) + V_t$ and as $Q^D{}_t = \alpha - \beta P_t + U_t$, respectively The stochastic variables V_t and U_t determine the levels of the current year's supply and demand curves They are known after harvest Producer's expected price, $E(P_t)$, is determined at planting At equilibrium, this is the rational price that equates total expected supply (the given carryin level plus expected production) with total expected demand (expected carryover plus expected current year demand) The current year's price, P_t, is determined after harvest by the carryover decision The optimal carryover level equates this price with next year's discounted expected price minus the marginal storage cost

The flow chart in the figure shows the steps in finding carryover and expected production in the current year given a particular level of carryin Following is a detailed explanation of the flow chart and of the other steps in Ippolito's algorithm The i index (i = 1,2, ,I) indicates the alternative levels of the difference between the levels of the supply curve and the demand curve The indexes for k and n (k,n = 1,2, ,K) indicate the alternative levels of carryover for the current year and carryin from the previous year As in the previous algorithms, the t index (t = T, T-1, T-2,) indicates the year, starting with the most distant year considered

Equation (13) in the figure shows that the current year's carryover, $C_{k,t}$, is found given the level of carryin, $C_{n,t-1}$, and given the level of the supply curve minus the level of the demand curve, $W_{i,t}$ The stochastic variable $W_{i,t}$ equals V_t minus U_t Each $W_{i,t}$ represents the probability associated with an interval on the probability distribution

Figure

Steps in Ippolito's Algorithm for Determining Carryover and Expected Production



of W_t The intervals for all the $W_{i,t}$'s define the probability distribution of W_t in discrete segments [7] As previously explained, the supply curve level minus the demand curve level (in this algorithm, $W_{i,t}$) is a determinant of carryover and price An assumed value of expected production, $EPROD_t$, is first used in equation (13) for a given level of carryin Improved estimates of expected production are calculated from the results of equation (13) for all the values of $W_{i,t}$ given a particular level of carryin The procedure for calculating improved estimates of expected production is explained later

Equation (13) is solved for carryin for all values of $W_{i,t}$ given a particular level of carryin, $C_{n,t-1}$, by use of expression (14) This expression differs from Gardner's expression (11 1) only in that specific alternative production levels are not specified prior to using the algorithm A trial-and-error search is made among the possible carryover levels, $C_{k,t}$, given the level of $C_{n,t-1} + EPROD_t + W_{i,t}$, to find the level that makes the current year's price in (14) equivalent to next year's discounted expected price minus the marginal storage cost This trial-and-error search is done for each value of $W_{i,t}$ One calculates the current year's price, $P_{i\,t}$, from the demand equation, $Q = \alpha - \beta P_t$, by using $C_{n,t-1} + E(PROD_t) + W_{i,t} - C_{k,t}$ as the value of Q

To start the algorithm, assume next year's carryover, year T, equals zero regardless of the carryover level from the current year, year T-1 This restriction allows us to find next year's expected price by setting carryin plus the supply equation equal to the demand equation and then by solving for expected price, $E(P_T \mid C_{k,T-1}) = (-C_{k,T-1} + \alpha - \gamma)/(\beta+\delta)$ The expected price is found for each level of carryover, $C_{k,T-1}$, from the current year (carryin for next year) Next year's expected production equals $E(PROD_T) = \gamma + \delta\ E(P_T \mid C_{k,T-1})$ Next year's expected price for each level of carryover from the current year, $E(P_T \mid C_{k,T-1})$, is the information we need to determine the optimal carryover level using expression (14)

The results from equation (13) for a given level of carryin are used to determine whether the value of expected production used in this equation is within an allowable limit of equating total expected supply with total expected demand First, as shown in (15a), the expected price, $E(P_t \mid C_{n\,t-1})$, is calculated from the results of equation (13) Next, the expected production, $E(PROD_t)$, and the expected current year demand, $E(D_t)$, implied by this expected price, are calculated in (15b) and (15c) Finally in (15d), the expected carryover, $E(C_t)$, is calculated from the results of equation (13) These expected production, $E(PROD_t)$, and the expected in, are used in equation (16) to determine the expected excess supply

This is the excess supply that rational producers would expect given the sum of carryin and the tentative value of expected production used in equation (13) Next, expression (17) determines if the absolute value of this expected excess supply is less than a small positive value, ϵ If not, we calculate an improved, but still tentative, estimate of expected production using (18a), (18b), and (18c) This improved estimate is then used in equation (13)

Ippolito showed that the expected excess supply decreases monotonically as total expected supply increases This decrease implies that there is a unique value of total expected supply which has a zero excess supply Ippolito also showed that the derivative of expected excess supply with respect to total expected supply lies between $-(\beta+\delta)/\beta$ and -1 [8] One finds an improved estimate of total expected supply by approximating the excess supply function by its linear tangent at the current estimate of total expected supply The improved estimate of total expected supply is calculated in equation (18b), it is the value that makes the linear approximation equal to zero Because carryin is given or fixed, the improved estimate of expected production equals the improved estimate of total expected supply minus

[7] It is convenient to make $W_{i,t}$ the midpoint of interval i and to make all the intervals of equal width

[8] The derivative of the expected excess supply function with respect to total expected excess supply equals $\frac{\alpha}{\beta}\left(\frac{\beta+\alpha}{\beta(r+1)+\alpha}\text{PROB} - 1\right) - 1$ The variable PROB is the probability that carryover for the current year is greater than zero and equals $\sum_{i=1}^{Z} W_{i,t}$ (where $W_{z,t}$ represents the smallest value of $W_{i,t}$ resulting in carryover greater than zero in year t) The values of $W_{i,t}$ in this summation decline in succession from the largest value to the smallest value associated with carryover greater than zero PROB increased with increasing sums of carryin and expected production

carryin as shown in equation (18c) The improved estimate of expected production is then used in equation (13)

Ippolito's algorithm proceeds by using improved estimates of expected production in equation (13) to calculate optimal carryover levels and by using the results of equation (13) to calculate improved estimates of expected production This iterative procedure is continued until the absolute value of the expected excess supply in expression (17) is less than the small positive value, ϵ At this point, the optimal carryover decision and the rational production decision are consistent for the given level of carryin Using this procedure, one calculates the rational production decision and the optimal carryover decision for each level of carryin in the current year

The current year's results from the algorithm include the current year's expected price for each level of carryin From the viewpoint of the previous year, next year's expected price is known for each level of carryover This information is used to solve equation (13) and expression (14) for carryover in the previous year Also, one finds improved estimates of expected production by iterating the results of equation (13) for a given level of carryin until the expected production is the rational production level—that is, until the absolute value of excess supply is less than the small positive value, ϵ

Additional years are considered until the optimal carryover converges to a particular value for each sum of carryin, expected production, and supply level minus demand level ($C_{n,t-1}$ + $E(PROD_t)$ + $W_{i,t}$) and until the rational production level converges to a particular level for each level of carryin At this point the influence of the zero carryover restriction in the last year, year T, has been completely dissipated

For an example of the use of Ippolito's algorithm, readers may check our paper (*4*) We used the algorithm to find optimal storage rules and rational production responses for the U S soybean market The efficiency of Ippolito's algorithm enabled us to find optimal storage rules and production responses under the additional constraint of a price band supported by buffer stocks We also performed stochastic simulations of several price-buffering grain storage policies, and we analyzed the resulting distributions of prices and production

Conclusions

The two dynamic programming algorithms involving rational producer response that Gardner and Ippolito developed are the most advanced in the optimal grain storage literature Both are logical extensions of Gustafson's original dynamic programming algorithms which are the foundation for this literature When these algorithms are described with the same notation, the close relationships among them become clear

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