Recent Developments in Applying Duality Theory

Denis Lawrence*

1. Introduction

An approach to measuring production response which has proven to be popular is the use of applied duality theory. For instance, the use of a profit function framework enables the modelling of producers' responses to market conditions given only the constraint of certain fixed inputs each time period. Given the level of fixed inputs available and prevailing prices for outputs and variable inputs, producers choose the quantities of outputs and variable inputs so as to maximise their profit each period. This framework, while being tightly defined theoretically, imposes few *a priori* restrictions on production relationships and is particularly suitable for modelling Australian agriculture where there are numerous producers all making production decisions on the basis of output and input prices over which they, as individuals, have no control. This formed the basis of the McKay, Lawrence and Vlastuin (1982, 1983) papers (hereafter MLV) where translog profit functions are used to model production relationships in the sheep industry.

Until now the usefulness of applied duality theory models has been limited, however, by the inability to impose curvature conditions required by economic theory without loss of flexibility properties when these conditions are not satisfied by the estimated models. More importantly, econometric techniques have also limited the degree of disaggregation which can be incorporated within these models. Thus, studies such as those of MLV have been criticised for being too aggregative in nature compared to the greater degree of commodity disaggregation possible in general equilibrium models such as ORANI (Dixon, Parmenter, Sutton and Vincent 1982). While large scale general equilibrium models are capable of producing more detailed results than many smaller scale models, they are typically based on relatively simplistic functional forms to enable their implementation, and many important elasticities are usually assumed rather than estimated within the models.

The purpose of this paper is to briefly outline some recent developments in applied duality techniques which enable imposition of the correct curvature conditions with minimal loss of flexibility properties and the incorporation of a much finer level of commodity disaggregation than previously possible. These developments make it possible to use the theoretically well defined duality models in a wider range of applications. To this end, the following section of this paper outlines the recently developed Generalised McFadden functional forms which enable curvature imposition, while the third section looks at the use of aggregator and semi-flexible functional forms which enable a finer degree of commodity disaggregation. Finally, the paper closes on a cautionary note emphasising the sensitivity of elasticity estimates to the data and estimation techniques used. No one set of estimates should be considered "correct". Rather, a range of estimates should always be used when examining the effects of policy changes and external shocks.

2. Curvature Conditions

Denoting the N variable net output quantites by the vector x (entries positive for outputs, negative for inputs), net output prices by the vector $p >> 0$, the M fixed input quantities by the vector z, fixed

* University of British Columbia

input shadow prices by the vector w and the production possibility set by T, the production technology can be represented by the following restricted profit function:

$$(1) \quad \pi(p;z) = \max_{x}. \{ p'x : (z;x) \text{ belongs to } T, \; p >> 0 \}$$

The restricted profit function (1) will be linearly homogeneous and convex in net output prices and monotonically increasing (decreasing) in the prices of variable outputs (inputs). It will be linearly homogeneous, concave and monotonically increasing in fixed input quantities. The properties of restricted profit functions are discussed in detail in Diewert (1974).

If the restricted profit function is differentiable with respect to p then the net output supply functions can be derived by applying Hotelling's (1932) Lemma:

$$(2) \quad x(p,z) = \nabla_p \pi(p;z)$$

Furthermore, if the restricted profit function is differentiable with respect to the fixed input quantities, z, then the inverse demand functions for the fixed inputs may be obtained by:

$$(3) \quad w(p,z) = \nabla_z \pi(p;z)$$

To simplify later presentation, capital (K) is assumed to be the only fixed input. Constant returns to scale are also assumed with respect to the capital stock. The restricted profit function (1) can then be represented by a unit profit function which represents the maximum amount of revenue a firm can produce from one unit of capital.

To implement the model empirically a functional form for the restricted profit function must be specified and estimation of the system of derived net output supply functions undertaken. The characteristics of the production technology and net output responses are obtained from the calculation of various elasticities derived from the estimated profit function.

Desirable characteristics of a functional form for the restricted profit function are that it be flexible (able to provide a second-order approximation to an arbitrary twice continuously differentiable profit function), parsimonious (have the minimal number of free parameters required for flexibility), and consistent with the required theoretical properties of a profit function. While the translog and Generalised Leontief forms have become popular because of their flexibility and relative ease of implementation, they often suffer in empirical applications from failure to satisfy the required curvature properties at all (or any) of the observation points. While the MLV studies failed to satisfy the price convexity property, all the estimated own net supply elasticities had the correct sign. In most cases of convexity failure, however, at least one of the own price net supply elasticities will have the wrong sign and the whole set of estimates will be rendered worthless.

Early attempts to overcome this problem concentrated on imposing curvature on the translog functional form. Jorgenson and Fraumeni (1981) showed that, in the case of the translog cost function, ensuring that the matrix of second-order coefficients was negative semi-definite was sufficient to ensure global concavity in prices provided that the estimated input shares were non-negative. However, Diewert and Wales (1987) pointed out that this procedure can introduce large biases in the estimated elasticities and, hence, destroys the constrained translog's flexibility. In the case where the true elasticities are close to zero, application of the Jorgenson and Fraumeni procedure will result in an upward bias of estimated input substitutability.

In response to this problem, recent developments in functional forms have led to the development of functions which are flexible and easily verified as satisfying curvature conditions globally. If the curvature conditions are not satisfied they can be imposed with minimal cost to flexibility properties, although non-linear regression techniques then have to be used.

The functional form for the unit profit function outlined here is the Symmetric Generalised McFadden (SGM) function of Diewert and Wales (1987). The N-variable net output SGM unit profit function is given by:

$$(4) \qquad \pi(p,K)/K = (1/2) \sum_{i=1}^{N} \sum_{j=1}^{N} s_{ij}\, p_i\, p_j \,/\, (\sum_{k=1}^{N} T_k\, p_k) + \sum_{i=1}^{N} b_{ii}\, p_i + \sum_{i=1}^{N} b_{it}\, p_i t + b_{tt} (\sum_{i=1}^{N} C_i\, p_i)\, t^2$$

where time superscripts have been deleted and the s_{ij}, b_{ii}, b_{it} and b_{tt} are parameters to be estimated subject to:

$$(5) \qquad s_{ij} = s_{ji} \qquad \textit{for all } i,j; \textit{ and}$$

$$(6) \qquad \sum_{i=1}^{N} s_{ij} = 0 \qquad \textit{for } j = 1,\ldots,N.$$

The variable t is an index representing technical progress and the exogenous parameters T_k and C_i are set equal to the average net output quantity per unit of capital input quantity for k, i = 1,..,N.

Diewert and Wales (1987) showed that the SGM form is flexible for a price vector $\tilde{p}$ satisfying $S\tilde{p} = 0_N$. While the non-symmetric Generalised McFadden function also put forward by Diewert and Wales has superior flexibility properties in that it is not restricted to being flexible at just one point, the results obtained are sensitive to the choice of the numeraire good which plays an asymmetric role. This sensitivity is eliminated by use of the SGM form.

Differentiating the unit profit function (4) with respect to the net output prices yields unit net output supply functions of the form:

$$(7) \qquad x_i/K = \sum_{j=1}^{N} s_{ij}\, p_j \,/\, (\sum_{k=1}^{N} T_k\, P_k) - T_i\, (\sum_{k=1}^{N} \sum_{j=1}^{N} s_{kj}\, p_k\, p_j) \,/\, [2\, (\sum_{k=1}^{N} T_k\, p_k)^2] + b_{ii} + b_{it} t + b_{tt} C_i t^2 + u_i; \qquad i = 1,\ldots,N$$

The variable net output quantity is divided by the quantity of the capital input to reduce heteroskedasticity problems and an error term is appended to each equation. The vectors of error terms for the observations are assumed to be independently distributed with a multivariate normal distribution with zero means and covariance matrix Ω.

The estimating system thus consists of (7) subject to the restrictions (5) and (6). The profit function (4) is not included in the estimating system as it adds no new information. Maximum likelihood estimates of the system of equations (7) can be obtained by using the iterative Zellner technique. If the matrix of estimated coefficients S is positive semi-definite then the restricted profit function can be shown to be globally convex in prices p.

If the estimated S matrix is not positive semi-definite then it can be reparameterised using a

technique due to Wiley, Schmidt and Bramble (1973) to ensure global convexity without any loss of flexibility properties (Diewert and Wales 1987, p. 53). This technique replaces the matrix $S=[s_{ij}]$ by the product of a lower triangular matrix and its transpose:

$$(8) \qquad S = AA' \text{ where } A = [a_{ij}]; \quad i,j = 1,\ldots,N; \quad \text{and } a_{ij} = 0 \text{ for } i < j.$$

Using a result due to Lau (1978), Diewert and Wales show that this is a general way of imposing positive semidefiniteness. Using this procedure the coefficients in the first three rows and columns of S would become:

$$(9) \qquad [s_{ij}] = \begin{bmatrix} a_{11}^2 & a_{11}a_{21} & a_{11}a_{31} \\ a_{11}a_{21} & a_{21}^2 + a_{22}^2 & a_{21}a_{31} + a_{32}a_{22} \\ a_{11}a_{31} & a_{21}a_{31} + a_{32}a_{22} & a_{31}^2 + a_{32}^2 + a_{33}^2 \end{bmatrix} \quad ; i,j = 1,2,3.$$

The Nth row and column of S are obtained from the summing restrictions (6). The reparameterised system imposing curvature can be estimated by using non-linear regression techniques such as the non-linear algorithm in the SHAZAM package (White 1978).

The Generalised McFadden restricted profit function is described in more detail in Diewert (1985) and applications can be found in the context of models of international trade elasticities in Lawrence (1987). Diewert and Wales (1987) present a comparison of the performance of different cost functions on a common data set. Use of the SGM functional form in either the profit function context or the more specific cost function case, which concentrates on input use relationships subject to a fixed output quantity (analogous to the earlier MLV (1980) work), would give estimates of production relationships in Australian agriculture from a wider range of configurations and data sources than previously possible, given past problems with the failure to satisfy curvature requirements.

3. Commodity Disaggregation

With the use of flexible functional forms, the explicit incorporation of many inputs and outputs in econometric studies rapidly exhausts the available degrees of freedom and creates significant multicollinearity problems which are difficult, if not impossible, to overcome. The increased computational burden is also an important consideration. Aggregation of input and output components is thus a necessary part of any empirical study to ensure tractability but the cost of this procedure is usually a loss of information. One solution to this problem is the use of aggregator functions as proposed by Fuss (1977) to accommodate many output and input components. While not new, use of this procedure to date has been hampered by the inability to impose curvature conditions on flexible functional forms at the various stages of the estimation procedure. The development of the SGM functional form has now made this procedure more tractable.

The theoretical basis for the use of aggregator functions comes from implying certain properties for the functional structure. This is the condition of homogeneous weak separability which assumes the profit function can be written as:

$$(10) \qquad \pi(p,z) = \pi^{*}(R,V)$$

where $R = (R_1,...,R_n,..)$, $V = (V_1,...,V_m,..)$, $R_n = R_n(p_n)$, $V_m = V_m(z_m)$ and p_n, z_n belong to p,z, respectively. $R_n(p_n)$ is a price index for the goods in group n while $V_m(z_m)$ is a quantity index for the fixed inputs in group m. The corresponding transformation function is:

$$(11) \qquad T(x,z) = T\sim(Y,V) = 0$$

where $Y = (Y_1,...,Y_n,..)$ and $Y_n(x_n)$ is the corresponding quantity index which is assumed to be linearly homogeneous. We then have:

$$(12) \qquad \max_{x_n} \{ p_n x_n : Y_n(x_n) = Y_n \}$$

$$= Y_n \max_{x_n/Y_n} \{ p_n x_n / Y_n : Y_n(x_n/Y_n) = 1 \} = Y_n R_n(p_n)$$

where $R_n(p_n)$ is a revenue or aggregator function. Thus:

$$(13) \qquad \pi(p,z) = \max. \{ \Sigma_n p_n x_n : T\sim(Y_1(x_n),\ldots,V) = 0 \}$$

$$= \max. \{ \Sigma_n R_n(p_n) Y_n : T\sim(Y,V) = 0 \}$$

$$= \pi\sim(R,V)$$

which is a valid profit function in the aggregates to which Hotelling's Lemma and the standard profit function properties can be applied (Woodland 1982, p.368).

The important implication of weak separability is that optimisation proceeds by a two-stage process. First, the optimal quantity of the aggregate is chosen and then the optimal mix of that aggregate quantity is chosen. The marginal rate of substitution between two components of one aggregate is independent of the quantities of the other aggregates. Thus, the mix of that aggregate is independent of both the level and the mix of the other aggregates. It is this aspect of weak separability which forms the basis of the use of aggregator functions. Thus, in the case of agriculture, it would be possible to work with a number of aggregated categories such as those used by MLV at the profit function level. If more information was desired on some of the components making up these aggregates (say different types of labour input or sheep product outputs), then the relevant aggregates could be further disaggregated by the use of aggregator functions.

The use of aggregator functions permits the use of flexible functional forms for the profit function at the aggregate level along with flexible aggregator functions. The response at the most disaggregated level can be obtained by;

$$(14) \qquad x_n = \partial\pi/\partial p_n = \partial\pi\sim/\partial R_n . \partial R_n/\partial p_n$$

Making use of the assumption that the $Y_n(x_n)$ functions are linearly homogeneous, the following Symmetric Generalised McFadden unit revenue functions could be used for the aggregate commodity on which more detailed information is required:

$$(15) \qquad R(p,X)/X = (1/2) \sum_{i=1}^{D} \sum_{j=1}^{D} s_{ij} p_i p_j / (\sum_{k=1}^{D} T_k p_k)$$

$$+ \sum_{i=1}^{D} b_{ii} p_i + \sum_{i=1}^{D} b_{it} p_i t + b_{tt} (\sum_{i=1}^{D} C_i p_i) t^2$$

where time superscripts have again been deleted, X represents the relevant aggregate commodity which has D components and the s_{ij}, b_{ii}, b_{it} and b_{tt} are parameters to be estimated subject to:

$$(16) \qquad s_{ij} = s_{ji} \qquad \textit{for all } i,j; \textit{ and}$$

$$(17) \qquad \sum_{i=1}^{D} s_{ij} = 0 \qquad \textit{for } j = 1,\ldots,D$$

The variable t is an index representing technical progress and the exogenous parameters T_k and C_i are set equal to the average X component quantity per unit of total X quantity for k, i = 1,..,D. Profit maximising behaviour implies that the X component quantities per unit of total X are given by:

$$(18) \qquad x_i / X = \sum_{j=1}^{D} s_{ij} p_j / (\sum_{k=1}^{D} T_k p_k) - T_i (\sum_{k=1}^{D} \sum_{j=1}^{D} s_{kj} p_k p_j) / [2 (\sum_{k=1}^{D} T_k p_k)^2]$$
$$+ b_{ii} + b_{it} t + b_{tt} C_i t^2 + u_i ; \qquad i = 1,\ldots,D$$

The quantity of total X would be derived as a Divisia index of the corresponding X component quantities. Convexity in prices can be imposed on the aggregator functions by reparameterising the S matrix along the same lines as (8) and (9). The vectors of error terms can again be assumed to be independently distributed with a multivariate normal distribution with zero means and covariance matrix Ω.

By estimating the system (18) and substituting the estimated parameters in (15), an estimate of the aggregate unit price would be obtained. A property of the two-stage optimisation procedure is that, although the prices of the individual components of the aggregate are exogenous, the price of the aggregate itself is not exogenous because the choice of input and output mix will determine the aggregate price. Thus, to implement the procedure empirically, an instrumental variable for the aggregate price is required. Fuss proposed the use of the estimated price of the aggregate obtained by substituting the parameters estimated in equations similar to (18) into the aggregator function. This is used as an instrumental variable for the aggregate price in the second stage of the estimation process.

The estimation procedure would be, thus, to first estimate the unit quantity equations (18) for the X components. Next, the parameter estimates obtained in the first stage are substituted in (15) to obtain instrumental variables for the aggregate X prices. The second stage of the estimation procedure would be the estimation of the net output supply equations (7) derived from the SGM restricted profit function using the instrumental variables for aggregate prices. Application of this conditional estimation procedure produces estimates which are full information maximum likelihood (Fuss 1977).

Two sets of elasticities are obtained for the individual components of X. In the case of cross-price elasticities between X components, for instance, from the first stage of estimation (equation (18)) cross-price supply elasticities are obtained given a fixed level of aggregate X. By extending equation (14) cross-price supply elasticities between X components, i and j are obtained subject to the constant fixed capital input quantity as follows:

$$(19) \qquad E_{ij}^{K} = E_{ij}^{X} + s_j E_{xx}^{K}$$

where E_{ij}^{X} is the cross-price elasticity between i and j given a constant level of aggregate X, s_j is the share of component j in total X and E_{XX}^{K} is the own-price elasticity of aggregate X for a given fixed capital input level. By extending (19) to the components of the other aggregates which have been disaggregated, price elasticities for all the components are obtained which are directly comparable with the price elasticities for the other net output categories obtained from the second stage of estimation.

Use of this aggregator function procedure combined with the recently developed functional forms such as the SGM, which allow curvature imposition while retaining flexibility, now provides a more general way of incorporating many output and input categories in small scale econometric studies. Output from such studies may provide improved input into larger scale general equilibrium models and also a check on the results obtained from the more elaborate general equilibrium models.

One criticism of the aggregator function procedure, however, is that it relies on acceptance of the property of weak separability. An alternative procedure for incorporating many outputs and inputs which does not rely on this assumption which interested readers may wish to pursue is the use of semi-flexible functional forms as proposed by Diewert and Wales (1986). This approach basically reduces the size of the two triangular matrices which are multiplied together in equation (8) to form the coefficient matrix in the SGM function. This reduces the total number of parameters to be estimated hence allowing more output and input components to be included but at the expense of achieving less than full flexibility.

4. Conclusions

This paper has outlined some recent developments in the field of applied duality modelling. These developments overcome some of the criticisms which have been levelled at small scale econometric production models making use of duality theory. Specifically, use of functional forms such as the Symmetric Generalised McFadden enables the correct curvature conditions to be imposed without loss of flexibility properties and use of these forms in conjunction with the aggregator function procedure make the incorporation of many output and input categories possible. Use of these techniques in the context of modelling production response in Australian agriculture will make the theoretically rigorous duality models a more attractive alternative.

However, when modelling production response the important point to consider is that no one set of estimates, be they derived from econometric or general equilibrium models, can be considered to be "correct". The estimates derived from all models are far too sensitive to the assumptions made, the data set used and the specification adopted to be the sole basis for predicting the effects of policy changes and external shocks. Rather, a range of elasticity estimates should be considered and the effects of the sensitivity analysis on the results stressed. While econometric and general equilibrium models are continually becoming more sophisticated, there remain many factors which influence economic decisions which models do not fully capture. Caution, thus, remains the key word when it comes to interpreting and comparing the results of alternative production response models.

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