SOME NOTES ON "DYNAMIC" LINEAR PROGRAMMING

C. D. THROSBY*

Economics Research Officer

SUMMARY

The most common application of linear programming in agricultural situations has been to the problem of resource allocation between competing farm activities. Given relevant input-output information for a specific farm, together with real or assumed price and cost patterns, the technique of linear programming enables calculation of the combination of enterprises which maximizes net profit, within the limitations imposed by the availability of farm resources.

It is necessary in some linear programming analyses to make explicit allowance for the peculiar influence of time on the structure of the system under study. Of the many ways in which this may be achieved, this article considers four, which have been, or are likely to be, of relevance in an agricultural context:

(i) Parametric programming, which allows consideration of resource or price variation between time periods;

(ii) extension of the time-span of an activity to cover a series of sequential processes, for example the treatment of rotational sequences as single activities;

(iii) the referencing of some resources and/or activities to specific time periods; a common example is the fragmentation of labour supply into months; and

(iv) the so-called "multi-stage" or "dynamic" linear programming where a single matrix is used to describe, in an orderly fashion, a system's structure over a time-span of several periods.

It is the latter with which we are primarily concerned here. In its simplest form a dynamic linear programming problem may be set up as a large matrix composed of a series of smaller matrices lying down the diagonal. In its more advanced form allowance can be made for interactions between resources and activities in different periods. In general, dynamic linear programming problems are characterized by large "sparse" matrices (i.e., matrices in which many coefficients are zero) and usually a "block diagonal" or "block triangular" pattern is evident. The size of such matrices is frequently forbidding; however, computational algorithms are available which allow overall solutions to be obtained by solving a series of smaller problems.

With the aid of a little ingenuity a great variety of time-dependent restrictions, resources, activities and opportunities can be accounted for in a dynamic linear programming analysis. From an agricultural economist's viewpoint it would not seem extravagant to claim that dynamic linear programming can be used to provide a more adequate analytical description of whole-farm situations over time than most other tools at present available in his kit.

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An example of the use of dynamic linear programming to study resource allocation on a farm over a four-year period is presented in the body of the article to illustrate some of the points made. The data are taken from an actual farm in Central-western New South Wales.

INTRODUCTION: GENERAL STATEMENT OF THE LINEAR PROGRAMMING PROBLEM

The way in which the linear programming problem is formulated and solved is now fairly widely known, but for the sake of completeness, and to standardize notation, a brief summary of the way in which it is applied to the study of farm situations is presented below.

The problem of linear programming is that of maximizing[1] a function subject to constraints where both the function to be maximized (the objective function), and the set of restrictions are linear. Stated mathematically, linear programming is used to determine a vector X composed of a series of values x_j which maximizes:

$$z = \sum_{j=1}^{n} c_j x_j \qquad (1)$$

subject to

$$b_i \geqq \sum_{j=1}^{n} a_{ij} x_j \qquad (i = 1, 2, \ldots, m) \qquad (2)$$

and

$$x_j \geqq 0 \qquad (j = 1, 2, \ldots, n) \qquad (3)$$

where the matrix A of coefficients a_{ij}, and the vectors B of components b_i and C of components c_j are given.

The most frequent application of this technique to agricultural situations has been to problems of maximizing the net profit of a farm firm engaged in several lines of production, and subject to limitations on the supply of resources used in the production processes. In such a context the above formal statement of the linear programming problem can be interpreted in its most general form as follows:—

(i) Suppose that the j-th activity on a farm is undertaken at a level of x_j units (acres of crop, head of cattle, etc.) and that the net returns from undertaking *one* unit of the j-th activity can be represented as c_j. The net profit accruing from the j-th activity is thus $c_j x_j$ and the *total* profit (z) over *all* the n activities possible on the farm can be represented as:

$$z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \qquad (4)$$

This is the objective function.

(ii) Suppose that the i-th resource available on the farm is available at a level of b_i units (£ of capital, acres of land, etc.) and that undertaking *one* unit of the j-th activity uses up a quantity a_{ij} of the resource, then the amount of that resource used up by operating the j-th activity at a level x_j is $a_{ij}x_j$ and the *total* amount of the i-th resource used up over all n activities is

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \qquad (5)$$

[1] The reader should remember that the technique of linear programming is as readily adaptable to minimization as to maximization of the objective function.

Since the total amount of the i-th resource to be used up cannot exceed the amount b_i which is available, we can write the expression shown in (5) as follows (at the same time putting a separate line for each resource available):

$$\begin{aligned} b_1 &\geqq a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ b_2 &\geqq a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ &\vdots \\ b_m &\geqq a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{aligned} \quad (6)$$

(iii) Suppose no enterprise can be undertaken at a negative level, although zero levels are quite feasible, *i.e.*

$$x_j \geqq 0 \quad (7)$$

(iv) Then, linear programming provides a means of finding levels $x_1, x_2, \ldots, x_n$ (*i.e.* a vector X) at which the various activities should be undertaken in order to maximize z in equation (4) subject to the limitations imposed by resource availabilities expressed in the equations of (6) and to the non-negativity requirement of equation (7).

A discussion of the mechanical means of solving linear programming problems is not relevant here. Suffice it to say that the most effective method is the simplex technique, an iterative process which commences with a feasible solution (set of values of x_j) then moves to alternative solutions which successively increase the value of the objective function within the area enclosed by the constraints, until a maximum is found in a finite number of iterations.

The above brief discussions should provide the reader not already acquainted with linear programming with sufficient basis for understanding the points made in the following pages, and we may proceed now to a consideration of the role played by time in linear programming analyses.

THE TIME ELEMENT IN LINEAR PROGRAMMING

In many applications of linear programming to real or hypothetical farm situations, time is not taken into account as a specific variable factor. In other words, the programme is usually formulated in terms of only one period, often a year. It is assumed that resources are available and are used, activities are undertaken, products are sold and monetary returns are received all within the span of one time period. Obviously this assumption is quite justifiable in many cases: products such as fat lambs, pigs, grains etc., whose production cycle takes a year or less are handled adequately by this approach.

A "static" or single-period analysis is rather unrealistic, however, for activities whose influence on the economic structure of a farm extends beyond one time period. A good example of such an activity is pasture improvement, one of whose essential features from an economic viewpoint is its time-dependent nature. An acre of improved pasture sown in year 1 will return varying amounts and will have varying resource requirements in years 1, 2, 3, . . . Further, the effects of a programme of pasture improvement on other farm activities and on the overall economic structure of a farm change drastically from year to year as adoption of the programme progresses. Another factor militating against single-period analyses is the notorious inconstancy of the exogenous economic climate: yearly fluctuations in product prices and factor costs must often be allowed for in empirical linear programming investigations if worthwhile results are to be obtained.

It is obvious then, that in many linear programming studies time should be included explicitly as a variable of great importance. In the following paragraphs we examine four ways in which this may be achieved. They are all similar in that they are all merely conceptual extensions of the general linear programming problem formulated above.

(i) Parametric Programming

The two most successful ways in which parametric programming has been applied to agricultural situations are "variable price programming" and "variable resource programming".

Variable price programming[2] enables a series of prices (net returns) to be placed on the activities in a programme, and hence optimal adjustments to an initial solution may be calculated for any real or assumed pattern of price changes between time periods. This application of parametric programming has been useful in two ways: (*a*) to compute the optimum sequential course of action for a firm operating under a constant input-output regime, but subject to a known or assumed sequence of price changes; and (*b*) to test the sensitivity of a given programming solution to changes in product prices.

Variable resource programming,[3] where adjustments to optimal plans are calculated for variations in the supply of a resource such as capital, is also a useful dynamic tool. One method of computing such a programme is to convert the problem to its *dual* formulation, whereby the resource supply column becomes a parametric objective function enabling solution by the techniques of variable price programming. Candler[4] has developed a modification of the simplex method for handling variable capital restrictions.

Investigations have suggested that " parameterization " of the *whole* of a linear programming problem (viz. of a_{ij}, b_i, c_j) is quite feasible but tends to become unwieldy from a computational viewpoint[5].

Parametric programming should prove useful as a dynamic weapon in two distinct time contexts. Firstly, it could be used to calculate optimal resource adjustments to *short-period* price fluctuations for continuously operating

[2]See E. O. Heady and W. Candler, *Linear Programming Methods* (Ames: Iowa State College Press, 1958), Chapter 8; also W. Candler, "A modified simplex solution for linear programming with variable prices", *Journal of Farm Economics,* Vol. 39, No. 2 (May, 1957), pp. 409-428.

[3]See Heady and Candler, *op. cit.,* Chapter 7.

[4]W. Candler, "A modified simplex solution for linear programming with variable capital restrictions", *Journal of Farm Economics,* Vol. 38, No. 4 (November, 1956), pp. 940-55.

[5]See, for example, T. L. Saaty, "Coefficient Perturbation of a Constrained Extremum", *Operations Research,* Vol. 7, No. 3 (May-June, 1959), p. 294.

enterprises. For example, if a monthly price pattern for products such as eggs or broiler chickens were known, variable price programming could be used with a basic input-output matrix to determine optimal resource adjustments for a farm between months. Secondly, taking a longer view, there would seem to be some potential for programming with a variable capital restriction in planning farm activities over a series of years. This presupposes that the input-output matrix and prices are unchanging from year to year. Whilst in long term analyses this *ceteris paribus* condition may become rather strained, it could well be that for *ex ante* programming the assumption that history will repeat itself is the best available.

(ii) COALITION OF ACTIVITIES

It is possible in a linear programming analysis to combine several activities undertaken in successive time periods into one activity. It is essential if this procedure is used that the component activities bear some fixed relationship with each other such that it is quite feasible for one unit of one activity undertaken in one period to necessitate the adoption of the same level of each of the other activities in the other periods. In an agricultural context this technique has been most useful in allowing consideration of rotational farming systems; the several land uses which go to make up the rotation sequence are subsumed under one activity.[6]

(iii) DATING OF SOME RESOURCES AND/OR ACTIVITIES

It is quite a common practice in general economic analysis to consider a commodity used or produced in different time periods as a series of different commodities. So too in linear programming can a resource which has different utilities or availabilities in different periods, or a product which can be produced at different times, be subdivided into more than one item for inclusion in the programming matrix. In some respects this is precisely the reverse of the amalagamation of time periods considered in the previous section. A common example of the fragmentation of a farm resource into smaller time units is the "dating" of the labour input by months to allow for varying seasonal supplies and demands.[7]

Dating of some activities may also follow a similar pattern; this is particularly relevant when time-of-market considerations might influence a farmer. For example, the optimum monthly sequence of production of fat lambs or vealers could be calculated by this means, if the pattern of prices prevailing over the year were known from past experience. Instead of just one column in the programming matrix for "fat lambs" or "vealers" we would have one for each month at which the animals might be sent to market: "lambs for January sale", "vealers for February sale", etc.

Pursuing this process to the point of dating resources *and* activities in the one matrix leads us directly into the next section.

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[6]An example of such an analysis may be found in E. O. Heady, R. Alexander and W. Shrader, *Combinations of Rotations and Fertilization to Maximize Crop Profits on Farms in North-Central Iowa.* (Iowa A.E.S., Res. Bull. No. 439, April, 1956.)

[7]An example which illustrates this is to be found in G. J. Tyler, "An Application of Linear Programming", *Journal of Agricultural Economics,* Vol. 13, No. 4 (January, 1960), pp. 473-86; see also E. R. Swanson, "Integrating Crop and Livestock Activities in Farm Management Activity Analysis", *Journal of Farm Economics*, Vol. 37, No. 5 (December, 1955), pp. 1249-58. In his analysis, Swanson dates both resources and activities.

(iv) Dynamic Linear Programming

The most versatile means of handling linear programming problems whose scope embraces more than one distinct time period is the general formulation known variously as "dynamic linear programming", "multi-stage linear programming", and "inter-temporal linear programming". It is wrong and misleading to refer to it as "dynamic programming".[8] Essentially any linear programming problem where *all* resources and *all* activities bear a time subscript[9] can be classed under the heading of dynamic linear programming. (This is neither a necessary nor a sufficient definition but is satisfactory as an explanatory basis.)

The simplest case of multi-period linear programming is that in which activities and resources are dated but are not made to bear any inter-temporal relationships with one another. Now, whilst the final solution to the matrix corresponding to this type of problem is independent of the geographical arrangement of the vectors within the matrix, there are conceptual advantages in observing the following convention for grouping the rows and columns, especially if the matrix is very large. All activities (columns) bearing a time subscript referring to period 1 should be grouped at the left hand side of the matrix, then proceeding in a rightwards direction should come period 2 columns, then period 3, and so on. Similarly period 1 resources or restrictions (rows) should be grouped at the top of the matrix followed in a downwards direction by period 2, period 3, etc.

For the simple no-intertemporal-relationships case already mentioned, it will be seen that such an arrangement of rows and columns leads to a matrix composed entirely of zeros except for a series of submatrices lying down the diagonal. If the problem under consideration concerns a firm operating under an input-output regime which is invariant from period to period, then these submatrices are identical. If, on the other hand, prices, resource supplies and input-output coefficients are assumed to vary from period to period, our original problem may be represented (for a four-period analysis) as:

$$\begin{array}{lllll} B_1 & \geqq & A_1X_1 & & & \\ B_2 & \geqq & & A_2X_2 & & \\ B_3 & \geqq & & & A_3X_3 & \\ B_4 & \geqq & & & & A_4X_4 \end{array} \tag{8}$$

with the objective function

$$z = C_1X_1 + C_2X_2 + C_3X_3 + C_4X_4 \tag{9}$$

where B, A, X and C have the same connotation as in equations (1), (2) and (3), except that now they are referenced throughout to a specific time period k. All elements on the right-hand side of equation (8), other than the block-diagonal elements, are zero.

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[8]Dynamic programming is a distinct discipline for studying multi-stage decision processes which was originally formalized by Bellman early in the 1950's. The basic text on this subject is Richard Bellman, *Dynamic Programming* (Princeton: Princeton University Press), 1957. The relationship between linear and dynamic programming is briefly mentioned at a later stage in the present article.

[9]i.e., Where every element of the matrix bears not only subscripts i and j but also a subscript $k = 1, 2, 3 \ldots$, which serves to reference the element to a specific period in time.

The solution to the large matrix in equation (8) can be deduced from the solutions to the diagonal submatrices.

From the viewpoint of dynamic analysis the problem contained in equations (8) and (9) is of little value since there is no intertemporal connection other than the fact that total profit is the sum of the individual profits earned in each discrete time period. We pass on, then, to a consideration of the more interesting case where resources and activities can bear relationships with each other between time periods. A wide range of problems can be treated by this method of formulation.[10]

It can be seen that increasing the number of stages in a dynamic linear programming problem will increase the matrix size, and hence the computational burden at a very rapid rate. A number of methods by which large matrices can be broken down so that a series of solutions to smaller problems can be used to yield an *overall* solution, have been proposed.[11] Variations of the simplex procedure which can solve large matrices intact in fewer iterations than the normal simplex would take, have also been put forward.[12] Dantzig and Wolfe [13] have recently devised a method for large matrix solution which involves solving a series of linear subprogrammes and a co-ordinating programme. If the only relationship between periods is a single vector (or, at the most, two or three vectors) at each stage, dynamic or parametric programming methods may be employed. The former technique is applicable when there is a column vector shared between one stage and the next: for example the closing inventory of stage k might be required as the opening inventory of stage $k + 1$. Morton[14] has examined a problem of this general nature with reference to a forestry production model. Parametric programming can be used, for example, when activities in period k generate a supply of a certain resource available for use in period $k + 1$. An illustration is the generation of money (profit) by activities in one period which is transferable to the capital supply of the next period. In this guise the problem is merely a special case of the variable resource programming mentioned earlier, as has been pointed out by Candler[15] in connection with a model derived by Loftsgard and Heady.[16]

We proceed finally to a consideration of the type of dynamic linear programming formulation which is likely to have greatest relevance to whole-farm planning over time. Using the same notation as previously, this problem may be described for a four period analysis as follows:

Maximize:

$$z = C_1X_1 + C_2X_2 + C_3X_3 + C_4X_4 \qquad (10)$$

[10] A detailed treatment of different multi-stage linear programming systems can be found in G. B. Dantzig, *On the Status of Multistage Linear Programming Problems* (The RAND Corporation), Paper no. P-1028, February 20, 1957.

[11] Dantzig, *op. cit., passim,* but especially p. 19ff.

[12] *Ibid.,* pp. 26-7.

[13] G. B. Dantzig and P. Wolfe, "Decomposition Principles for Linear Programs", *Operations Research,* Vol. 8, No. 1 (Jan.-Feb. 1960) pp. 101-111.

[14] G. Morton, "An Application of Dynamic Programming", *Proc. Conference on Linear Programming,* May 1954, (Ferranti Ltd., *mimeo,* 1954) pp. 22-30. See also Dantzig, *op. cit.,* pp. 5-8.

[15] W. Candler, "Reflections on 'Dynamic Programming Models'," *Journal of Farm Economics,* Vol. 42, No. 4 (November, 1960), pp. 920-26.

[16] L. D. Loftsgard and E. O. Heady, "Application of Dynamic Programming Models for Farm and Home Plans", *ibid.,* Vol. 41, No. 1 (February, 1959), pages 51-62. Note that the authors of this paper err in their use of the term "dynamic programming".

subject to

$$\begin{aligned} B_1 &\geqq A_{11}X_1 \\ B_2 &\geqq A_{12}X_1 + A_{22}X_2 \\ B_3 &\geqq A_{13}X_1 + A_{23}X_2 + A_{33}X_3 \\ B_4 &\geqq A_{14}X_1 + A_{24}X_2 + A_{34}X_3 + A_{44}X_4 \end{aligned} \qquad (11)$$

The two subscripts to the submatrices A_{kk} reference it to the time periods of the activity block and to the time period of the resource supply block respectively. Thus A_{23}, for example, describes the interactions between activities undertaken in period 2 with resources available in period 3. The matrix elements to the right and above the diagonal submatrices are all zeros. The coefficient matrix is thus block triangular; for large problems this characteristic can allow short-cut procedures to be used in computation.

We shall illustrate this system with an example, using data collected during a survey conducted by the author during 1960-61. Unless otherwise stated, the figures are all averages from the eight-year period ending June, 1960.

The farm in question is a 900-acre wheat-sheep holding in the Cowra district of mid-western New South Wales. Prior to the introduction of improved pasture in the late 1940's, wheat growing was the predominant activity on the property. Since that time the emphasis has shifted further towards livestock production, although cropping still plays an integral part in the farm structure.

For the purposes of this illustration we assume that the farmer has available 900 acres of unimproved land and that he desires to determine the optimal allocations of this land between wheat, improved pasture and unimproved pasture over a four-year period. He may establish improved pasture by one of two methods, viz., he may sow improved pasture species directly into a prepared seedbed, or he may sow them under a cover crop of wheat.[17] The farmer's programme is restricted by the amounts of land and capital he has available, and the capacity of his labour force at sowing and harvesting time.

Thus the principal activities which may appear in his programme at each year are wheat, wheat plus improved pasture, improved pasture, and unimproved pasture; and the chief limiting resources are land, capital and labour.

Let us now define the resource requirements and supplies.

(a) Capital

Each activity requires capital expenditure in year 1 which includes the costs of cultivation, fertilizer and seed (if any) and miscellaneous running expenses which are dependent on the number of stock carried. These latter expenses are assessed at £0.28 per head and are multiplied by the appropriate stocking rate (see below) for inclusion in the capital restriction. In

[17] Under this system, pasture and wheat are planted in the same ground at the same time; after the wheat crop has been harvested, the improved pasture sward is available for grazing.

addition, the activities involving improved pasture will incur maintenance expenditure for topdressing, etc., in years 2, 3 and 4. A summary of capital requirements is given in Table 1. The farmer is assumed to have £2,000 of working capital on hand at the start of the programme.

TABLE 1

Capital Requirements per Acre of Principal Activities in Years 1, 2, 3 and 4: £ per Acre

Year*	Expenditure Item	Activity			
		Wheat	Wheat + Improved Pasture	Improved Pasture	Un-improved Pasture
		£	£	£	£
1	Wheat seed	0·50	0·25	..	..
	Pasture seed	..	0·71	1·42	..
	Superphosphate	0·34	0·34	0·34	..
	Cultivations	3·50	3·50	1·50	..
	Sowing	0·75	0·75	0·75	..
	Miscellaneous stock expenses	..	0·10	0·20	0·20
	TOTAL	5·09	5·65	4·21	0·20
2	Superphosphate	..	0·34	0·34	..
	Spreading	..	0·38	0·38	..
	Miscellaneous stock expenses	..	0·39	0·39	..
	TOTAL	..	1·11	1·11	..
3	Superphosphate	..	0·34	0·34	..
	Spreading	..	0·38	0·38	..
	Miscellaneous stock expenses	..	0·53	0·53	..
	TOTAL	..	1·25	1·25	..
4	Superphosphate	..	0·34	0·34	..
	Spreading	..	0·38	0·38	..
	Miscellaneous stock expenses	..	0·67	0·67	..
	TOTAL	..	1·39	1·39	..

* In Tables 1, 2, and 3 " Year 1 " refers to the Establishment Year with respect to Wheat, Wheat + Improved Pasture, and Improved Pasture. The Unimproved Pasture activity only lasts for one year before the land which it occupies becomes re-available for allocation to alternative uses.

(*b*) *Land*

An unchanging quantity of land, 900 acres, is available at each period. Since the resource requirements of the land-using activities are drawn up on a per acre basis, the entries in the land row of the programming matrix for these activities will be unity.

(c) Labour

There are two periods when the farmer's labour supply limits his activities, viz., at sowing and harvesting. We are supposing that he cannot obtain extra labour at these times. It is assumed that the total labour supply at the establishment time for wheat, wheat + improved pasture, and improved pasture is 300 hours and that each acre of each of these three activities uses up 1 hour of this labour. The total area which may be established in any period is thus 300 acres. At harvesting time we assume that the farmer has sufficient machinery and labour to strip a maximum of 200 acres.

(d) Stocking Rates

The following table shows the carrying capacities of the different pasture types in each year:—

TABLE 2

Carrying Capacities of Pasture Types in Years 1, 2, 3 *and* 4: *Sheep/Acre*

Year	Activity		
	Wheat + Improved Pasture	Improved Pasture	Unimproved Pasture
1	0·35	0·70	0·63
2	1·40	1·40	....
3	1·90	1·90	....
4	2·40	2·40	....

The farmer commences the programme with 400 sheep.

(e) Fixed Financial Commitments

The farmer is assumed to be faced with a fixed yearly commitment of £2,500 out of which he has to meet household expenses, fixed costs, unavoidable investments, repairs to plant and structures, etc.

Let us turn now to the objective function. The following productivity data are used:

On unimproved land one sheep produces 0.66 lambs and 11.14 lb. of wool each year.

On improved pasture one sheep produces 0.66 lambs and 12.87 lb. of wool per year.

Of these 0.66 lambs, 0.33 are available for sale and 0.33 are available for incorporation into the breeding flock (These figures are adjusted for mortality).

With lambs selling at £4.57 per head and wool at £0.30 per lb., one sheep grazing unimproved land returns £4.81, whilst on improved pasture a sheep returns £5.32. (It is assumed that a lamb or a pound of wool sells for the same price, irrespective of whether it was grown on improved or unimproved pasture).

The above figures are multiplied by stocking rates (from Table 2) to give gross returns per acre of each sheep-carrying activity.

In addition the gross return per acre of wheat is assessed at £12.49. (This farm is a consistently high wheat producer, with an annual average in excess of 20 bushels/acre.) When improved pasture is sown with wheat it is assumed that the wheat yield is lowered by 25 per cent.

The above figures are summarized in Table 3.

TABLE 3

Gross Returns for Years 1, 2, 3 *and* 4 *from Various Activities*: £ *per Acre*

Year	Activity					
	Wheat	Wheat + Improved Pasture		Improved Pasture		Unimproved Pasture
	Gross Return in Year	Gross Return in Year	Cumulative Total	Gross Return in Year	Cumulative Total	Gross Return in Year
	£	£	£	£	£	£
1	12·49	11·23	11·23	3·72	3·72	3·03
2	..	7·45	18·68	7·45	11·17	....
3	..	10·11	28·79	10·11	21·28	....
4	..	12·77	41·56	12·77	34·05	....

The figures in Table 3 are placed directly in the objective function. The justification for using "gross" profit as the criterion to be maximized is that the capital supply is assumed in a formulation such as this to be committed already, and to have no alternative use. Thus we are maximizing "profit" given amongst other things that the capital supply is to be allocated in the best possible way.[18]

It is now convenient to refer to the complete programming matrix, which is shown in Table 4, in order to explain some further features of dynamic linear programming exemplified in this analysis.

(*a*) *The committing of a resource for several periods*

Since an acre of improved pasture lasts for at least four years from the date of sowing, the land it occupies is "tied up" or committed for this length of time, and is thus not available in subsequent periods for reallocation to fresh uses. Hence for example, unity is included in the "land" rows for years 2, 3, 4 (vectors 37, 44, 50) for the improved pasture activities commenced in year 1, to indicate that any land established in the first year is not to be used for any other purpose subsequently. If the programme covered a longer time period than four years, the pasture land could be made to become re-available on expiry of the existing sward.

[18] cf. a discussion in J. D. Stewart, "Farm Operating as a Constraint—a Problem in the Application of Linear Programming", *The Farm Economist*, Vol. 9, No. 10 (1961). pp. 463-71.

TABLE 4

Dynamic Linear Programming Matrix for optimizing Farm Activities over a Four-year Period

(Identity matrix not included)

VECTOR No. →					1	2	3	4	5	6	7	8
↓	YEAR → ↓	ACTIVITY → / RESOURCE		b_i	Year 1: Wheat	Wheat + Improved Pasture	Improved Pasture	Un-improved Pasture	Sheep Purchase	House-hold, etc.	Borrow from Year 2	Borrow from Year 3
					ac.	ac.	ac.	ac.	No.	£	£	£
29	1	Capital	£	2,000	5·09	5·65	4·21	0·20	3·11	1	— 1	— 1
30		Land	ac.	900	1	1	1	1	0	0	0	0
31		Household, etc.	£	2,500	0	0	0	0	0	1	0	0
32		Sheep	No.	400	0	0·35	0·70	0·63	0·63	0	0	0
33		Harvest Restriction	ac.	200	1	1	0	0	0	0	0	0
34		Borrowing Restriction	£	3,000	0	0	0	0	0	0	1	1
35		Establishment Restriction	hr.	300	1	1	1	0	0	0	0	0
36	2	Capital	£	0	—12·49	—10·12	— 2·61	— 3·03	0	0	1·05	0
37		Land	ac.	900	0	1	1	0	0	0	0	0
38		Household, etc.	£	2,500	0	0	0	0	0	0	0	0
39		Sheep	No.	533	0	1·40	1·40	0	— 1·33	0	0	0
40		Harvest Restriction	ac.	200	0	0	0	0	0	0	0	0
41		Borrowing Restriction	£	3,000	0	0	0	0	0	0	0	0
42		Establishment Restriction	hr.	300	0	0	0	0	0	0	0	0
43	3	Capital	£	0	0	— 6·20	— 6·20	0	0	0	0	1·10
44		Land	ac.	900	0	1	1	0	0	0	0	0
45		Household, etc.	£	2,500	0	0	0	0	0	0	0	0
46		Sheep	No.	711	0	1·90	1·90	0	— 1·78	0	0	0
47		Harvest Restriction	ac.	200	0	0	0	0	0	0	0	0
48		Establishment Restriction	hr.	300	0	0	0	0	0	0	0	0
49	4	Capital	£	0	0	— 8·72	— 8·72	0	0	0	0	0
50		Land	ac.	900	0	1	1	0	0	0	0	0
51		Household, etc.	£	2,500	0	0	0	0	0	0	0	0
52		Sheep	No.	948	0	2·40	2·40	0	— 2·37	0	0	0
53		Harvest Restriction	ac.	200	0	0	0	0	0	0	0	0
54		Establishment Restriction	hr.	300	0	0	0	0	0	0	0	0
		c_j →			12·49	41·56	34·05	3·03	0	80·00	0	0

VECTOR No. →					9	10	11	12	13	14	15	16
	YEAR →							2				
↓	↓	ACTIVITY →			Wheat	Wheat + Improved Pasture	Improved Pasture	Un-improved Pasture	Sheep Purchase	House-hold, etc.	Borrow from Year 3	Borrow from Year 4
		RESOURCE		b_i								
					ac.	ac.	ac.	ac.	No.	£	£	£
29		Capital	£	2,000	..	..	..	..	..	..	..	..
30		Land	ac.	900	..	..	..	..	..	..	..	..
31	1	Household, etc.	£	2,500	..	..	..	..	..	..	..	..
32		Sheep	No.	400	..	..	..	..	..	..	..	..
33		Harvest Restriction	ac.	200	..	..	..	..	..	..	..	..
34		Borrowing Restriction	£	3,000	..	..	..	..	..	..	..	..
35		Establishment Restriction	hr.	300	..	..	..	..	..	..	..	..
36		Capital	£	0	5·09	5·65	4·21	0·20	3·11	1	— 1	— 1
37		Land	ac.	900	1	1	1	1	0	0	0	0
38	2	Household, etc.	£	2,500	0	0	0	0	0	1	0	0
39		Sheep	No.	533	0	0·35	0·70	0·63	— 1	0	0	0
40		Harvest Restriction	ac.	200	1	1	0	0	0	0	0	0
41		Borrowing Restriction	£	3,000	0	0	0	0	0	0	1	1
42		Establishment Restriction	hr.	300	1	1	1	0	0	0	0	0
43		Capital	£	0	— 12·49	— 10·12	— 2·61	— 3·03	0	0	1·05	0
44		Land	ac.	900	0	1	1	0	0	0	0	0
45	3	Household, etc.	£	2,500	0	0	0	0	0	0	0	0
46		Sheep	No.	711	0	1·40	1·40	0	— 1·33	0	0	0
47		Harvest Restriction	ac.	200	0	0	0	0	0	0	0	0
48		Establishment Reaction	hr.	300	0	0	0	0	0	0	0	0
49		Capital	£	0	0	— 6·20	— 6·20	0	0	0	0	1·10
50		Land	ac.	900	0	1	1	0	0	0	0	0
51	4	Household, etc.	£	2,500	0	0	0	0	0	0	0	0
52		Sheep	No.	948	0	1·90	1·90	0	— 1·78	0	0	0
53		Harvest Restriction	ac.	200	0	0	0	0	0	0	0	0
54		Establishment Restriction	hr.	300	0	0	0	0	0	0	0	0
		c_j →			12·49	28·79	21·28	3·03	0	80·00	0	0

VECTOR No. →					17	18	19	20	21	22
	YEAR →				3					
↓	↓	ACTIVITY → / RESOURCE		b_i	Wheat	Wheat + Improved Pasture	Improved Pasture	Unimproved Pasture	Sheep Purchase	Household, etc.
					ac.	ac.	ac.	ac.	No.	£
29	1	Capital	£	2,000	..	..	..	..	..	..
30	1	Land	ac.	900	..	..	..	..	..	..
31	1	Household, etc.	£	2,500	..	..	..	..	..	..
32	1	Sheep	No.	400	..	..	..	..	..	..
33	1	Harvest Restriction	ac.	200	..	..	..	..	..	..
34	1	Borrowing Restriction	£	3,000	..	..	..	..	..	..
35	1	Establishment Restriction	hr.	300	..	..	..	..	..	..
36	2	Capital	£	0	..	..	..	..	..	..
37	2	Land	ac.	900	..	..	..	..	..	..
38	2	Household, etc.	£	2,500	..	..	..	..	..	..
39	2	Sheep	No.	533	..	..	..	..	..	..
40	2	Harvest Restriction	ac.	200	..	..	..	..	..	..
41	2	Borrowing Restriction	£	3,000	..	..	..	..	..	..
42	2	Establishment Restriction	hr.	300	..	..	..	..	..	..
43	3	Capital	£	0	5·09	5·65	4·21	0·20	3·11	1
44	3	Land	ac.	900	1	1	1	1	0	0
45	3	Household, etc.	£	2,500	0	0	0	0	0	1
46	3	Sheep	No.	711	0	0·35	0·70	0·63	—1	0
47	3	Harvest Restriction	ac.	200	1	1	0	0	0	0
48	3	Establishment Restriction	hr.	300	1	1	1	0	0	0
49	4	Capital	£	0	—12·49	—10·12	—2·61	—3·03	0	0
50	4	Land	ac.	900	0	1	1	0	0	0
51	4	Household, etc.	£	2,500	0	0	0	0	0	0
52	4	Sheep	No.	948	0	1·40	1·40	0	—1·33	0
53	4	Harvest Restriction	ac.	200	0	0	0	0	0	0
54	4	Establishment Restriction	hr.	300	0	0	0	0	0	0
		c_j →			12·49	18·68	11·17	3·03	0	80·00

VECTOR No. →					23	24	25	26	27	28
	YEAR →				4					
↓	↓	ACTIVITY →			Wheat	Wheat + Improved Pasture	Improved Pasture	Unimproved Pasture	Sheep	Household, etc.
		RESOURCE		b_i						
					ac.	ac.	ac.	ac.	No.	£
29		Capital	£	2,000	..	..	..	..	..	..
30		Land	ac.	900	..	..	..	..	..	..
31	1	Household, etc.	£	2,500	..	..	..	..	..	..
32		Sheep	No.	400	..	..	..	..	..	..
33		Harvest Restriction..	ac.	200	..	..	..	..	..	..
34		Borrowing Restriction	£	3,000	..	..	..	..	..	..
35		Establishment Restriction	hr.	300	..	..	..	..	..	..
36		Capital	£	0	..	..	..	..	..	..
37		Land	ac.	900	..	..	..	..	..	..
38	2	Household, etc.	£	2,500	..	..	..	..	..	..
39		Sheep	No.	533	..	..	..	..	..	..
40		Harvest Restriction..	ac.	200	..	..	..	..	..	..
41		Borrowing Restriction	£	3,000	..	..	..	..	..	..
42		Establishment Restriction	hr.	300	..	..	..	..	..	..
43		Capital	£	0	..	..	..	..	..	..
44		Land	ac.	900	..	..	..	..	..	..
45	3	Household, etc.	£	2,500	..	..	..	..	..	..
46		Sheep	No.	711	..	..	..	..	..	..
47		Harvest Restriction..	ac.	200	..	..	..	..	..	..
48		Establishment Restriction	hr.	300	..	..	..	..	..	..
49		Capital	£	0	5·09	5·65	4·21	0·20	3·11	1
50		Land	ac.	900	1	1	1	1	0	0
51	4	Household, etc.	£	2,500	0	0	0	0	0	1
52		Sheep	No.	948	0	0·35	0·70	0·63	—1	0
53		Harvest Restriction..	ac.	200	1	1	0	0	0	0
54		Establishment Restriction	hr.	300	1	1	1	0	0	0
		c_j →			12·49	11·23	3·72	3·03	0	80·00

VECTOR No. →				17	18	19	20	21	22
↓	YEAR →			3					
	↓	ACTIVITY →		Wheat	Wheat + Improved Pasture	Improved Pasture	Unimproved Pasture	Sheep	Household, etc.
		OPTIMUM LEVEL →		200 ac.	0	100 ac.	0	0	£2,500
		RESOURCE		ac.	ac.	ac.	ac.	No.	£
29		Capital	£	..	..	..	..	..	..
30		Land	ac.	..	..	..	..	..	..
31		Household, etc.	£	..	..	..	..	..	..
32	1	Sheep	No.	..	..	..	..	..	..
33		Harvest Restriction	ac.	..	..	..	..	..	..
34		Borrowing Restriction	£	..	..	..	..	..	..
35		Establishment Restriction ..	hr.	..	..	..	..	..	..
36		Capital	£	..	..	..	..	..	..
37		Land	ac.	..	..	..	..	..	..
38		Household, etc.	£	..	..	..	..	..	..
39	2	Sheep	No.	..	..	..	..	..	..
40		Harvest Restriction	ac.	..	..	..	..	..	..
41		Borrowing Restriction	£	..	..	..	..	..	..
42		Establishment Restriction ..	hr.	..	..	..	..	..	..
43		Capital	£	1,018	..	421	..	..	2,500
44		Land	ac.	200	..	100	..	..	..
45		Household, etc.	£	..	..	..	..	..	2,500
46	3	Sheep	No.	..	..	70	..	..	..
47		Harvest Restriction	ac.	200	..	..	..	..	..
48		Establishment Restriction ..	hr.	200	..	100	..	..	..
49		Capital	£	—2,498	..	—261	..	..	..
50		Land	ac.	..	..	100	..	..	..
51		Household, etc.	£	..	..	..	..	..	..
52	4	Sheep	No.	..	..	140	..	..	..
53		Harvest Restriction	ac.	..	..	..	..	..	..
54		Establishment Restriction ..	hr.	..	..	..	..	..	..
"TRUE PROFIT"			£	2,498	..	1,117	..	..	..
"FALSE PROFIT"			£	..	..	..	..	..	200,000

VECTOR No. →				23	24	25	26	27	28		
↓	YEAR → ↓			4							
		ACTIVITY →		Wheat	Wheat + Improved Pasture	Improved Pasture	Unimproved Pasture	Sheep	Household, etc.	SLACK	*b*
		OPTIMUM LEVEL →		200 ac.	0	0	0	0	£2,500		
		RESOURCE		ac.	ac.	ac.	ac.	No.	£		
29	1	Capital	£	..	..	..	..	..	..	..	2,000
30		Land	ac.	..	..	..	..	..	..	..	900
31		Household, etc.	£	..	..	..	..	..	..	..	2,500
32		Sheep	No.	..	..	..	..	..	..	..	400
33		Harvest Restriction	ac.	..	..	..	..	..	..	..	200
34		Borrowing Restriction	£	..	..	..	..	..	..	462	3,000
35		Establishment Restriction	hr.	..	..	..	..	..	..	..	300
36	2	Capital	£	..	..	..	..	..	..	..	0
37		Land	ac.	..	..	..	..	..	..	..	900
38		Household, etc.	£	..	..	..	..	..	..	..	2,500
39		Sheep	No.	..	..	..	..	..	..	55	533
40		Harvest Restriction	ac.	..	..	..	..	..	..	..	200
41		Borrowing Restriction	£	..	..	..	..	..	..	..	3,000
42		Establishment Restriction	hr.	..	..	..	..	..	..	..	300
43	3	Capital	£	..	..	..	..	..	..	1,087	0
44		Land	ac.	..	..	..	..	..	..	..	900
45		Household, etc.	£	..	..	..	..	..	..	..	2,500
46		Sheep	No.	..	..	..	..	..	..	13	711
47		Harvest Restriction	ac.	..	..	..	..	..	..	..	200
48		Establishment Restriction	hr.	..	..	..	..	..	..	..	300
49	4	Capital	£	1,018	..	..	..	..	2,500	417	0
50		Land	ac.	200	..	..	..	..	..	..	900
51		Household, etc.	£	..	..	..	..	..	2,500	..	2,500
52		Sheep	No.	..	..	..	..	..	..	..	948
53		Harvest Restriction	ac.	200	..	..	..	..	..	..	200
54		Establishment Restriction	hr.	200	..	..	..	..	..	100	300
" TRUE PROFIT "			£	2,498	..	..	..	..	..	..	28,443
" FALSE PROFIT "			£	..	..	..	..	..	200,000	..	800,000

TABLE 6

Presentation of Linear Programming Results

VECTOR No. →				1	2	3	4	5	6	7	8
↓	YEAR →						1				
	↓	ACTIVITY →		Wheat	Wheat + Improved Pasture	Improved Pasture	Unimproved Pasture	Sheep	Household, etc.	Borrow from Year 2	Borrow from Year 3
		OPTIMUM LEVEL →		0 ac.	200 ac.	100 ac.	600 ac.	118	£2,500	£2,513	£25
		RESOURCE		ac.	ac.	ac.	ac.	No.	£	£	£
29		Capital	£	..	1,130	421	120	367	2,500	—2,513	—25
30		Land	ac.	..	200	100	600	..	..	..	..
31		Household, etc.	£	..	..	..	..	..	2,500	..	..
32	1	Sheep	No.	..	70	70	378	—118	..	..	..
33		Harvest Restriction	ac.	..	200	..	..	..	..	..	..
34		Borrowing Restriction	£	..	..	..	..	..	..	2,513	25
35		Establishment Restriction	hr.	..	200	100	..	..	..	..	..
36		Capital	£	..	—2,024	—261	—1,818	..	..	2,638	..
37		Land	ac.	..	200	100	..	..	..	..	..
38		Household, etc.	£	..	..	..	..	..	..	..	..
39	2	Sheep	No.	..	280	140	..	—157	..	..	..
40		Harvest Restriction	ac.	..	..	..	..	..	..	..	..
41		Borrowing Restriction	£	..	..	..	..	..	..	..	..
42		Establishment Restriction	hr.	..	..	..	..	..	..	..	..
43		Capital	£	..	—1,240	—620	..	..	..	..	28
44		Land	ac.	..	200	100	..	..	..	..	..
45		Household, etc.	£	..	..	..	..	..	..	..	..
46	3	Sheep	No.	..	380	190	..	—210	..	..	..
47		Harvest Restriction	ac.	..	..	..	..	..	..	..	..
48		Establishment Restriction	hr.	..	..	..	..	..	..	..	..
49		Capital	£	..	—1,744	—872	..	..	..	..	..
50		Land	ac.	..	200	100	..	..	..	..	..
51		Household, etc.	£	..	..	..	..	..	..	..	..
52	4	Sheep	No.	..	480	240	..	—280	..	..	..
53		Harvest Restriction	ac.	..	..	..	..	..	..	..	..
54		Establishment Restriction	hr.	..	..	..	..	..	..	..	..
"TRUE PROFIT"			£	..	8,312	3,405	1,818	..	..	..	..
"FALSE PROFIT"			£	..	..	..	..	..	200,000	..	..

VECTOR No. →				9	10	11	12	13	14	15	16
↓	YEAR →			2							
	↓	ACTIVITY →		Wheat	Wheat + Improved Pasture	Improved Pasture	Unimproved Pasture	Sheep	Household, etc.	Borrow from Year 3	Borrow from Year 4
		OPTIMUM LEVEL →		0 ac.	200 ac.	100 ac.	300 ac.	114	£2,500	0	£3,000
		RESOURCE		ac.	ac.	ac.	ac.	No.	£	£	£
29	1	Capital	£	..	..	..	..	..	..	..	..
30		Land	ac.	..	..	..	..	..	..	..	..
31		Household, etc.	£	..	..	..	..	..	..	..	..
32		Sheep	No.	..	..	..	..	..	..	..	..
33		Harvest Restriction	ac.	..	..	..	..	..	..	..	..
34		Borrowing Restriction	£	..	..	..	..	..	..	..	..
35		Establishment Restriction	hr.	..	..	..	..	..	..	..	..
36	2	Capital	£	..	1,130	421	60	354	2,500	..	—3,000
37		Land	ac.	..	200	100	300	..	..	..	..
38		Household, etc.	£	..	..	..	..	..	2,500	..	..
39		Sheep	No.	..	70	70	189	—114	..	..	..
40		Harvest Restriction	ac.	..	200	..	..	..	..	..	..
41		Borrowing Restriction	£	..	..	..	..	..	..	..	3,000
42		Establishment Restriction	hr.	..	200	100	..	..	..	..	..
43	3	Capital	£	..	—2,024	—261	—909	..	..	..	..
44		Land	ac.	..	200	100	..	..	..	..	..
45		Household, etc.	£	..	..	..	..	..	..	..	..
46		Sheep	No.	..	280	140	..	—152	..	..	..
47		Harvest Restriction	ac.	..	..	..	..	..	..	..	..
48		Establishment Restriction	hr.	..	..	..	..	..	..	..	..
49	4	Capital	£	..	—1,240	—620	..	..	..	..	3,300
50		Land	ac.	..	200	100	..	..	..	..	..
51		Household, etc.	£	..	..	..	..	..	..	..	..
52		Sheep	No.	..	380	190	..	—202	..	..	..
53		Harvest Restriction	ac.	..	..	..	..	..	..	..	..
54		Establishment Restriction	hr.	..	..	..	..	..	..	..	..
"TRUE PROFIT"			£	..	5,758	2,128	909	..	..	..	..
"FALSE PROFIT"			£	..	..	..	..	..	200,000	..	..

(b) Capital transfer between periods

The operating capital for use in years 2, 3, 4 is derived from the profits from preceding years' activities. Provision must be made, then, for the transfer of money from one year to the next. This is achieved by placing a negative coefficient at the intersection of the *k-th* year's activity columns and the $(k + 1)$th year's capital row. For example a coefficient of —1 at the intersection of the j-th column in year 1 and the capital row of year 2 would indicate that each unit of the j-th activity undertaken in year 1 would add £1 to the capital supply of year 2. This should become more obvious if reference is made back to equations (6) and (11). In addition improved pasture activities undertaken in year k will "donate" capital to years $k + 2$ and $k + 3$ as well as $k + 1$ and will also require maintenance expenditure in these years. The capital transfer figures for activity j in year k in Table 4 are thus given as the maintenance expenditure in year k (where applicable) *minus* the gross returns in year $k - 1$, with respect to the *j-th* activity. The reader may verify this using figures from Tables 1 and 3.

(c) Borrowing

In years 1 and 2 the farmer has been given the opportunity of borrowing money, to be paid back in a lump sum one year hence at a simple 5 per cent interest or two years hence at a simple 10 per cent interest. For example vector 8 (borrowing in year 1 from year 3) includes —1 at its intersection with vector 29 and 1.10 at vector 43 to show that each unit of this activity "donates" £1 to the capital supply of year 1 at a cost of £1·10 in year 3. A restriction specifying that the farmer is not willing to go further than £3,000 into debt in any year is included as vectors 34 and 41.

(d) Fixed financial commitments

Since the fixed monetary obligations mentioned earlier must be met regardless of the optimum programme, the "Household etc." activity for each year (vectors 6, 14, 22 and 28) must be "forced" to appear in the final plan. This is achieved by placing an artificially high c_j, value on this activity; for our matrix, a value of 80 was chosen. These capital withdrawals could have been specified as a single column vector, with a coefficient 2,500 at the intersection with each capital row, and with a very high c_j value. However, the scaling requirement of the computer programme on which this matrix was to be solved made this approach infeasible.

(e) Sheep number increases between years

A column vector has been included in each year to permit extra sheep to be purchased at £3·11 per head if required. It was noted above that each sheep on hand in period k produced 0.33 sheep for inclusion in the breeding flock in period $k + 1$. This is allowed for in the matrix by causing the original sheep numbers (400) to increase by one-third in period 2, a further one-third in period 3, and again in period 4 (see b column, rows 32, 39, 46 and 52). In addition the extra sheep purchased are made to increase through time in a similar fashion (see, for example, column 5, rows 32, 39, 46 and 52).

The entire 28 × 26 matrix in Table 4 was solved by electronic computer using the simplex procedure. The problem required 44 iterations to reach a solution, which took the computer three minutes.

The b column of the matrix at the end of the final iteration appeared as in Table 5.

The criterion which was maximized, z, includes an amount of £$(4 \times 2{,}500 \times 80) =$ £800,000 due to the "forcing in" of four years' "Household, etc." expenditures. Thus the correct figure for z at the bottom of Table 5 should be £(828,443 — 800,000) = £28,443.

TABLE 5

b Column from Optimized Linear Programming Matrix

Vector No.	Activity or Resource	Year	Optimum Level
6	Household, etc.	1	£2,500
10	Wheat + Improved Pasture ..	2	200 acres
23	Wheat	4	200 acres
13	Sheep (activity)	2	114 head
14	Household, etc.	2	£2,500
49	Capital	4	£417
8	Borrow year 3	1	£25
7	Borrow year 2	1	£2,513
5	Sheep (activity)	1	118 head
4	Unimproved pasture	1	600 acres
17	Wheat	3	200 acres
34	Borrow Restriction	1	£462
43	Capital	3	£1,087
39	Sheep (resource)	2	55 head
22	Household, etc.	3	£2,500
20	Unimproved pasture	3	0
12	Unimproved pasture	2	300 acres
46	Sheep (resource)	3	13 head
16	Borrow year 3	2	£3,000
18	Wheat + Improved Pasture ..	3	0
28	Household, etc.	4	£2,500
3	Improved Pasture	1	100 acres
2	Wheat + Improved Pasture ..	1	200 acres
11	Improved Pasture	2	100 acres
19	Improved Pasture	3	100 acres
54	Establishment Restriction ..	4	100 acres
0	" Profit "	..	£828,443

To make the dynamics of the solution more understandable, each element a_{ij} in the original matrix has been multiplied by the optimum level x_j of each activity. The results, shown in Table 6, indicate the total amounts of each resource used up at each stage by each activity, and the amount added to resource supplies by various activities in various periods (the latter distinguished by being negative). Zero elements are not shown. The b column in Table 6 has been transferred to the right-hand side to enable the interested reader to work across each row, verifying at the end that the resource use tallies with its supply. Sub-totals can be calculated readily from the Table to give year by year profit, capital expenditure, etc.

It is not proposed here to discuss the numerical solution in detail. It should be noted that both sowing and harvesting restrictions are met in full in years 1, 2 and 3, which would suggest that expansion of the labour force at these times might be worthwhile. However, it would be difficult to incorporate larger acreages of wheat into the rotational programme; perhaps then additional land is also required. Two ways out of these dilemmas suggest themselves: firstly rotational activities as discussed in part (iii) above might be incorporated into a dynamic linear programming format; secondly, the strain on the linearity assumption when high wheat acreages, etc., are involved might be relaxed by introducing a non-linear element via conventional non-linear programming techniques.

It should now have become apparent that a great many facilities are available with dynamic linear programming for characterising farm systems over time. To conclude, some extensions which might be applied to the above numerical analysis are suggested, which indicate further the scope of this technique.

(*a*) A capital selling, as well as a capital buying, activity could be included to dispose of the assumption of there being no alternative opportunities for employing capital. One would have to be prepared, however, for linear programming solutions which advise the farmer to retire from active agricultural production and invest all his working capital elsewhere!

(*b*) In multiperiod analyses it is generally desirable to discount future prices and costs relative to the first time period under consideration. Discounting was not applied to our above problem in order to keep the figures relatively simple.

(*c*) Provisions for borrowing might be extended to encompass repayment by instalments if such were feasible. In any case, since dynamic linear programming can tell one not only how much it is profitable to borrow, but also the best mixture of borrowing strategies amongst the available alternatives, it would seem desirable to include in an analysis a fairly representative range of borrowing possibilities.

(*d*) It would be more realistic in transferring wheat revenue to spread it out over several periods instead of donating it in a lump sum to the next period. The payment pattern for wheat could be made to simulate the actual sequence of Wheat Pool repayments experienced by the farmer.

(*e*) A more realistic characterization of our particular farm could have been achieved by increasing the number of activities (and restrictions) in the programme. For example, further crop and livestock opportunities could have been included, allowance could have been made for fodder conservation, sale, storage and feeding out, etc. In addition the farmers' planning horizon might have been extended to cover seven or eight periods instead of four. The analysis as it is presented here assumes that whatever might happen in the 5th, 6th and subsequent periods is quite irrelevant to the problem in hand.

(*f*) For the sake of simplicity, our example assumed a largely repetitive pattern of input-output coefficients, prices and resource supplies between periods. For more descriptive accuracy, all these coefficients could be allowed to vary over time.

CONCLUSION

In this article an attempt has been made to present a brief outline of some ways in which time may be incorporated into linear programming analyses in agriculture. Although differing somewhat in detail, the four methods examined are basically merely conceptual and mechanical extensions of simple "static" linear programming. It is suggested that in the future the dynamic linear programming formulation which is characterized, as in our example, by a block-triangular matrix could evolve as the most powerful of the techniques discussed here, both from a descriptive and an analytical viewpoint.