

AJAE Appendix for “Crop Yield Skewness Under Law of the Minimum Technology”

Author: David A. Hennessy

Date: 5/15/2008

Note: The material contained herein is supplementary to the article named in the title and published in the American Journal of Agricultural Economics (AJAE).

Uniform Distribution; Heterogeneity in Means:

The moments are

$$\mu_y = 0.5(0) + 0.25\tau + 0.25(1) = \frac{\tau + 1}{4},$$

$$\begin{aligned}\sigma_y^2 &= 0.5\left(0 - \frac{(\tau + 1)}{4}\right)^2 + 0.25\left(\tau - \frac{(\tau + 1)}{4}\right)^2 + 0.25\left(1 - \frac{(\tau + 1)}{4}\right)^2 \\ &= \frac{(\tau + 1)^2}{32} + \frac{(3\tau - 1)^2}{64} + \frac{(3 - \tau)^2}{64} = \frac{2\tau^2 + 4\tau + 2 + 9\tau^2 - 6\tau + 1 + 9 - 6\tau + \tau^2}{64} \\ &= \frac{12\tau^2 - 8\tau + 12}{64} = \frac{3\tau^2 - 2\tau + 3}{16},\end{aligned}$$

$$\begin{aligned}\zeta_y &= 0.5\left(0 - \frac{(\tau + 1)}{4}\right)^3 + 0.25\left(\tau - \frac{(\tau + 1)}{4}\right)^3 + 0.25\left(1 - \frac{(\tau + 1)}{4}\right)^3 \\ &= -\frac{(\tau + 1)^3}{128} + \frac{(3\tau - 1)^3}{256} + \frac{(3 - \tau)^3}{256} \\ &= -\frac{2\tau^3 + 6\tau^2 + 6\tau + 2}{256} + \frac{27\tau^3 - 27\tau^2 + 9\tau - 1}{256} + \frac{27 - 27\tau + 9\tau^2 - \tau^3}{256} \\ &= \frac{24\tau^3 - 24\tau^2 - 24\tau + 24}{256} = \frac{3(\tau^3 - \tau^2 - \tau + 1)}{32} = \frac{3(1 - \tau)^2(1 + \tau)}{32},\end{aligned}$$

$$\gamma_y = \frac{(3/32)(1 - \tau)^2(1 + \tau)}{\left(\frac{3\tau^2 - 2\tau + 3}{16}\right)^{3/2}} = \frac{6(1 - \tau)^2(1 + \tau)}{(3\tau^2 - 2\tau + 3)^{3/2}} > 0.$$

So, for (9),

$$\begin{aligned}
\frac{d\gamma_y}{d\tau} &= \frac{(3\tau^2 - 2\tau + 3)^{3/2} \left[6[(1-\tau)^2 - 2(1-\tau)(1+\tau)] - 9(1-\tau)^2(1+\tau)(6\tau - 2) \right] (3\tau^2 - 2\tau + 3)^{1/2}}{(3\tau^2 - 2\tau + 3)^3} \\
&= \frac{(3\tau^2 - 2\tau + 3)6(1-\tau)[-1 - 3\tau] - 9(1-\tau)^2(1+\tau)(6\tau - 2)}{(3\tau^2 - 2\tau + 3)^{2.5}} \\
&= -6(1-\tau) \frac{(3\tau^2 - 2\tau + 3)[1 + 3\tau] + 3(1-\tau)(1+\tau)(3\tau - 1)}{(3\tau^2 - 2\tau + 3)^{2.5}} \\
&= -6(1-\tau) \frac{3\tau^2 - 2\tau + 3 + 9\tau^3 - 6\tau^2 + 9\tau + 3(1-\tau^2)(3\tau - 1)}{(3\tau^2 - 2\tau + 3)^{2.5}} = -\frac{6(1-\tau)(16\tau)}{(3\tau^2 - 2\tau + 3)^{2.5}} < 0.
\end{aligned}$$

Uniform Distribution; Heterogeneity in Variances:

The moments are

$$\mu_y = 0.5(0) + 0.25\tau + 0.25(1-\tau) = 0.25,$$

$$\begin{aligned}
\sigma_y^2 &= 0.5(0 - 0.25)^2 + 0.25(\tau - 0.25)^2 + 0.25(1 - \tau - 0.25)^2 \\
&= \frac{2}{16} + \frac{(4\tau - 1)^2}{16} + \frac{(3 - 4\tau)^2}{16} = \frac{2 + 16\tau^2 - 8\tau + 1 + 9 - 24\tau + 16\tau^2}{64} \\
&= \frac{12 + 32\tau^2 - 32\tau}{64} = \frac{8\tau^2 - 8\tau + 3}{16}.
\end{aligned}$$

$$\begin{aligned}
\zeta_y &= 0.5(0 - 0.25)^3 + 0.25(\tau - 0.25)^3 + 0.25(1 - \tau - 0.25)^3 \\
&= \frac{-2}{64} + \frac{(4\tau - 1)^3}{64} + \frac{(3 - 4\tau)^3}{64} \\
&= \frac{-2 + (64\tau^3 - 48\tau^2 + 12\tau - 1) + (27 - 108\tau + 144\tau^2 - 64\tau^3)}{256} \\
&= \frac{3 - 12\tau + 12\tau^2}{32} = \frac{3(1 - 2\tau)^2}{32}.
\end{aligned}$$

Turning to (10), use $\gamma_y = 6(1 - 2\tau)^2(8\tau^2 - 8\tau + 3)^{-3/2}$ to obtain

$$\begin{aligned}
\frac{d\gamma_y}{d\tau} &= 6 \left[\frac{-4(8\tau^2 - 8\tau + 3)^{3/2}(1-2\tau) - 1.5(1-2\tau)^2(16\tau - 8)(8\tau^2 - 8\tau + 3)^{1/2}}{(8\tau^2 - 8\tau + 3)^3} \right] \\
&= 6(1-2\tau) \left[\frac{12(1-2\tau)^2 - 4(8\tau^2 - 8\tau + 3)}{(8\tau^2 - 8\tau + 3)^{2.5}} \right] = 24(1-2\tau) \left[\frac{3(1-2\tau)^2 - (8\tau^2 - 8\tau + 3)}{(8\tau^2 - 8\tau + 3)^{2.5}} \right] \\
&= 24(1-2\tau) \left[\frac{-4\tau + 4\tau^2}{(8\tau^2 - 8\tau + 3)^{2.5}} \right] = \frac{96(2\tau - 1)\tau(1-\tau)^{\text{sign}}}{(3 - 8\tau + 8\tau^2)^{2.5}} = \tau - 0.5 \leq 0.
\end{aligned}$$

Dependence in Uniform Distribution:

Use (11) to obtain the moments as

$$\mu_y = 0.25(1+\rho)(0) + 0.25(1-\rho)(0) + 0.25(1-\rho)(0) + 0.25(1+\rho)(1) = 0.25(1+\rho),$$

$$\begin{aligned}
\sigma_y^2 &= 0.25(1+\rho)[0 - 0.25(1+\rho)]^2 + 0.25(1-\rho)[0 - 0.25(1+\rho)]^2 \\
&+ 0.25(1-\rho)[0 - 0.25(1+\rho)]^2 + 0.25(1+\rho)[1 - 0.25(1+\rho)]^2 \\
&= 0.25(3-\rho)[0.25(1+\rho)]^2 + 0.25(1+\rho)[0.75 - 0.25\rho]^2 \\
&= 0.25(1+\rho)(3-\rho) \left\{ (0.25)^2(1+\rho) + (0.25)^2(3-\rho) \right\} \\
&= \frac{(3-\rho)(1+\rho)}{16},
\end{aligned}$$

$$\begin{aligned}
\zeta_y &= 0.25(1+\rho)[0 - 0.25(1+\rho)]^3 + 0.25(1-\rho)[0 - 0.25(1+\rho)]^3 \\
&+ 0.25(1-\rho)[0 - 0.25(1+\rho)]^3 + 0.25(1+\rho)[1 - 0.25(1+\rho)]^3 \\
&= -0.25(3-\rho)[0.25(1+\rho)]^3 + 0.25(1+\rho)[0.75 - 0.25\rho]^3 \\
&= 0.25 \left\{ (1+\rho)(0.25)^3 [3-\rho]^3 - (3-\rho)(0.25)^3 (1+\rho)^3 \right\} \\
&= (0.25)^4 (1+\rho)(3-\rho) \left\{ (3-\rho)^2 - (1+\rho)^2 \right\} \\
&= (0.25)^4 (1+\rho)(3-\rho) \{8 - 8\rho\} = (0.25)^3 (1+\rho)(3-\rho)2(1-\rho) \\
&= (3-\rho)(1+\rho)(1-\rho)/32,
\end{aligned}$$

$$\gamma_y = \frac{(3-\rho)(1+\rho)(1-\rho)/32}{[(3-\rho)(1+\rho)]^{3/2}/(16)^{3/2}} = \frac{2(1-\rho)}{(3-\rho)^{1/2}(1+\rho)^{1/2}}.$$

Use this last expression to establish (12),

$$\begin{aligned}
\frac{d\gamma_y}{d\rho} &= \frac{(3-\rho)^{1/2}(1+\rho)^{1/2}(-2) - 2(1-\rho) \left\{ 0.5 \frac{(3-\rho)^{1/2}}{(1+\rho)^{1/2}} - 0.5 \frac{(1+\rho)^{1/2}}{(3-\rho)^{1/2}} \right\}}{(3-\rho)(1+\rho)} \\
&= \frac{(3-\rho)^{1/2}(1+\rho)^{1/2}(-2) - \frac{2(1-\rho)^2}{(1+\rho)^{1/2}(3-\rho)^{1/2}}}{(3-\rho)(1+\rho)} \\
&= -2 \left\{ \frac{(3-\rho)(1+\rho) + (1-\rho)^2}{(1+\rho)^{3/2}(3-\rho)^{3/2}} \right\} = -\frac{8}{(3-\rho)^{3/2}(1+\rho)^{3/2}} < 0.
\end{aligned}$$

Moments of Distribution with Density:

$$(A1) \quad g(y) = \frac{\text{Cos}(y)[1 - \text{Sin}(y)]}{2}.$$

Integrate

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} yg(y)dy &= 0.5 \int_{-\pi/2}^{\pi/2} y \text{Cos}(y)[1 - \text{Sin}(y)]dy \\
&= \left[0.5 \text{Cos}(x) + 0.125x \text{Cos}(2x) + 0.5x \text{Sin}(x) - 0.0625 \text{Sin}(2x) \right]_{-\pi/2}^{\pi/2} \\
(A2) \quad &= \left[0.125x \text{Cos}(2x) + 0.5x \text{Sin}(x) \right]_{-\pi/2}^{\pi/2} \\
&= 0.125 \left[(\pi/2) \text{Cos}(\pi) + 2\pi \text{Sin}(\pi/2) - (-\pi/2) \text{Cos}(-\pi) - (-2\pi) \text{Sin}(-\pi/2) \right] \\
&= 0.125 \left[(\pi/2)(-1) + 2\pi(1) + (\pi/2)(-1) + 2\pi(-1) \right] = 0.125[-\pi] = -0.3927.
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} y^2 g(y)dy &= 0.5 \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}(y)[1 - \text{Sin}(y)]dy \\
&= \left[x \text{Cos}(x) + 0.0625(2x^2 - 1) \text{Cos}(2x) + 0.5(x^2 - 2) \text{Sin}(x) - 0.125x \text{Sin}(2x) \right]_{-\pi/2}^{\pi/2} \\
&= \left[0.0625(2x^2 - 1) \text{Cos}(2x) + 0.5(x^2 - 2) \text{Sin}(x) \right]_{-\pi/2}^{\pi/2} \\
(A3) \quad &= 0.0625 \left[\begin{aligned} &\left[\left(\frac{\pi^2}{2} - 1 \right) \text{Cos}(\pi) + 8 \left(\frac{\pi^2}{4} - 2 \right) \text{Sin}(\pi/2) \right. \\ &\left. - \left(\frac{\pi^2}{2} - 1 \right) \text{Cos}(-\pi) - 8 \left(\frac{\pi^2}{4} - 2 \right) \text{Sin}(-\pi/2) \right] \end{aligned} \right] \\
&= 0.0625 \left[\left(\frac{\pi^2}{2} - 1 \right) (-1) + 8 \left(\frac{\pi^2}{4} - 2 \right) (1) - \left(\frac{\pi^2}{2} - 1 \right) (-1) - 8 \left(\frac{\pi^2}{4} - 2 \right) (-1) \right] \\
&= 0.5 \left[\left(\frac{\pi^2}{4} - 2 \right) + \left(\frac{\pi^2}{4} - 2 \right) \right] = \left(\frac{\pi^2}{4} - 2 \right) = 0.467397.
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} y^3 g(y) dy &= 0.5 \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}(y) [1 - \text{Sin}(y)] dy \\
&= \left[1.5(x^2 - 2)\text{Cos}(x) + 0.0625x(2x^2 - 3)\text{Cos}(2x) \right. \\
&\quad \left. + 0.5x(x^2 - 6)\text{Sin}(x) - 0.09375(2x^2 - 1)\text{Sin}(2x) \right]_{-\pi/2}^{\pi/2} \\
&= \left[0.0625x(2x^2 - 3)\text{Cos}(2x) + 0.5x(x^2 - 6)\text{Sin}(x) \right]_{-\pi/2}^{\pi/2} \\
\text{(A4)} \quad &= 0.0625 \left[\begin{aligned} &\frac{\pi}{2} \left(\frac{\pi^2}{2} - 3 \right) \text{Cos}(\pi) + 4\pi \left(\frac{\pi^2}{4} - 6 \right) \text{Sin}(\pi/2) \\ &+ \frac{\pi}{2} \left(\frac{\pi^2}{2} - 3 \right) \text{Cos}(-\pi) + 4\pi \left(\frac{\pi^2}{4} - 6 \right) \text{Sin}(-\pi/2) \end{aligned} \right] \\
&= 0.0625 \left[\frac{\pi}{2} \left(\frac{\pi^2}{2} - 3 \right) (-1) + \frac{\pi}{2} \left(\frac{\pi^2}{2} - 3 \right) (-1) \right] \\
&= -0.0625\pi \left(\frac{\pi^2}{2} - 3 \right) = -0.379896.
\end{aligned}$$

$$\begin{aligned}
\gamma_y &= \frac{E[(y - \mu_y)^3]}{\sigma_y^3} = \frac{E[y^3] - 3\mu_y E[y^2] + 2\mu_y^3}{\sigma_y^3} \\
\text{(A5)} \quad &= \frac{-0.3799 - 3(-0.3927)(0.4674) + 2(-0.3927)^3}{[0.4674 - (-0.3927)^2]^{3/2}} = \frac{0.17074 - 2(0.3927)^3}{0.175269} = 0.283.
\end{aligned}$$

Moments of Distribution with Density:

$$\text{(A6)} \quad g(y) = \frac{4\text{Cos}^2(y) \{0.5 - ([y + \text{Cos}(y)\text{Sin}(y)] / \pi)\}}{\pi}.$$

Integrate

$$\begin{aligned}
\text{(A7)} \quad \int_{-\pi/2}^{\pi/2} yg(y) dy &= \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} y \text{Cos}^2(y) \left\{ 0.5 - \frac{[y + \text{Cos}(y)\text{Sin}(y)]}{\pi} \right\} dy \\
&= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} y \text{Cos}^2(y) dy - \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^2(y) dy - \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) \text{Sin}(y) dy.
\end{aligned}$$

Now

$$\text{(A8)} \quad \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} y \text{Cos}^2(y) dy = \frac{1}{4\pi} \left[2x^2 + \text{Cos}(2x) + 2x\text{Sin}(2x) \right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi} [x\text{Sin}(2x)]_{-\pi/2}^{\pi/2} = 0;$$

$$\begin{aligned}
& \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^2(y) dy = \frac{4}{\pi^2} \left[\frac{x^3}{6} + \frac{x \text{Cos}(2x)}{4} + \frac{(2x^2 - 1) \text{Sin}(2x)}{8} \right]_{-\pi/2}^{\pi/2} \\
\text{(A9)} \quad & = \frac{4}{\pi^2} \left[\frac{x^3}{6} + \frac{x \text{Cos}(2x)}{4} \right]_{-\pi/2}^{\pi/2} = \frac{4}{\pi^2} \left[\frac{\pi^3}{24} + \frac{\pi \text{Cos}(\pi)}{8} + \frac{\pi \text{Cos}(-\pi)}{8} \right] \\
& = \frac{4}{\pi^2} \left[\frac{\pi^3}{24} - \frac{\pi}{4} \right] = \frac{\pi^2 - 6}{6\pi}.
\end{aligned}$$

$$\begin{aligned}
& \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) \text{Sin}(y) dy = \frac{4}{\pi^2} \left[\frac{\text{Sin}(4x)}{128} - \frac{x \text{Cos}(4x)}{32} + \frac{\text{Sin}(2x)}{16} - \frac{x \text{Cos}(2x)}{8} \right]_{-\pi/2}^{\pi/2} \\
\text{(A10)} \quad & = -\frac{1}{8\pi^2} [x \text{Cos}(4x) + 4x \text{Cos}(2x)]_{-\pi/2}^{\pi/2} \\
& = -\frac{1}{8\pi^2} \left[\frac{\pi}{2} \text{Cos}(2\pi) + \frac{\pi}{2} \text{Cos}(-2\pi) + 2\pi \text{Cos}(\pi) + 2\pi \text{Cos}(-\pi) \right] \\
& = -\frac{1}{8\pi^2} \left[\frac{\pi}{2} + \frac{\pi}{2} + 2\pi(-1) + 2\pi(-1) \right] = \frac{3}{8\pi}.
\end{aligned}$$

So

$$\begin{aligned}
\text{(A11)} \quad & \int_{-\pi/2}^{\pi/2} yg(y) dy = -\frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^2(y) dy - \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) \text{Sin}(y) dy \\
& = -\frac{\pi^2 - 6}{6\pi} - \frac{3}{8\pi} = \frac{1}{2\pi} \left(\frac{5}{4} - \frac{\pi^2}{3} \right) = -0.3247.
\end{aligned}$$

Also,

$$\begin{aligned}
\text{(A12)} \quad & \int_{-\pi/2}^{\pi/2} y^2 g(y) dy = \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^2(y) \left\{ 0.5 - \frac{[y + \text{Cos}(y) \text{Sin}(y)]}{\pi} \right\} dy \\
& = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^2(y) dy - \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^2(y) dy - \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^3(y) \text{Sin}(y) dy.
\end{aligned}$$

$$\begin{aligned}
& \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^2(y) dy = \frac{2}{\pi} \left[\frac{x^3}{6} + \frac{x \text{Cos}(2x)}{4} + \frac{(2x^2 - 1) \text{Sin}(2x)}{8} \right]_{-\pi/2}^{\pi/2} \\
\text{(A13)} \quad & = \frac{2}{\pi} \left[\frac{x^3}{6} + \frac{x \text{Cos}(2x)}{4} \right]_{-\pi/2}^{\pi/2} = \frac{2}{\pi} \left[\frac{\pi^3}{24} + \frac{\pi \text{Cos}(\pi)}{8} + \frac{\pi \text{Cos}(-\pi)}{8} \right] \\
& = \frac{2}{\pi} \left[\frac{\pi^3}{24} - \frac{\pi}{8} - \frac{\pi}{8} \right] = \frac{\pi^2 - 6}{12}.
\end{aligned}$$

$$\begin{aligned}
\text{(A14)} \quad \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^2(y) dy &= \frac{4}{\pi^2} \left[\frac{x^4}{8} + \frac{3(2x^2 - 1)\text{Cos}(2x)}{16} + \frac{x(2x^2 - 3)\text{Sin}(2x)}{8} \right]_{-\pi/2}^{\pi/2} \\
&= \frac{4}{\pi^2} \left[\frac{3(2x^2 - 1)\text{Cos}(2x)}{16} \right]_{-\pi/2}^{\pi/2} = \frac{3}{4\pi^2} \left[(2x^2 - 1)\text{Cos}(2x) \right]_{-\pi/2}^{\pi/2} = 0.
\end{aligned}$$

$$\begin{aligned}
\text{(A15)} \quad \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^3(y) \text{Sin}(y) dy \\
&= \frac{4}{\pi^2} \left[\frac{x\text{Sin}(2x)}{8} - \frac{(2x^2 - 1)\text{Cos}(2x)}{16} + \frac{x\text{Sin}(4x)}{64} - \frac{(8x^2 - 1)\text{Cos}(4x)}{256} \right]_{-\pi/2}^{\pi/2} \\
&= -\frac{4}{\pi^2} \left[\frac{(2x^2 - 1)\text{Cos}(2x)}{16} + \frac{(8x^2 - 1)\text{Cos}(4x)}{256} \right]_{-\pi/2}^{\pi/2} = 0.
\end{aligned}$$

So

$$\text{(A16)} \quad \int_{-\pi/2}^{\pi/2} y^2 g(y) dy = \frac{\pi^2 - 6}{12} = 0.322466.$$

Also,

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} y^3 g(y) dy &= \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^2(y) \left\{ 0.5 - \frac{[y + \text{Cos}(y)\text{Sin}(y)]}{\pi} \right\} dy \\
\text{(A17)} \quad &= \overbrace{\frac{2}{\pi} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^2(y) dy}^{0, \text{ odd function}} - \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y^4 \text{Cos}^2(y) dy - \frac{4}{\pi^2} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) \text{Sin}(y) dy \\
&= -\frac{4}{\pi^2} \left\{ \int_{-\pi/2}^{\pi/2} y^4 \text{Cos}^2(y) dy + \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) \text{Sin}(y) dy \right\}.
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} y^4 \text{Cos}^2(y) dy &= \left[\frac{x^5}{10} + \frac{x(2x^2 - 3)\text{Cos}(2x)}{4} + \frac{(2x^4 - 6x^2 + 3)\text{Sin}(2x)}{8} \right]_{-\pi/2}^{\pi/2} \\
&= \left[\frac{x^5}{10} + \frac{x(2x^2 - 3)\text{Cos}(2x)}{4} \right]_{-\pi/2}^{\pi/2} \\
\text{(A18)} \quad &= \left[\frac{(\pi/2)^5}{5} + \frac{(\pi/2)(2(\pi/2)^2 - 3)\text{Cos}(\pi)}{4} - \frac{(-\pi/2)(2(\pi/2)^2 - 3)\text{Cos}(-\pi)}{4} \right] \\
&= \frac{(\pi/2)^5}{5} - \frac{(\pi/2)(2(\pi/2)^2 - 3)}{4} - \frac{(\pi/2)(2(\pi/2)^2 - 3)}{4} \\
&= \frac{\pi^5}{160} - \frac{\pi((\pi^2/2) - 3)}{4} = 1.9126149 - 1.51958 = 0.393033.
\end{aligned}$$

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) \text{Sin}(y) dy \\
&= \left[-\frac{x(2x^2-3)\text{Cos}(2x)}{16} + \frac{3(2x^2-1)\text{Sin}(2x)}{32} \right]_{-\pi/2}^{\pi/2} \\
&= \left[-\frac{x(8x^2-3)\text{Cos}(4x)}{256} + \frac{3(8x^2-1)\text{Sin}(4x)}{128(8)} \right]_{-\pi/2}^{\pi/2} \\
&= - \left[\frac{x(2x^2-3)\text{Cos}(2x)}{16} + \frac{x(8x^2-3)\text{Cos}(4x)}{256} \right]_{-\pi/2}^{\pi/2} \\
&= - \left[\frac{(\pi/2)(2(\pi/2)^2-3)\text{Cos}(\pi)}{16} + \frac{(\pi/2)(8(\pi/2)^2-3)\text{Cos}(2\pi)}{256} \right. \\
&\quad \left. - \frac{(-\pi/2)(2(\pi/2)^2-3)\text{Cos}(-\pi)}{16} - \frac{(-\pi/2)(8(\pi/2)^2-3)\text{Cos}(-2\pi)}{256} \right] \\
&= - \left[\frac{(\pi/2)(2(\pi/2)^2-3)(-1)}{16} + \frac{(\pi/2)(8(\pi/2)^2-3)}{256} \right. \\
&\quad \left. - \frac{(-\pi/2)(2(\pi/2)^2-3)(-1)}{16} - \frac{(-\pi/2)(8(\pi/2)^2-3)}{256} \right] \\
\text{(A19)} &= - \left[-\frac{\pi(2(\pi/2)^2-3)}{16} + \frac{\pi(8(\pi/2)^2-3)}{256} \right] = \frac{\pi(2(\pi/2)^2-3)}{16} - \frac{\pi(8(\pi/2)^2-3)}{256} \\
&= \frac{\pi(\pi^2/2-3)}{16} - \frac{\pi(2\pi^2-3)}{256} = 0.379895 - 0.20542 = 0.17447.
\end{aligned}$$

So

$$\text{(A20)} \quad \int_{-\pi/2}^{\pi/2} y^3 g(y) dy = -\frac{4}{\pi^2} \{0.393033 + 0.17447\} = -\frac{2.270012}{\pi^2} = -0.23000691.$$

$$\begin{aligned}
\gamma_y &= \frac{E[(y - \mu_y)^3]}{\sigma_y^3} = \frac{E[y^3] - 3\mu_y E[y^2] + 2\mu_y^3}{\sigma_y^3} \\
\text{(A21)} &= \frac{-0.23000691 - 3(-0.3247)(0.322466) + 2(-0.3247)^3}{\left[(0.322466) - (-0.3247)^2 \right]^{3/2}} \\
&= \frac{0.08410722 - 2(0.3247)^3}{0.10111076} = 0.15624 > 0.
\end{aligned}$$

Moments of Distribution with Density $0.75\text{Cos}^3(\varepsilon_i)$:

When the RA density function is $0.75\text{Cos}^3(\varepsilon_i)$, then the cumulative distribution function is

$0.5 + (9/16)\text{Sin}(\varepsilon_i) + (1/16)\text{Sin}(3\varepsilon_i)$, and yield density is

$$(A22) \quad g(y) = \frac{3}{2} \text{Cos}^3(y) \left[0.5 - \frac{9}{16} \text{Sin}(y) - \frac{1}{16} \text{Sin}(3y) \right].$$

Integrate

$$(A23) \quad \begin{aligned} \int_{-\pi/2}^{\pi/2} yg(y)dy &= \frac{3}{2} \int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) \left[0.5 - \frac{9}{16} \text{Sin}(y) - \frac{1}{16} \text{Sin}(3y) \right] dy \\ &= \frac{3}{4} \overbrace{\int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) dy}^{0, \text{ odd function}} - \frac{27}{32} \int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) \text{Sin}(y) dy - \frac{3}{32} \int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) \text{Sin}(3y) dy \\ &= -\frac{27}{32} \int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) \text{Sin}(y) dy - \frac{3}{32} \int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) \text{Sin}(3y) dy. \end{aligned}$$

$$(A24) \quad \begin{aligned} \int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) \text{Sin}(y) dy &= \left[\frac{\text{Sin}(2x)}{16} - \frac{x \text{Cos}(2x)}{8} + \frac{\text{Sin}(4x)}{8(16)} - \frac{x \text{Cos}(4x)}{32} \right]_{-\pi/2}^{\pi/2} \\ &= - \left[\frac{x \text{Cos}(2x)}{8} + \frac{x \text{Cos}(4x)}{32} \right]_{-\pi/2}^{\pi/2} \\ &= - \left[\frac{(\pi/2) \text{Cos}(\pi)}{8} + \frac{(\pi/2) \text{Cos}(2\pi)}{32} - \frac{(-\pi/2) \text{Cos}(-\pi)}{8} - \frac{(-\pi/2) \text{Cos}(-2\pi)}{32} \right] \\ &= - \left[-\frac{(\pi/2)}{8} + \frac{(\pi/2)}{32} - \frac{(\pi/2)}{8} + \frac{(\pi/2)}{32} \right] = - \left[\frac{\pi}{32} - \frac{\pi}{8} \right] = \frac{3\pi}{32}. \end{aligned}$$

$$(A25) \quad \begin{aligned} \int_{-\pi/2}^{\pi/2} y \text{Cos}^3(y) \text{Sin}(3y) dy &= \left[\frac{3 \text{Sin}(2x)}{32} - \frac{3x \text{Cos}(2x)}{16} + \frac{3 \text{Sin}(4x)}{8(16)} - \frac{3x \text{Cos}(4x)}{32} + \frac{\text{Sin}(6x)}{8(36)} - \frac{x \text{Cos}(6x)}{48} \right]_{-\pi/2}^{\pi/2} \\ &= -\frac{1}{48} \left[9x \text{Cos}(2x) + \frac{9x \text{Cos}(4x)}{2} + x \text{Cos}(6x) \right]_{-\pi/2}^{\pi/2} \\ &= -\frac{1}{48} \left[\frac{9\pi}{2} \text{Cos}(\pi) + \frac{9\pi}{4} \text{Cos}(2\pi) + \frac{\pi}{2} \text{Cos}(3\pi) \right. \\ &\quad \left. + \frac{9\pi}{2} \text{Cos}(-\pi) + \frac{9\pi}{4} \text{Cos}(-2\pi) + \frac{\pi}{2} \text{Cos}(-3\pi) \right] \\ &= -\frac{1}{48} \left[-\frac{9\pi}{2} + \frac{9\pi}{4} - \frac{\pi}{2} - \frac{9\pi}{4} + \frac{9\pi}{4} - \frac{\pi}{2} \right] = -\frac{1}{48} \left[-9\pi + \frac{9\pi}{2} - \pi \right] = \frac{11\pi}{96}. \end{aligned}$$

So

$$(A26) \quad \int_{-\pi/2}^{\pi/2} yg(y)dy = -\frac{27}{32} \left(\frac{3\pi}{32} \right) - \frac{3}{32} \left(\frac{11\pi}{96} \right) = - \left[27 + \frac{11}{3} \right] \frac{3\pi}{(32)^2} = -0.28225.$$

Also,

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} y^2 g(y) dy &= \frac{3}{2} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^3(y) \left[0.5 - \frac{9}{16} \text{Sin}(y) - \frac{1}{16} \text{Sin}(3y) \right] dy \\
&= \frac{3}{4} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^3(y) dy - \frac{27}{32} \overbrace{\int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^3(y) \text{Sin}(y) dy}^{0, \text{ odd function}} - \frac{3}{32} \overbrace{\int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^3(y) \text{Sin}(3y) dy}^{0, \text{ odd function}} \\
&= \frac{3}{4} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^3(y) dy \\
&= \frac{3}{4} \left[\frac{3x \text{Cos}(x)}{2} + \frac{x \text{Cos}(3x)}{18} + \frac{3(x^2 - 2) \text{Sin}(x)}{4} + \frac{(9x^2 - 2) \text{Sin}(3x)}{108} \right]_{-\pi/2}^{\pi/2} \\
&= \frac{3}{4} \left[\frac{3(x^2 - 2) \text{Sin}(x)}{4} + \frac{(9x^2 - 2) \text{Sin}(3x)}{108} \right]_{-\pi/2}^{\pi/2} \\
&= \frac{3}{4} \left[\frac{3((\pi/2)^2 - 2)}{2} - \frac{(9(\pi/2)^2 - 2)}{54} \right] = \frac{3}{4} \left[\frac{3(\pi/2)^2}{2} - 3 - \frac{9(\pi/2)^2}{54} + \frac{2}{54} \right] \\
\text{(A27)} \quad &= \frac{3}{4} \left[\frac{4(\pi/2)^2}{3} - \frac{80}{27} \right] = (\pi/2)^2 - \frac{20}{9} = 0.245174709.
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} y^3 g(y) dy &= \frac{3}{2} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) \left[0.5 - \frac{9}{16} \text{Sin}(y) - \frac{1}{16} \text{Sin}(3y) \right] dy \\
\text{(A28)} \quad &= \frac{3}{4} \overbrace{\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) dy}^{0, \text{ odd function}} - \frac{27}{32} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) \text{Sin}(y) dy - \frac{3}{32} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) \text{Sin}(3y) dy \\
&= -\frac{27}{32} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) \text{Sin}(y) dy - \frac{3}{32} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) \text{Sin}(3y) dy.
\end{aligned}$$

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) \text{Sin}(y) dy \\
&= \left[\frac{3(2x^2 - 1)\text{Sin}(2x)}{32} - \frac{x(2x^2 - 3)\text{Cos}(2x)}{16} + \frac{3(8x^2 - 1)\text{Sin}(4x)}{8(128)} - \frac{x(8x^2 - 3)\text{Cos}(4x)}{8(32)} \right]_{-\pi/2}^{\pi/2} \\
&= - \left[\frac{x(2x^2 - 3)\text{Cos}(2x)}{16} + \frac{x(8x^2 - 3)\text{Cos}(4x)}{8(32)} \right]_{-\pi/2}^{\pi/2} \\
\text{(A29)} \quad &= - \left[\frac{(\pi/2)(2(\pi/2)^2 - 3)}{16} + \frac{(\pi/2)(8(\pi/2)^2 - 3)}{8(32)} \right] \\
&\quad + \left[\frac{(-\pi/2)(2(\pi/2)^2 - 3)}{16} - \frac{(-\pi/2)(8(\pi/2)^2 - 3)}{8(32)} \right] \\
&= - \left[-\frac{\pi(2(\pi/2)^2 - 3)}{16} + \frac{\pi(8(\pi/2)^2 - 3)}{8(32)} \right] = \frac{\pi}{16} \left[2(\pi/2)^2 - 3 - \frac{(8(\pi/2)^2 - 3)}{16} \right] \\
&= \frac{\pi}{16} \left[2(\pi/2)^2 - 3 - \frac{(\pi/2)^2}{2} + \frac{3}{16} \right] = \frac{\pi}{16} \left[\frac{3(\pi/2)^2}{2} - \frac{45}{16} \right] = 0.174475.
\end{aligned}$$

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^3(y) \text{Sin}(3y) dy \\
&= \left[\frac{9(2x^2 - 1)\text{Sin}(2x)}{64} - \frac{3x(2x^2 - 3)\text{Cos}(2x)}{32} \right]_{-\pi/2}^{\pi/2} \\
&\quad + \left[\frac{9(8x^2 - 1)\text{Sin}(4x)}{8(128)} - \frac{3x(8x^2 - 3)\text{Cos}(4x)}{8(32)} \right]_{-\pi/2}^{\pi/2} \\
&\quad + \left[\frac{(18x^2 - 1)\text{Sin}(6x)}{8(216)} - \frac{x(6x^2 - 1)\text{Cos}(6x)}{8(36)} \right]_{-\pi/2}^{\pi/2} \\
&= - \left[\frac{3x(2x^2 - 3)\text{Cos}(2x)}{32} \right]_{-\pi/2}^{\pi/2} - \left[\frac{3x(8x^2 - 3)\text{Cos}(4x)}{8(32)} \right]_{-\pi/2}^{\pi/2} - \left[\frac{x(6x^2 - 1)\text{Cos}(6x)}{8(36)} \right]_{-\pi/2}^{\pi/2} \\
&= - \left[-\frac{3(\pi/2)(2(\pi/2)^2 - 3)}{32} + \frac{3(-\pi/2)(2(\pi/2)^2 - 3)}{32} \right] \\
&\quad - \left[\frac{3(\pi/2)(8(\pi/2)^2 - 3)}{8(32)} - \frac{3(-\pi/2)(8(\pi/2)^2 - 3)}{8(32)} \right] \\
&\quad - \left[-\frac{(\pi/2)(6(\pi/2)^2 - 1)}{8(36)} + \frac{(-\pi/2)(6(\pi/2)^2 - 1)}{8(36)} \right] \\
\text{(A30)} \quad &= \frac{3\pi(2(\pi/2)^2 - 3)}{32} - \frac{3\pi(8(\pi/2)^2 - 3)}{8(32)} + \frac{\pi(6(\pi/2)^2 - 1)}{8(36)} \\
&= 0.569843 - 0.61626 + 0.1505823 = 0.1041653.
\end{aligned}$$

So

$$(A31) \quad \int_{-\pi/2}^{\pi/2} y^3 g(y) dy = -\frac{27}{32}(0.174475) - \frac{3}{32}(0.1041653) = -0.147213281 - 0.0097655 \\ = -0.156978781.$$

Finally,

$$(A32) \quad \gamma_y = \frac{E[(y - \mu_y)^3]}{\sigma_y^3} = \frac{E[y^3] - 3\mu_y E[y^2] + 2\mu_y^3}{\sigma_y^3} \\ = \frac{-0.156978781 - 3(-0.28225)(0.2451747) + 2(-0.28225)^3}{[(0.2451747) - (-0.28225)^2]^{3/2}} \\ = \frac{0.050622307 - 2(0.28225)^3}{0.0673312} = 0.083934 > 0.$$

Moments of Distribution with Density $[8/(3\pi)]\text{Cos}^4(\varepsilon_i)$:

When the RA density function is $[8/(3\pi)]\text{Cos}^4(\varepsilon_i)$, then the cumulative distribution function is

$0.5 + (\varepsilon_i / \pi) + [2/(3\pi)]\text{Sin}(2\varepsilon_i) + [1/(12\pi)]\text{Sin}(4\varepsilon_i)$, and yield density is

$$(A33) \quad g(y) = \frac{16}{3\pi} \text{Cos}^4(y) \left[0.5 - \frac{y}{\pi} - \frac{2}{3\pi} \text{Sin}(2y) - \frac{1}{12\pi} \text{Sin}(4y) \right].$$

Integrate

$$(A34) \quad \int_{-\pi/2}^{\pi/2} yg(y) dy = \frac{16}{3\pi} \int_{-\pi/2}^{\pi/2} y \text{Cos}^4(y) \left[0.5 - \frac{y}{\pi} - \frac{2}{3\pi} \text{Sin}(2y) - \frac{1}{12\pi} \text{Sin}(4y) \right] dy \\ = \frac{8}{3\pi} \overbrace{\int_{-\pi/2}^{\pi/2} y \text{Cos}^4(y) dy}^{0, \text{ odd function}} - \frac{16}{3\pi^2} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y) dy - \frac{32}{9\pi^2} \int_{-\pi/2}^{\pi/2} y \text{Cos}^4(y) \text{Sin}(2y) dy \\ - \frac{4}{9\pi^2} \int_{-\pi/2}^{\pi/2} y \text{Cos}^4(y) \text{Sin}(4y) dy \\ = -\frac{16}{3\pi^2} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y) dy - \frac{32}{9\pi^2} \int_{-\pi/2}^{\pi/2} y \text{Cos}^4(y) \text{Sin}(2y) dy \\ - \frac{4}{9\pi^2} \int_{-\pi/2}^{\pi/2} y \text{Cos}^4(y) \text{Sin}(4y) dy.$$

Integrate

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y) dy \\
&= \left[\frac{x^3}{8} + \frac{x \text{Cos}(2x)}{4} + \frac{x \text{Cos}(4x)}{64} + \frac{(2x^2 - 1) \text{Sin}(2x)}{8} + \frac{(8x^2 - 1) \text{Sin}(4x)}{256} \right]_{-\pi/2}^{\pi/2} \\
&= \left[\frac{x^3}{8} + \frac{x \text{Cos}(2x)}{4} + \frac{x \text{Cos}(4x)}{64} \right]_{-\pi/2}^{\pi/2} \\
\text{(A35)} \quad &= \frac{(\pi/2)^3}{8} + \frac{(\pi/2)(-1)}{4} + \frac{(\pi/2)(1)}{64} - \frac{(-\pi/2)^3}{8} - \frac{(-\pi/2)(-1)}{4} - \frac{(-\pi/2)(1)}{64} \\
&= \frac{2(\pi/2)^3}{8} + \frac{2(\pi/2)(-1)}{4} + \frac{2(\pi/2)(1)}{64} = \frac{\pi^3}{32} - \frac{\pi}{4} + \frac{\pi}{64} \\
&= \frac{\pi}{32} (\pi^2 - 7.5) = 0.2326335.
\end{aligned}$$

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} y \text{Cos}^4(y) \text{Sin}(2y) dy \\
&= 2 \left[\frac{5 \text{Sin}(2x)}{128} - \frac{5x \text{Cos}(2x)}{64} + \frac{\text{Sin}(4x)}{8(16)} - \frac{x \text{Cos}(4x)}{32} + \frac{\text{Sin}(6x)}{32(36)} - \frac{x \text{Cos}(6x)}{32(6)} \right]_{-\pi/2}^{\pi/2} \\
\text{(A36)} \quad &= -2 \left[\frac{5x \text{Cos}(2x)}{64} + \frac{x \text{Cos}(4x)}{32} + \frac{x \text{Cos}(6x)}{32(6)} \right]_{-\pi/2}^{\pi/2} \\
&= -2 \left[\frac{5(\pi/2)(-1)}{64} + \frac{(\pi/2)(1)}{32} + \frac{(\pi/2)(-1)}{32(6)} - \frac{5(-\pi/2)(-1)}{64} - \frac{(-\pi/2)(1)}{32} - \frac{(-\pi/2)(-1)}{32(6)} \right] \\
&= -2 \left[-\frac{5\pi}{64} + \frac{\pi}{32} - \frac{\pi}{32(6)} \right] = -\frac{2\pi}{32} \left[-\frac{5}{2} + 1 - \frac{1}{6} \right] = \frac{5\pi}{48}.
\end{aligned}$$

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} y \cos^4(y) \sin(4y) dy \\
&= \left[\frac{\sin(2x)}{16} - \frac{x \cos(2x)}{8} + \frac{3 \sin(4x)}{8(16)} - \frac{3x \cos(4x)}{32} \right]^{\pi/2} \\
& \quad + \left[\frac{\sin(6x)}{4(36)} - \frac{x \cos(6x)}{24} + \frac{\sin(8x)}{16(64)} - \frac{x \cos(8x)}{16(8)} \right]_{-\pi/2}^{\pi/2} \\
&= - \left[\frac{x \cos(2x)}{8} + \frac{3x \cos(4x)}{32} + \frac{x \cos(6x)}{24} + \frac{x \cos(8x)}{16(8)} \right]_{-\pi/2}^{\pi/2} \\
&= - \left[\frac{(\pi/2) \cos(\pi)}{8} + \frac{3(\pi/2) \cos(2\pi)}{32} + \frac{(\pi/2) \cos(3\pi)}{24} + \frac{(\pi/2) \cos(4\pi)}{16(8)} \right. \\
& \quad \left. + \frac{(\pi/2) \cos(-\pi)}{8} + \frac{3(\pi/2) \cos(-2\pi)}{32} + \frac{(\pi/2) \cos(-3\pi)}{24} + \frac{(\pi/2) \cos(-4\pi)}{16(8)} \right] \\
&= - \left[\frac{(\pi/2)(-1)}{8} + \frac{3(\pi/2)(1)}{32} + \frac{(\pi/2)(-1)}{24} + \frac{(\pi/2)(1)}{16(8)} \right. \\
& \quad \left. + \frac{(\pi/2)(-1)}{8} + \frac{3(\pi/2)(1)}{32} + \frac{(\pi/2)(-1)}{24} + \frac{(\pi/2)(1)}{16(8)} \right] \\
(A37) \quad &= - \left[-\frac{\pi}{8} + \frac{3\pi}{32} - \frac{\pi}{24} + \frac{\pi}{16(8)} \right] = \pi \left[\frac{1}{32} + \frac{1}{24} - \frac{1}{16(8)} \right] = \frac{\pi}{8} \left[\frac{3}{16} + \frac{1}{3} \right] = \frac{5\pi}{192}.
\end{aligned}$$

So

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} yg(y) dy = -\frac{16}{3\pi^2} \int_{-\pi/2}^{\pi/2} y^2 \cos^4(y) dy - \frac{32}{9\pi^2} \int_{-\pi/2}^{\pi/2} y \cos^4(y) \sin(2y) dy \\
& \quad - \frac{4}{9\pi^2} \int_{-\pi/2}^{\pi/2} y \cos^4(y) \sin(4y) dy \\
(A38) \quad &= -\frac{16}{3\pi^2} () - \frac{32}{9\pi^2} () - \frac{4}{9\pi^2} () = -\frac{4}{9\pi^2} \left[12() + 8() + () \right] \\
&= -\frac{4}{9\pi^2} \left[12(0.2326335) + 8\left(\frac{5\pi}{48}\right) + \left(\frac{5\pi}{192}\right) \right] \\
&= -\frac{4}{9\pi^2} [2.791602 + 2.617991667 + 0.081812239] = -0.247287416.
\end{aligned}$$

Integrate

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} y^2 g(y) dy &= \frac{16}{3\pi} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y) \left[0.5 - \frac{y}{\pi} - \frac{2}{3\pi} \text{Sin}(2y) - \frac{1}{12\pi} \text{Sin}(4y) \right] dy \\
&= \frac{8}{3\pi} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y) dy - \frac{16}{3\pi^2} \overbrace{\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y) dy}^{0, \text{ odd function}} - \frac{32}{9\pi^2} \overbrace{\int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y) \text{Sin}(2y) dy}^{0, \text{ odd function}} \\
&\quad - \frac{4}{9\pi^2} \overbrace{\int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y) \text{Sin}(4y) dy}^{0, \text{ odd function}} \\
&= \frac{8}{3\pi} \int_{-\pi/2}^{\pi/2} y^2 \text{Cos}^4(y) dy = \frac{8(0.2326335)}{3\pi} = 0.197465614.
\end{aligned}
\tag{A39}$$

And

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} y^3 g(y) dy &= \frac{16}{3\pi} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y) \left[0.5 - \frac{y}{\pi} - \frac{2}{3\pi} \text{Sin}(2y) - \frac{1}{12\pi} \text{Sin}(4y) \right] dy \\
&= \frac{8}{3\pi} \overbrace{\int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y) dy}^{0, \text{ odd function}} - \frac{16}{3\pi^2} \int_{-\pi/2}^{\pi/2} y^4 \text{Cos}^4(y) dy - \frac{32}{9\pi^2} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y) \text{Sin}(2y) dy \\
&\quad - \frac{4}{9\pi^2} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y) \text{Sin}(4y) dy \\
&= -\frac{16}{3\pi^2} \int_{-\pi/2}^{\pi/2} y^4 \text{Cos}^4(y) dy - \frac{32}{9\pi^2} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y) \text{Sin}(2y) dy \\
&\quad - \frac{4}{9\pi^2} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y) \text{Sin}(4y) dy.
\end{aligned}
\tag{A40}$$

Now

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} y^4 \text{Cos}^4(y) dy &= \left[\frac{3x^5}{40} + \frac{(2x^2 - 3)x \text{Cos}(2x)}{4} + \frac{(8x^2 - 3)x \text{Cos}(4x)}{256} \right. \\
&\quad \left. + \frac{(2x^4 - 6x^2 + 3)\text{Sin}(2x)}{8} + \frac{(32x^4 - 24x^2 + 3)\text{Sin}(4x)}{1024} \right]_{-\pi/2}^{\pi/2} \\
&= \left[\frac{3x^5}{40} + \frac{(2x^2 - 3)x \text{Cos}(2x)}{4} + \frac{(8x^2 - 3)x \text{Cos}(4x)}{256} \right]_{-\pi/2}^{\pi/2} \\
&= \frac{3(\pi/2)^5}{40} + \frac{(2(\pi/2)^2 - 3)(\pi/2)\text{Cos}(2x)}{4} + \frac{(8(\pi/2)^2 - 3)(\pi/2)\text{Cos}(4x)}{256} \\
&\quad + \frac{3(\pi/2)^5}{40} - \frac{(2(\pi/2)^2 - 3)(-\pi/2)\text{Cos}(2x)}{4} - \frac{(8(\pi/2)^2 - 3)(-\pi/2)\text{Cos}(4x)}{256} \\
&= \frac{3(\pi/2)^5}{20} + \frac{(2(\pi/2)^2 - 3)(\pi/2)(-1)}{4} + \frac{(8(\pi/2)^2 - 3)(\pi/2)}{256} \\
&\quad + \frac{(2(\pi/2)^2 - 3)(\pi/2)(-1)}{4} + \frac{(8(\pi/2)^2 - 3)(\pi/2)}{256} \\
&= \frac{3(\pi/2)^5}{20} - \frac{(2(\pi/2)^2 - 3)\pi}{4} + \frac{(8(\pi/2)^2 - 3)\pi}{256} \\
\text{(A41)} \quad &= 1.434461214 - 1.519582264 + 0.205420414 = 0.120299364.
\end{aligned}$$

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y) \text{Sin}(2y) dy = 2 \left[\frac{15(2x^2 - 1) \text{Sin}(2x)}{8(32)} - \frac{5x(2x^2 - 3) \text{Cos}(2x)}{128} \right]_{-\pi/2}^{\pi/2} \\
& + 2 \left[\frac{3(8x^2 - 1) \text{Sin}(4x)}{8(128)} - \frac{x(8x^2 - 3) \text{Cos}(4x)}{8(32)} \right]_{-\pi/2}^{\pi/2} \\
& + 2 \left[\frac{(18x^2 - 1) \text{Sin}(6x)}{216(32)} - \frac{x(6x^2 - 1) \text{Cos}(6x)}{36(32)} \right]_{-\pi/2}^{\pi/2} \\
& = 2 \left[-\frac{5x(2x^2 - 3) \text{Cos}(2x)}{128} \right]_{-\pi/2}^{\pi/2} + 2 \left[-\frac{x(8x^2 - 3) \text{Cos}(4x)}{8(32)} \right]_{-\pi/2}^{\pi/2} \\
& + 2 \left[-\frac{x(6x^2 - 1) \text{Cos}(6x)}{36(32)} \right]_{-\pi/2}^{\pi/2} \\
& = 2 \left[-\frac{5(\pi/2)(2(\pi/2)^2 - 3)(-1)}{128} - \frac{5(\pi/2)(2(\pi/2)^2 - 3)(-1)}{128} \right] \\
& + 2 \left[-\frac{(\pi/2)(8(\pi/2)^2 - 3)(1)}{8(32)} - \frac{(\pi/2)(8(\pi/2)^2 - 3)(1)}{8(32)} \right] \\
& + 2 \left[-\frac{(\pi/2)(6(\pi/2)^2 - 1)(-1)}{36(32)} - \frac{(\pi/2)(6(\pi/2)^2 - 1)(-1)}{36(32)} \right] \\
\text{(A42)} \quad & = 2 \left[\frac{5\pi(2(\pi/2)^2 - 3)}{128} \right] - \frac{\pi(8(\pi/2)^2 - 3)}{4(32)} + \frac{\pi(6(\pi/2)^2 - 1)}{18(32)} \\
& = 0.474869457 - 0.410840829 + 0.075291158 = 0.139319786.
\end{aligned}$$

$$\begin{aligned}
\int_{-\pi/2}^{\pi/2} y^3 \cos^4(y) \sin(4y) dy &= \left[\frac{3(2x^2 - 1)\sin(2x)}{32} - \frac{x(2x^2 - 3)\cos(2x)}{16} \right]_{-\pi/2}^{\pi/2} \\
&+ \left[\frac{9(8x^2 - 1)\sin(4x)}{8(128)} - \frac{3x(8x^2 - 3)\cos(4x)}{8(32)} \right]_{-\pi/2}^{\pi/2} \\
&+ \left[\frac{(18x^2 - 1)\sin(6x)}{4(216)} - \frac{x(6x^2 - 1)\cos(6x)}{4(36)} \right]_{-\pi/2}^{\pi/2} \\
&+ \left[\frac{3(32x^2 - 1)\sin(8x)}{16(2048)} - \frac{x(32x^2 - 3)\cos(8x)}{16(256)} \right]_{-\pi/2}^{\pi/2} \\
&= - \left[\frac{x(2x^2 - 3)\cos(2x)}{16} \right]_{-\pi/2}^{\pi/2} - \left[\frac{3x(8x^2 - 3)\cos(4x)}{8(32)} \right]_{-\pi/2}^{\pi/2} \\
&- \left[\frac{x(6x^2 - 1)\cos(6x)}{4(36)} \right]_{-\pi/2}^{\pi/2} - \left[\frac{x(32x^2 - 3)\cos(8x)}{16(256)} \right]_{-\pi/2}^{\pi/2} \\
&= - \left[\frac{(\pi/2)(2(\pi/2)^2 - 3)(-1)}{16} + \frac{(\pi/2)(2(\pi/2)^2 - 3)(-1)}{16} \right] \\
&- \left[\frac{3(\pi/2)(8(\pi/2)^2 - 3)(1)}{8(32)} + \frac{3(\pi/2)(8(\pi/2)^2 - 3)(1)}{8(32)} \right] \\
&- \left[\frac{(\pi/2)(6(\pi/2)^2 - 1)(-1)}{4(36)} + \frac{(\pi/2)(6(\pi/2)^2 - 1)(-1)}{4(36)} \right] \\
&- \left[\frac{(\pi/2)(32(\pi/2)^2 - 3)(1)}{16(256)} + \frac{(\pi/2)(32(\pi/2)^2 - 3)(1)}{16(256)} \right] \\
\text{(A43)} \quad &= \frac{\pi(2(\pi/2)^2 - 3)}{16} - \frac{3\pi(8(\pi/2)^2 - 3)}{8(32)} + \frac{\pi(6(\pi/2)^2 - 1)}{4(36)} - \frac{\pi(32(\pi/2)^2 - 3)}{16(256)} \\
&= \frac{\pi}{16} \left\{ (2(\pi/2)^2 - 3) - \frac{3(8(\pi/2)^2 - 3)}{16} + \frac{(6(\pi/2)^2 - 1)}{9} - \frac{(32(\pi/2)^2 - 3)}{256} \right\} \\
&= \frac{\pi}{16} \{ 1.934793864 - 3.138595398 + 1.533820177 - 0.296705866 \} = 0.0065409428.
\end{aligned}$$

So

$$\begin{aligned}
& \int_{-\pi/2}^{\pi/2} y^3 g(y) dy = -\frac{16}{3\pi^2} \int_{-\pi/2}^{\pi/2} y^4 \text{Cos}^4(y) dy - \frac{32}{9\pi^2} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y) \text{Sin}(2y) dy \\
& - \frac{4}{9\pi^2} \int_{-\pi/2}^{\pi/2} y^3 \text{Cos}^4(y) \text{Sin}(4y) dy \\
\text{(A44)} \quad & = -\left[\frac{16}{3\pi^2} (0.120299364) + \frac{32}{9\pi^2} (0.139319786) + \frac{4}{9\pi^2} (0.0065409428) \right] \\
& = -\frac{4}{9\pi^2} [12(0.120299364) + 8(0.139319786) + (0.0065409428)] \\
& = -\frac{4}{9\pi^2} [1.443592368 + 1.114558288 + 0.0065409428] = -\frac{4(2.564691598)}{9\pi^2} \\
& = -0.115492456.
\end{aligned}$$

Finally,

$$\begin{aligned}
\gamma_y &= \frac{E[(y - \mu_y)^3]}{\sigma_y^3} = \frac{E[y^3] - 3\mu_y E[y^2] + 2\mu_y^3}{\sigma_y^3} \\
\text{(A45)} \quad &= \frac{-0.115492456 - 3(-0.247287416)(0.197465614) + 2(-0.247287416)^3}{\left[(0.197465614) - (-0.247287416)^2 \right]^{3/2}} \\
&= \frac{0.030999828 - 2(0.247287416)^3}{0.050328421} = 0.015022322 > 0.
\end{aligned}$$

Demonstrating eqn. (21):

$$\begin{aligned}
\gamma_y &= \frac{E[y^3 | z] - 3E[y | z]E[y^2 | z] + 2(E[y | z])^3}{\left(E[y^2 | z] - (E[y | z])^2 \right)^{1.5}}; \\
&= \frac{\frac{1+3z^4}{4} - 3\frac{1+z^2}{2}\frac{1+2z^3}{3} + 2\left(\frac{1+z^2}{2}\right)^3}{\left(\frac{1+2z^3}{3} - \left(\frac{1+z^2}{2}\right)^2 \right)^{1.5}} \\
\text{(A46)} \quad &= \frac{\frac{1+3z^4}{4} - \frac{(1+z^2)(1+2z^3)}{2} + \frac{(1+z^2)^3}{4}}{\left(\frac{1+2z^3}{3} - \frac{(1+z^2)^2}{4} \right)^{1.5}} \\
&= \frac{(1-4z+6z^2-4z^3+z^4)z^2 6\sqrt{3}}{(1-6z^2+8z^3-3z^4)^{1.5}}.
\end{aligned}$$

Demonstrating eqn. (22):

$$\begin{aligned}
 \gamma_y &= \frac{\frac{1+3\lambda z^4}{4} - \frac{(1+\lambda z^2)(1+2\lambda z^3)}{2} + \frac{(1+\lambda z^2)^3}{4}}{\left(\frac{1+2\lambda z^3}{3} - \left(\frac{1+\lambda z^2}{2}\right)^2\right)^{1.5}} \\
 \text{(A47)} &= \frac{\lambda z^2 - 4\lambda z^3 + 3\lambda^2 z^4 + 3\lambda z^4 - 4\lambda^2 z^5 + \lambda^3 z^6}{4\left(\frac{1+8\lambda z^3 - 6\lambda z^2 - 3\lambda^2 z^4}{12}\right)^{1.5}} \\
 &= 6\sqrt{3}\lambda z^2 \frac{(1-4z + 3\lambda z^2 + 3z^2 - 4\lambda z^3 + \lambda^2 z^4)}{(1+8\lambda z^3 - 6\lambda z^2 - 3\lambda^2 z^4)^{1.5}}.
 \end{aligned}$$