Fuzzy Logic and Compromise Programming in Portfolio Management

By

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Abstract: The objective of this paper is to develop a portfolio optimization technique that is simple enough for an individual with little knowledge of economic theory to systematically determine his own optimized portfolio. A compromise programming approach and a fuzzy logic approach are developed as alternatives to the traditional EV model.

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Introduction

The objective of this paper is to develop a portfolio optimization technique that is simple enough for an average individual with little or no knowledge of economic theory to systematically determine their own optimized portfolio. Such a technique would allow farmers to chose their own asset allocation or small investors to interactively determine with the portfolio that is optimal for them. Ultimately, such a technique, packaged into an ergonomic software, would be a way to familiarize individuals with the risk/return trade-off as well as shift part of the risk involved with choosing an optimal portfolio to individual investors. Shifting the decision process to consumers is consistent with marketing concepts such as one-on-one marketing and product customization.

In order for such a technique to be successful, it has to be based on a very simple concept. Portfolio optimization using a compromise programming approach relies on the simple concept of selecting a portfolio by attributing weights to different conflicting objectives of the user, in order to select the most attractive portfolio for an individual among a set of efficient portfolios. The literature on decision-making processes strongly argue that individuals make decisions by making compromises of competing objectives, which makes compromise programming a natural and intuitive way to choose an optimal portfolio. Another alternative is to develop a fuzzy logic approach, that would allow the processing of words describing an ideal portfolio into fuzzy constraints that could then be used to solve a fuzzy optimization problem resulting in the portfolio that most closely matches the linguistic description. This approach is more intuitive than the compromise programming approach since coefficients are not needed from the individual. Both approaches would be implemented in an interactive environment to make the portfolio selection even more consistent with the human decision making-process[1].

Although compromise programming and fuzzy logic have been widely applied in many operational research and industrial control problems, it is still relatively unknown in economics (Ballestero and Romero, 1991). Therefore, the first two sections present an overview of compromise programming and fuzzy logic, respectively. In the remaining of the paper, both techniques are implemented to develop portfolio optimization models, a formal relationship between the weights of compromise programming and the risk aversion coefficient is derived, and a simple comparative numerical example is provided.

The Compromise Programming Approach

Compromise programming is a linear multiobjective programming (MOP) technique originally introduced by Zeleny (1974; Page 167-182). This technique allows one to find the complete set of efficient solutions from simultaneous optimization of two or more objective functions and then to select the most appropriate solution from this set of efficient solutions. The efficient set includes all feasible non-dominated solutions, i.e. all the pareto-optimal solutions such that no better outcome can be achieved without making at least one objective worse-off.

An ideal solution is then specified with coordinates given by the optimum values for each objective. Using this solution as a reference, the compromise programming

[1] The decision theory literature also reports that individuals go back and forth between decisions and objectives before making a final decision. Thus, making the process of choosing a portfolio interactive is important, in part because it recognizes that the truly optimal portfolio is not known. Here we recognize that the way the information is presented may be as important as the mathematical method implemented for reaching an optimum individualized decision.

technique allows selection of the compromise set, i.e. the set of feasible efficient solutions closest to the ideal solution. The optimal solution depends on the distance function used and the compromise set, is composed of the optimal solutions from the following minimization problem:

$$\underset{x}{Min}\ L_i = \left[\sum_{j=1}^{N} \left(a_j \frac{Z_j^* - Z_j(x)}{Z_j^* - Z_j^-} \right)^i \right]^{\frac{1}{i}} \text{s.t. } x \in S \quad i = 1,2,...\infty$$

where,

i indicates the distance measure

N is the number of objectives

Z_j^* is the ideal solution of objective j

Z_j^- is the anti-ideal solution of objective j

Yu (1973) proved that the compromise solutions of L_1 and L_{inf} metrics characterizes the bounds of the compromise set and that the compromise solutions of all other L-metrics fall in between these two solutions.

Compromise programming has been introduced in the agricultural economics literature by Romero and Rehman (1984; 1985) and Romero et al. (1987). The literature on portfolio optimization using a compromise programming is limited to two recent articles in the operations research literature (Ballestero and Romero, 1996; Ballestero, 1998). This paper clarifies some of their results by deriving a formal relationship between the weights of compromise programming and the risk aversion coefficient widely used in economics. In addition, it couches the work in fuzzy logic.

The Fuzzy Logic Approach

Fuzzy set theory is a generalization of traditional set theory in the sense that the domain of the characteristic function is extended from the discrete set {0,1} to the closed real interval [0,1]. Formally, a fuzzy set A of some universe X is represented by a generalized membership function $m_A : X \rightarrow [0,1]$. Fuzzy sets and fuzzy logic allows one to mimic the human decision-making process by modeling the lexical uncertainty associated with using words rather than number to reach a solution (Von Altrock, 1995).

Economic research relying on fuzzy set theory or fuzzy optimization techniques is very scarce. A few attempts have been made to integrate fuzzy concepts into microeconomic theory (see for example: Billot, 1995; Greenhut et al., 1995; Mansur, 1995). The only two published fuzzy logic applications in agricultural economics are an analysis of goals and objectives of organic producers in Canada (Molder et al., 1991) and a fuzzy production planning model for fresh tomato packing (Miller, 1997).

Fuzzy set theory and fuzzy arithmetic have been developed to model lexical uncertainty and give a mathematical representation of words or linguistic variables. Thus, fuzzy logic is an appropriate tool to determine the portfolio that would best satisfy an individual describing his ideal portfolio with terms such as “low risk” and “high return”.

The Portfolio Choice Models

The traditional E-V objective function is

$$\underset{x}{Max}\ R(x) - l V(x) \tag{1}$$

where R(x) is expected return, V(x) is variance of returns, λ is the trade-off between mean and variance (Markowitz). l may be interpreted as ½ of the Pratt-Arrow risk aversion coefficient when the utility function is negative exponential and the returns are normally distributed. Solutions to (1) satisfy the following first-order condition:

$$\mathrm{R}_{\mathrm{x}} = l \mathrm{V}_{\mathrm{x}} \tag{2}$$

The compromise-programming approach to a mean-variance portfolio optimization problem yields:

$$\underset{\mathrm{x}}{Min}\ L_a = \left[\left(w_R \frac{R^* - R\ (x)}{R^* - R^-} \right)^a + \left(w_V \frac{V^* - V\ (x)}{V^* - V^-} \right)^a \right]^{\frac{1}{a}} \text{s.t. } \mathrm{x} \in \mathrm{S} \quad a = 1,2,...\infty \tag{3}$$

where a indicates the distance measure, R^* is the mean return of the asset with the highest mean return considered for the portfolio, R^- is the mean return of the asset with the lowest mean return considered for the portfolio, V^* is the variance of the asset with the smallest variance, and V^- is the variance of the asset with the greatest variance. As noted before, all compromise solutions are bounded by the solution to the L_1 and the L_{inf} problem.

The L_1 compromise-programming problem (3) can be re-formulated into a traditional EV problem plus a constant C as follow:

$$Max\ L_1 = R(x) + \left(\frac{w_R(R^*\text{-}R^-)}{w_V(V^-\text{-}V^*)} \right) V(x) + C \tag{4}$$

Solutions to (4) satisfy the following first order condition:

$$R_x = -\ddot{o} V_x, \qquad \ddot{o} = -\frac{w_R(R^*\text{-}R^-)}{w_V(V^-\text{-}V^*)}, \tag{5}$$

Equations (2) and (5) are equivalent when $j = l$. The equivalence between (2) and (5) demonstrate that all L_1 solutions lie on the EV frontier. This result implies that the EV frontier is the set of all efficient, non-denominated solutions as defined earlier. Under the traditional assumption of the EV model, an average individual who puts equal weight on the risk and return factors would thus have a risk aversion coefficient equal to half the ratio of return variability over variance variability in the portfolio of assets under investigation.

We are now investigating the other bond of the compromise set, i.e. the solution to the L_{inf} compromise programming problem. Following Zeleny (1982), the solution to the L_{inf} problem is characterized by:

$$\begin{aligned} & Min D_{\infty} \\ & s.t. \\ & w_R\left(\frac{R^*\text{-}R(x)}{R^*\text{-}R^-}\right) \le D_{\infty}, \qquad w_V\left(\frac{V^*\text{-}V(x)}{V^*\text{-}V^-}\right) \le D_{\infty} \end{aligned} \tag{6}$$

or

$$w_R\left(\frac{R^*\text{-}R(x)}{R^*\text{-}R^-}\right) = w_V\left(\frac{V^*\text{-}V(x)}{V^*\text{-}V^-}\right) \tag{7}$$

Differentiating (7) with respect to x, we find:

$$\mathrm{R}_x = dV_x, \qquad d > 0, \tag{8}$$

which implies that the L_{inf} solution is also on the EV efficient frontier. By varying the weights on the objective in (7), the EV frontier can be traced out as in figure 1.

L_{inf} and L_1 thus bond the compromise solution set, which is itself part of the EV efficient set. Because of the complexity of a typical problem, the compromise programming method “concentrates” on eliminating “obviously bad” solutions rather than on identifying the best ones. Once the “obviously bad” solutions have been eliminated, the decision-maker can pick a solution within the smaller set of solutions (i.e. the compromise set). Note that the decision maker need only specify a safety coefficient and a return coefficient between zero and one to get his personal compromise set.

The fuzzy logic approach provides an alternative to specifying coefficients. In the context of our portfolio problem, we can define two linguistic variables: “low risk” and “high return”, which are the two linguistic variables that every individual might include in the description of their optimal portfolio. Each linguistic variable can be defined by a fuzzy set and its associated membership function as follows:

$$m_{LV}(x) = \frac{\mathrm{V}^{-} - \mathrm{V(x)}}{\mathrm{V}^{-} - \mathrm{V}^{*}} \quad (9), \qquad m_{HR}(x) = \frac{\mathrm{R(x)} - \mathrm{R}^{-}}{\mathrm{R}^{*} - \mathrm{R}^{-}} \quad (10)$$

where $m_{LR}(x)$ is the membership function of “low risk” and $m_{HR}(x)$ is the membership function of “high risk”. The higher the expected return of the portfolio the closer to 1 the membership value associated with $m_{HR}(x)$ and the lower the variance of the portfolio, the closer to 1 the membership value associated with $m_{LR}(x)$[2].

An individual expressing his desire for a portfolio with “high return and low risk” may perhaps put equal emphasis on return and safety. The optimal portfolio is equally weighted between “low risk” than it is “high return” and is characterized by the following fuzzy optimization problem:

$$\begin{aligned} &Max\ \ l \\ &s.t. \qquad l \le m_{LR}(x), \quad l \le m_{HR}(x) \end{aligned} \qquad (11)$$

[2] The portfolio with the highest expected return is given a “high return” value of 1 and the portfolio with the lowest expected variance is given a “low risk” value of 1.

where l equals the degree of membership of our portfolio to the fuzzy sets that characterize our linguistic variables (i.e. the membership values). Again, the solution to such a problem is an efficient portfolio located on the traditional EV frontier and equal to the L_{inf} bond of the compromise set when the weights are equal. Indeed, (6) can be easily rewritten as (11).

We can further refine the fuzzy optimization problem to account for more linguistic terms. Some individuals may add adverbs to the two basic linguistic variables. Adverbs are often referred to as modifiers and hedges (Lakoff, 1973) because they modify the membership functions of the linguistic term (generally adjectives) which they have received as argument. Schmuker (1984) separates adverbs into three groups depending on how they modify the original membership function: concentration, dilution or intensification of the original membership function. Adding VERY in front of "low risk" has a concentration effect on the membership function "low risk", i.e. $m_{VLR}(x) = m_{LR}(x)^2$. Adding FAIRLY in front of "high return" has a dilution effect on the membership function "high return", i.e. $m_{VLR}(x) = m_{LR}(x)^{.5}$.

Example

A farmer is considering whether he should produce sorghum, wheat, soybean, or simply rent his land. He accesses the Kansas agricultural extension homepage and selects the "What Should You Produce?" page. He is then asked to select his county as well as the activities he is considering. At this point, the set of efficient portfolio (EV frontier) is computed and shown on a graph similar to that of figure 1. In a window below the graph, the farmer is asked to pick safety and return coefficients on a self-explanatory scale.

Once the farmer has selected weights, the compromise programming problem is solved and the farmer is given the solution to the L_1 and the L_{inf} problem as well as an updated EV frontier showing the entire compromise set.

The results of the L_1 and the L_{inf} problems are reported in table 2 for a variety of safety and return coefficients. For comparison purposes, results of the EV model for different coefficients of risk aversion are also reported[3]. Figure 1 confirms the fact that the EV frontier and the efficient set are one and the same. It also shows the relative position of EV solutions for different risk aversion coefficients in compromise sets. For example, an individual with a negative exponential utility function and a coefficient of risk aversion of .04 (this coefficient may be interpreted as "relatively strong risk aversion") would choose the pair of weights (.2,.8). However, an individual with a coefficient of risk aversion of .02 would choose the pair of weights (.7,.3), indicating that, even though he is risk averse, he gives relatively more importance to return than to safety[4]. These examples also demonstrate the "non-intuitive" aspect of the choice of a risk-aversion coefficient[5].

Alternatively to choosing coefficients of safety and risks, the farmer could be asked to describe his portfolio using the following list of words: {low, high, very, fairly,

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[3] All results obtained using the nonlinear programming solver MINOS5 in GAMS.

[4] Note that, in compromise programming, the individual can be only risk averse($w_v > 0$) or risk neutral ($w_v = 0$), but never risk lover ($w_v < 0$ not allowed).

[5] In this example, the portfolio choice changes when theta changes from .02 to .01. Cochran (1986) found that the coefficient of risk aversion used vary between -.005 and +.002 for "almost risk neutral", and from +0.00004 to infinity for "strongly risk averse" individuals. Researchers have sometimes used elicitation methods to find the range of the risk aversion coefficient, but these elicitation techniques have often shown that individuals have difficulty in specifying preferences.

risk, return, and, or, ...}. The farmers' description would then be automatically processed into fuzzy constraints so that the appropriate fuzzy optimization problem may be solved. Table 2 shows the portfolio suggested by the fuzzy logic approach that correspond to various linguistic description. For example, the fuzzy approach suggests that an individual willing to invest in a very low risk but fairly high return portfolio invest in a portfolio P with an expected return of 57.74 and a standard deviation of 20.06. The membership value associated with this description is .642, which can be roughly interpreted as follow: the optimal portfolio suggested fit the linguistic description "very low risk and fairly high return" at the 60% level.

As you can see in table 2, the same portfolio P is suggested for someone looking for a "very very low risk and high return" portfolio but the membership value is much lower. The suggested portfolio fits the linguistic description at a 40% level only. This is because the individual wants a portfolio with both relatively low levels of risk and relatively high levels of return, which is less feasible since risk and return are negatively related. Low membership values, say below .5, indicate that the set of assets at hand cannot result in a portfolio that is likely to fully satisfy the individual. In our example, an individual describing his portfolio as "fairly low risk and fairly high return" is the most likely to be completely satisfied.

The farmer can pick the suggested portfolios or either adjust his return and safety coefficients or change his portfolio description, and run the program again until he finds a suitable portfolio. Using this interactive approach allows the farmer to reach his own personal decision while insuring the decision remains reasonably efficient from a mean variance perspective. Because of all the uncertainty associated with reaching a portfolio decision, the "do-it-yourself" aspect of portfolio optimization is very important.

Conclusion

The techniques presented in this paper are examples of how multicriteria decision making might be performed without direct assessment of the utility function. Though the utility maximization is not easily observed, individuals do assign priority weights no matter how imperfectly, fuzzily or temporarily. As a result, objective weighing as suggested in compromise programming is an intuitive approach that is not necessarily inferior to utility maximization. Individuals can also use words as subjective categories to process information and reach solutions. Imitating this process using fuzzy logic is easy, intuitive, and yields results similar to those of compromise programming and the traditional EV model.

The compromise programming and the fuzzy logic approach are computationally efficient and consistent with the EV model but do not require any understanding of utility theory. As such, they may be the most suitable approaches for solving popular optimization problems interactively. Compromise programming and fuzzy optimization can be applied to more complex problems. Indeed, it would be easy to introduce other objectives in addition to variance minimization and return maximization. However, direct assessment of weights may become more difficult and less intuitive as the number of objective grows. Moving from compromise programming to fuzzy optimization may then be appropriate.

Figure 1 – The EV Frontier of the Set of Pareto-optimal Portfolios

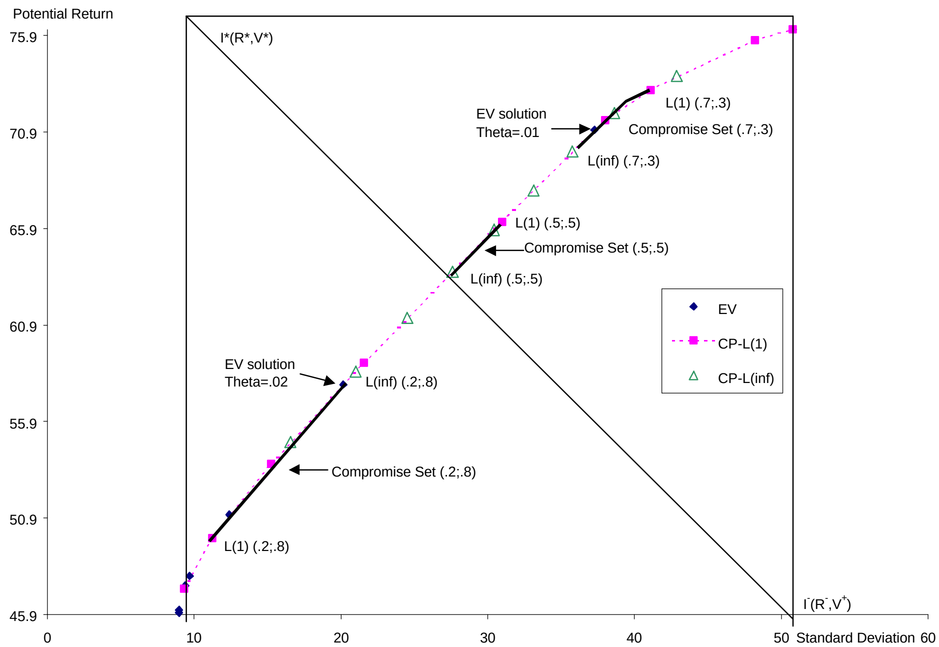

Table 1 – Solutions to the Compromise Programming Problem

		Compromise set		*Solutions to the EV Model*		
Return coeff.	Risk coeff.	Return range	STD range	theta	Expected return	STD
0.1	0.9	47-55	9-17	-0.0001	76.21	50.784
0.2	0.8	50-58	11-21	0	76.21	50.784
0.3	0.7	54-61	15-25	0.0001	76.21	50.784
0.4	0.6	59-64	22-28	0.01	71.011	37.266
0.5	0.5	66	30-31	0.02	57.816	20.153
0.6	0.4	68-72	33-38	0.04	51.075	12.383
0.7	0.3	70-73	36-41	0.08	47.907	9.685
0.8	0.2	72-76	39-48	0.1	47.403	9.384
0.9	0.1	74-76	43-51	0.5	46.148	8.974

Table 2 – Correspondence Between Linguistic Terms and Risk and Return of Optimal Portfolios

RISK / RETURN	Very very low	Very low	Low	Fairly low
Fairly high	54.19±15.78, *l=.491*	57.74±20.06, *l=.642*	61.75±25.13, *l=.781*	65.82±30.41, *l=.887*
High	57.74±20.06, *l=.402*	61.75±25.13, *l=.532*	65.82±30.41, *l=.663*	69.54±35.31, *l=.784*
Very high	61.75±25.13, *l=.316*	65.82±30.41, *l=.420*	69.54±35.31, *l=.535*	72.39±39.60, *l=.648*
Very very high	65.82±30.41, *l=.238*	69.54±35.31, *l=.318*	72.39±39.60, *l=.407*	74.01±43.40, *l=.483*

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