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Linear Response Stochastic Plateau Functions

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Abstract

A method of estimating a linear response stochastic plateau function is developed and used to determine optimal levels of nitrogen fertilization for wheat. Under the stochastic plateau model, optimizing behavior implies different nitrogen levels than the linear plateau model, depending on the output/input price ratio relative to a threshold level.

Key Words: stochastic plateau, von Liebig response, wheat, nitrogen, fertilizer

Introduction

Estimating crop yield response to fertilizer and determining economically optimal levels of fertilizer has been of interest for many decades. Early efforts to estimate crop yield response functions recognized the intuitive appeal of plateau type functional forms (Spillman; Johnson; Heady and Pesek; Heady, Pesek, and Brown). Spillman developed and applied what has come to be known as the Spillman functional form to reflect the von Liebig law of the minimum. Heady and Dillon wrote that "… most production functions probably have a von Liebig point…" (Heady and Dillon, p. 10). Heady and his associates used Spillman (plateau) functional forms in some of their multiple region models (Nicol, Heady, and Madsen, p. 216).

Waugh, Cate, and Nelson developed a graphical procedure for estimating a linear response and plateau model (LRP). Perrin and Lanzer and Paris both concluded that the LRP functional form performed as well or better than polynomial specifications. Grimm, Paris, and Williams concluded that the LRP explained crop response to fertilizer at least as well as if not better than polynomial forms.

Data obtained in a 1952 experiment and published by Heady, Pesek, and Brown have been used by a number of researchers who conclude that a plateau function is a more appropriate fit than polynomial specifications (Paris; Frank, Beattie, and Embleton; Chambers and Lichtenberg). Polynomial specifications that have been used in traditional farm management textbooks do not exhibit a plateau and generally result in nitrogen recommendations greater than a plateau model.

While a consensus may have developed regarding the existence of a plateau and the need to use plateau specifications to make nitrogen application recommendations, one important issue remains to be resolved. That has to do with variability of crop response to nitrogen and why farmers apply too much or too little nitrogen.

Johnson recognized the importance of variable weather and reported both a "good" weather and a "bad" weather response function. Ryan and Perrin concluded that Peruvian farmers were applying too little nitrogen. In their discussion of factors that might inhibit fertilizer use they include "…risks perceived by producers and their reactions to these risks…" (Ryan and Perrin, p. 343). de Janvry also recognized the importance of "noncontrolable climatic events" (de Janvry, p. 1). He concluded that risk aversion could reduce the level of fertilizer application. On the other hand, Babcock writes that "…It seems …that typical U.S. farmers apply more nitrogen…than would be warranted by the equation of marginal product and factor costs …" (Babcock, p. 271). Babcock used simulation to show that the uncertainty about future growing conditions causes farmers to apply more nitrogen on corn than if growing conditions were known.

Much of the previous work on estimation of plateau response functions makes two major simplifications. First, the year effect is either treated as a fixed effect and specified with year dummy variables (e.g. Sumelius) or ignored completely. Second, the plateau is assumed to be nonrandom, in spite of its determinants being stochastic (Ackello-Ogutu, Paris and Williams; Llewelyn and Featherstone; Paris and Knapp; Cerrato and Blackmer; Cox, 1992; Cox, 1996).

Assuming each of the inputs, including the plateau, has a normal random element, Berck and Helfand and Paris (BHP) estimated switching regression versions of the LRP using an extension of Maddala and Nelson's approach. Babcock's framework is closer to ours than that of Berck and Helfand, but Babcock assumes the response is stochastic rather than the plateau. Further, Babcock presents no empirical estimates.

In this article, an LRP function with a stochastic plateau is estimated by directly maximizing its marginal log-likelihood function. With this approach, it is not necessary for the function to be cast in a switching regression format and it is straight forward to include year as a random effect and to let the plateau shift only by year. The BHP model and our model are estimated under vastly different assumptions regarding the error terms and while they may appear similar, they actually are quite different, with neither model being a special case of the other.

The linear response stochastic plateau function is used to determine economically optimal levels of nitrogen fertilization for wheat. The conditions for maximizing expected returns when the objective function includes a response function with a stochastic plateau are derived. BHP derived no conditions for determining optimal levels. Pautsch, Babcock, and Breidt derived the optimality conditions for a stochastic response with a linear plateau, which is a very different problem than the stochastic plateau considered here. Wheat grain yield response to nitrogen is estimated using data from a long-term experiment. The dataset is both larger and considerably more recent than the 1952 dataset used in previous research.

A Linear Response Stochastic Plateau Model

In general a univariate LRP function may be expressed as

(1)
$$y = \begin{cases} \beta_0 + \beta_1 x + \varepsilon, & \text{if } y_m > \beta_0 + \beta_1 x \\ y_m + \varepsilon, & \text{otherwise,} \end{cases}$$

where y is the response variable (in this case, yield), x is the level of the limiting input, ε is a random error term, and y_m is the maximum yield, also referred to as the plateau yield. The parameters β_0 and β_1 are intercept and slope, respectively. To ensure continuity at the threshold, maximum yield is often defined as

(2)
$$y_m = \beta_0 + \beta_1 x_m,$$

where x_m is the level of the input necessary to reach the plateau. Thus, we can define (x_m, y_m) as the knot point at which the response and plateau portions are splined.

In the traditional LRP function, x_m and y_m are treated as parameters. The implicit assumption in this formulation is that all the factors that define the plateau are fixed and completely controllable. While this assumption is helpful in simplifying the problem, it is often too simplistic. In reality, most of the extraneous factors tend to vary randomly. Observations of weather conditions at any point in time, for example, constitute only a random sample from a population of all possible outcomes. Similarly, soil nutrient composition in a field tends to vary stochastically from site to site. Even management parameters, which are under deliberate control, are subject to measurement error, human error, and several other sources of variation. Therefore, it seems more realistic to visualize the knot as an expectation that is conditioned on realized values of these random factors.

In a linear response stochastic plateau model the plateau itself becomes a random variable. So while (1) and (2) still hold, we have that $y_m \sim \left(\beta_0 + \beta_1 E(x_m), \sigma_m^2\right)$.

Since y_m and x_m are linearly related, if one is random, so is the other. With y_m assumed stochastic, for example, it can be shown, by rewriting equation (2) as

(3) $$x_m = \frac{y_m}{\beta_1} - \frac{\beta_0}{\beta_1},$$

that x_m is also random with mean $E(x_m) = \dfrac{E(y_m) - \beta_0}{\beta_1}$, and variance $\mathrm{Var}(x_m) = \dfrac{\mathrm{Var}(y_m)}{\beta_1^2}$ As an example, if $y_m \sim N\left(\mu_m, \sigma_m^2\right)$, then, by substitution, $x_m \sim N\left(\dfrac{\mu_m - \beta_0}{\beta_1}, \dfrac{\sigma_m^2}{\beta_1^2}\right)$.

Figure 1 provides a visual demonstration of the behavior of equation (1) under conditions of a linear response stochastic plateau function. Suppose the knot is initially at $(\bar{x}_m, \bar{y}_m)$. Any changes that make the extraneous factors more limiting, such as unfavorable weather, will exert downward pressure on the maximum yield, say to $y_m^{/}$. This will, in turn, make *x* less limiting, with the effect of pushing the critical level downward, in this case to $x_m^{/}$. Favorable changes in environmental conditions, such as the occurrence of good weather, will have an opposite effect, as is demonstrated by the shifting of the knot to $\left(x_m^{//}, \mathrm{y}_\mathrm{m}^{//}\right)$.

In addition to affecting the position of the knot the stochastic factors may also affect the overall response relationship. Weather conditions, for example, will affect the incidence of disease, pests, weeds, and other physical and phenotypic determinants of yield (e.g. hail). These effects can be reflected in the random error term of the relationship. Thus, the error term in equation (1) can be defined as

(4) $$\varepsilon = u + \varepsilon^*,$$

where ε^* is an independently and identically distributed random error term with mean zero and constant variance $\sigma_{\varepsilon^*}^2$ and u, the year random effect, has mean zero and variance σ_u^2. In empirical specification, independence is assumed between the two variance components, $\sigma_{\varepsilon^*}^2$ and σ_u^2. Under this assumption, the variance of the overall error term, $\mathrm{Var}(\varepsilon)$, is $\sigma_\varepsilon^2 = \sigma_u^2 + \sigma_{\varepsilon^*}^2$.

Determining Profit-Maximizing Level of the Input

Figure 1 shows that a LRP function will exhibit constant positive marginal product when $y_m > \beta_0 + \beta_1 x$. Under the usual assumption of a nonstochastic plateau, this implies that only the input-output price ratio will matter in choosing the optimum level of the input. If the value of the marginal product (VMP) is equal to or less than marginal factor cost (MFC), the decision-maker would optimally apply no input (i.e. $x = 0$). However, if VMP > MFC, it is beneficial to continue increasing the level of the input until the maximum yield is attained. Increasing x beyond x_m will generate negative marginal returns, equal in absolute terms to the price of the input. Therefore, with the nonstochastic linear response plateau function, there are only two possibilities with respect to optimum input level. That is

(5) $$x = \begin{cases} x_m, & \text{if VMP > MFC,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus, if the plateau, y_m, and the level of the input at the threshold, x_m, are treated as parameters, determining the optimum input level is a trivial problem.

The problem is not so straightforward if the plateau is treated as a random variable, such as is the case in Figure (1). In this case, the farmer's decision will be influenced by his/her expectations about y_m and the distribution of y_m is central in representing this decision process. Derivation of a technique for determining the optimum input level under the assumption of a stochastic plateau follows.

Keeping equation (2) in mind and assuming a stochastic plateau, equation (1) can be expressed as

$$y = \{(\beta_0 + \beta_1 x)[1 - I_{(-\infty,\, \beta_0+\beta_1 x)}(y_m)] + y_m I_{(-\infty,\, \beta_0+\beta_1 x)}(y_m)\} + \varepsilon \,, \tag{6}$$

where $I_{(-\infty,\, \beta_0+\beta_1 x)}(y_m)$ is an indicator function, defined as

$$I_{(-\infty,\, \beta_0+\beta_1 x)}(y_m) = \begin{cases} 1, & \text{if } \mathrm{y}_m \le \beta_0 + \beta_1 x, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Taking expectations of equation (6) yields

$$E(y \mid x) = \{(\beta_0 + \beta_1 x)E(1 - I_{(-\infty,\, \beta_0+\beta_1 x)}(y_m)) + E(y_m I_{(-\infty,\, \beta_0+\beta_1 x)}(y_m))\} \tag{8}$$

where $E(\cdot)$ is the expectations operator. By definition,

$$E(I_{(-\infty,\, \beta_0+\beta_1 x)}(y_m)) = \mathrm{Prob}(\mathrm{y}_m \le \beta_0 + \beta_1 x)\,, \tag{9}$$

where $\mathrm{Prob}(y_m \le \beta_0 + \beta_1 x)$ is the probability that $y_m \le \beta_0 + \beta_1 x$, which is the cumulative distribution function (cdf) of y_m evaluated at $\beta_0 + \beta_1 x$, or $F(\beta_0 + \beta_1 x)$. By substitution, equation (8) can, thus, be expressed as

$$E(y \mid x) = (\beta_0 + \beta_1 x)[1 - F(\beta_0 + \beta_1 x)] + \int_{-\infty}^{\beta_0+\beta_1 x} y_m f(y_m)\, dy_m \,, \tag{10}$$

where $f(y_m)$ is the probability density function (pdf) of y_m. Note that the last right hand side term is $E(y_m / y_m \le \beta_0 + \beta_1 x)$.

Suppose the decision-maker is risk neutral and that his/her behavior can be adequately described by expected profit-maximization. Such a decision-maker's objective function can be expressed as

$$E(\pi \mid x) = pE(y) - rx \,, \tag{11}$$

where p and r are output and input price, respectively, and $E(\pi)$ is expected profit. Substituting (10) into (11) yields

$$(12)\qquad E(\pi \mid x) = p\left((\beta_0 + \beta_1 x)[1 - F(\beta_0 + \beta_1 x)] + \int_{-\infty}^{\beta_0+\beta_1 x} y_m f(y_m) dy_m \right) - rx$$

Equation (12) describes the profit-maximizing decision-maker's utility function under conditions of a linear response stochastic plateau function.

The first-order condition for profit maximization can be obtained by differentiating (12) with respect to x. That is,

$$(13)\qquad \frac{\partial E(\pi \mid x)}{\partial x} = p\left(\frac{\partial(\beta_0 + \beta_1 x)(1 - F(\beta_0 + \beta_1 x))}{\partial x} + \frac{\partial}{\partial x}\int_{-\infty}^{\beta_0+\beta_1 x} y_m f(y_m) dy_m \right) - r = 0 .$$

By the chain rule, the first term in the parenthesis reduces to

$$(14)\qquad \frac{\partial(\beta_0 + \beta_1 x)(1 - F(\beta_0 + \beta_1 x))}{\partial x} = \beta_1[1 - F(\beta_0 + \beta_1 x)] - \beta_1(\beta_0 + \beta_1 x) f(\beta_0 + \beta_1 x).$$

The second term in the parenthesis of (13) requires a special rule for differentiating under the integral sign. For any differentiable function $G : [a_2, b_2] \to \Re$, defined as $G(x_2) = \int_{\lambda(x_2)}^{\theta(x_2)} g(x_1, x_2) dx_1$, a generalization of the Liebnitz integral rule states that

$$(15)\qquad \frac{\partial G}{\partial x_2} = \int_{\lambda(x_2)}^{\theta(x_2)} \frac{\partial g(x_1, x_2)}{\partial x_2} dx_1 + \theta^{/}(x_2) g(\theta(x_2), x_2) - \lambda^{/}(x_2) g(\lambda(x_2), x_2),$$

where $g(\cdot)$ is some continuous function of x_1 and x_2, $\lambda(x_2)$ and $\theta(x_2)$ are lower and upper limits of the integral, as continuous functions of x_2, and $\lambda^{/}(x_2)$ and $\theta^{/}(x_2)$ are their respective first derivatives (Khuri, p. 302, Theorem 7.10.2). By using (15) and defining G as $G(x) = \int_{\lambda(x)}^{\theta(x)} y_m f(y_m) dy_m$, where $\lambda(x) = -\infty$ and $\theta(x) = \beta_0 + \beta_1 x$, it can be shown that

$$(16)\qquad \frac{\partial}{\partial x}\int_{-\infty}^{\beta_0+\beta_1 x} y_m f(y_m) dy_m = \beta_1(\beta_0 + \beta_1 x) f(\beta_0 + \beta_1 x).$$

Substituting (14) and (16) into (13) and simplifying produces

(17) $$\frac{\partial E(\pi \mid x)}{\partial x} = p\beta_1\{1 - F(\beta_0 + \beta_1 x^*)\} - r = 0.$$

According to (17), the decision-maker determines the level of input to apply by equating the value of marginal expected product, $p\beta_1\{1 - F(\beta_0 + \beta_1 x)\}$, to the input price, *r*. Notice that, at the solution, the second-order condition,

(18) $$\frac{\partial^2 E(\pi \mid x)}{\partial x^2} = -p\beta_1^2 f(\beta_0 + \beta_1 x) < 0,$$

is satisfied for expected profit maximization. This implies that imposing a stochastic knot on a LRP function transforms it into a strictly concave function. Berck and Helfand drew a similar conclusion in regard to their model.

Rearranging terms in (17) produces

(19) $$F(\beta_0 + \beta_1 x^*) = 1 - \frac{r}{p\beta_1}.$$

Because $0 \leq F(\beta_0 + \beta_1 x^*) \leq 1$, (19) applies only if the condition

(20) $$\beta_1 \geq \frac{r}{p}$$

is satisfied. From (19), the optimum level of the input can be expressed as

(21) $$x^* = \frac{1}{\beta_1}\left[F^{-1}\left(1 - \frac{r}{p\beta_1}\right) - \beta_0\right],$$

where $F^{-1}(\cdot)$ is the inverse of the cdf. To complete the computation, *r* and *p* can be replaced with data from input and output markets, and the parameters, β_0 and β_1, can be replaced by their statistical estimates. Because *x* cannot be negative and $\beta_1 \geq 0$, equation (21) is valid only if

(22) $$F^{-1}\left(1 - \frac{r}{p\beta_1}\right) - \beta_0 \geq 0.$$

Thus, both conditions (20) and (22) need to hold for a nonnegative level of the input to be applied. Pautsch, Babcock and Breidt's (p.349) solution is not the same because they assume the response to be stochastic due to uncertainty about carry over nitrogen while our model assumes the plateau is stochastic due primarily to differences in weather across years.

Under what circumstances will the optimum nitrogen level be equal for both, the stochastic and non-stochastic model? In other words, when will x^* equal the deterministic parameter x_m, which corresponds to $E(x_m)$ in the stochastic case? First note that (21) is similar to (3). From comparing these equations it is clear that if $F^{-1}\left(1-\frac{r}{p\beta_1}\right)$ were equal to $E(y_m)$, then x^* would equal $E(x_m)$. So, now again we can rephrase the question to say, when will the inverse CDF of y_m equal its expected value? In the case of a symmetric distribution where mean and median coincide, this will occur when $1-\frac{r}{p\beta_1}$ equals 0.5. For values below this level, i.e., when $r / p\beta_1 > 0.5$, the optimum level of nitrogen under the stochastic plateau model will be lower than the one obtained with a non-stochastic plateau model. Conversely, if $r / p\beta_1 < 0.5$ our model predicts that there will be a tendency to apply more nitrogen than what the non-stochastic plateau model dictates. And this is in fact what has been observed. Given the current output/input price ratios, farmers as expected, apply more nitrogen than what the certainty case predicts.

From equation (21), the optimum level of the input can be determined analytically if a unique inverse exists for the prescribed cdf. This is true for distributions such as the exponential and uniform. For many other distributions, however, the distribution function cannot be expressed in an easily invertible form. Examples include normal, gamma, and beta distributions. For these distributions, tables matching probabilities and random variables are available based on numerical integration techniques (Wackerly, Mendenhall and Scheaffer).

If $y_m \sim N\left(\mu_m, \sigma_m^2\right)$, for example, standard normal probability tables available in standard statistics and econometrics textbooks can be used to approximate the inverse in (21). This can be done by converting $E(y_m \mid x_m = x)$ into a standard normal variate Z_α. That is,

$$Z_\alpha = \frac{\beta_0 + \beta_1 x - \mu_m}{\sigma_m}, \tag{23}$$

where $\alpha = 1 - F(\beta_0 + \beta_1 x) = r / p\beta_1$ is the observed probability in the right-hand tail of the $N(0, 1)$ distribution and $F(\beta_0 + \beta_1 x)$, the cdf of y_m evaluated at $\beta_0 + \beta_1 x$, is as defined in (19). The optimum input level can then be determined by rearranging terms in

$$x = \frac{1}{\beta_1}\left(\mu_m + Z_\alpha \sigma_m - \beta_0\right), \tag{24}$$

indicating, among other things, that the optimum input level increases in variance. Notice that (24) is really (21) with $F^{-1}\left(1 - \frac{r}{p\beta}\right) \approx \mu_m + Z_\alpha \sigma_m$. Alternatively, for most continuous distributions, an $F^{-1}(\cdot)$ approximation can be obtained directly by using functions available in spreadsheet software.

Estimation

Estimation of equation (1) has been a focus of discussion and empirical work by production economists and agronomists for several decades. Many have implemented the model under various assumptions about the behavior of the plateau but few have had success with the more logically consistent stochastic-plateau specifications.

This section gives an overview of an alternative and straightforward procedure that involves maximizing the marginal log-likelihood function directly using the theory of nonlinear mixed effects models (SAS Institute Inc, 2000; Wolfinger, 1993; Wolfinger, 1999). First, a brief overview is given of procedures used to estimate other more restrictive versions of plateau specifications common in the literature. With cross-section time-series data and multiple inputs, the model to be estimated can be written as

(25) $$y_{jt} = \min\{h_1(\boldsymbol{x}_{1jt},\ \beta_1),\ldots,h_n(\boldsymbol{x}_{nt},\ \beta_n),\ y_{mt}\} + \varepsilon_{jt}$$

where $\beta_i = (\beta_{i0}, \beta_{i1}, \ldots, \beta_{iK})'$ is a $(K+1)$ x 1 vector of parameters and $\mathbf{x}_{ijt} = (1, x_{ijt1}, \ldots, x_{ijtK})'$ is a $(K+1)$ x 1 vector of predictors, $\forall$ i = 1, 2, …, n factors. Subscripts t and j index the year and the cross-sectional unit for each factor i. In general, the function $h_i(\mathbf{x}_{ijt}, \boldsymbol{\beta}_i)$ can take any form and can be linear or nonlinear in the parameters. An LRP such as equation (1) is a special case where $h_i(\mathbf{x}_{ijt}, \boldsymbol{\beta}_i)$ is linear in the parameters, i.e. $h_i(\mathbf{x}_{ijt}, \boldsymbol{\beta}_i) = \boldsymbol{\beta}_i'\mathbf{x}_{ijt}$, $\forall$ i. Thus, the equation estimated is:

(26) $$y_{jt} = \min\{\beta_1'\mathbf{x}_{ijt},\ldots,\beta_n'\mathbf{x}_{njt},y_{mt}\} + \varepsilon_{jt}$$

Equation (26) possesses several features that will make ordinary least squares inappropriate as an estimation technique. With year random effects constituting part of the error term, the need to estimate the variance components jointly with $\boldsymbol{\beta} = (\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_n)'$ calls for a maximum likelihood-based mixed effects procedure. However, estimation of (26) is further complicated by the assumptions made about plateau yield. That is, choice of an appropriate estimation procedure will depend on whether the plateau is treated as a known parameter, an unknown parameter, or a stochastic variable.

Estimation When the Plateau is Known

If the researcher uses prior knowledge from literature and/or expert opinion to place a numerical value, $\bar{y}_m$, on the plateau, then $y_{mt} = \bar{y}_m$, $\forall$ t, and equation (26) reduces to

(27) $$y_{jt} = \min\{\boldsymbol{\beta}_1'\mathbf{x}_{1jt},\ldots,\boldsymbol{\beta}_n'\mathbf{x}_{njt}, \bar{y}_m\} + \varepsilon_{jt}$$

where the minimum of $\boldsymbol{\beta}_i'\mathbf{x}_{ijt}$, once estimated, can be joined to the known plateau, $\bar{y}_m$. If the disturbances, ε_{jt}, are spherical and there is no interest in estimating the year random effect, equation (27) can be estimated with standard multiple linear regression or ordinary least squares using those data points that fall below the plateau. Otherwise, the estimation can be done more efficiently with maximum likelihood techniques. Estimated generalized least squares, $\hat{\hat{\boldsymbol{\beta}}} = (y - \mathbf{x}\boldsymbol{\beta})' \hat{\mathbf{V}}^{-1}(y - \mathbf{x}\boldsymbol{\beta})$, presents a viable option if ε_{jt} is normally distributed, but poses the challenge of finding a reasonable estimate of the covariance matrix, V (Littell et al.).

Estimation When the Plateau is an Unknown Parameter

If the plateau is assumed to be an unknown parameter, the need to estimate y_m jointly with the other model parameters renders equation (26) intrinsically nonlinear in the parameters. Because the normal equations (first-order derivatives) are also likely to be nonlinear, an iterative solution technique is required to complete the estimation process. Kennedy argues that running ordinary least squares (OLS) on a nonlinear function would lead to biased and meaningless parameter estimates. Either nonlinear least squares (NLS) or maximum likelihood (ML) estimation can provide consistent estimates (Judge et al.).

Because maximum likelihood estimators have several desirable statistical properties (Wackerly, Mendenhall and Scheaffer), such as consistency and asymptotic efficiency, they are often preferred to nonlinear least squares estimates. Most past estimates of nonstochastic plateau models also assume $\varepsilon_{jt} \sim N\left(0, \sigma_{\varepsilon}^{2} I\right)$ which means there is no year random effect (eg. Llewelyn and Featherstone; Cox, 1992; Cox, 1996; Cerrato and Blackmer; Bock and Sikora; Bullock and Bullock).

In general, however, the OLS desirable properties will not carry over to nonlinear least squares. Also, if the year random effect is of interest and needs to be estimated, the model with an unknown plateau becomes a nonlinear mixed effects model with fixed effects entering nonlinearly and random effects (year) entering linearly. Asymptotically efficient estimates can be obtained with maximum likelihood-based nonlinear mixed effects techniques.

Estimation When the Plateau is a Stochastic Variable

A stochastic plateau can be expressed as $y_{mt} = \mu_m + v_t$, where μ_m is the mean and v_t is the random effect associated with the plateau. Under this assumption, equation (26) becomes a nonlinear mixed-effects model and can be more generally expressed as

(28) $$y_{jt} = \min\{\beta_1' \mathbf{x}_{ijt}, \ldots, \beta_n' \mathbf{x}_{\mathrm{njt}},\ \mu_{\mathrm{m}} + v_t\} + \varepsilon_{jt}$$

Let $\boldsymbol{\varphi}$ represent a vector of fixed-effects parameters, i.e. $\boldsymbol{\varphi} = (\boldsymbol{\beta}, \mu_m)'$. Unlike the case of an unknown nonstochastic plateau with year random effects, equation (28) has both fixed effects ($\boldsymbol{\varphi}$) and a random effect, v_t, entering nonlinearly. Due to this nonlinearity in the random effect, there is no closed form solution for the marginal distribution of y. Thus, while more realistic in its assumptions about the behavior of plateau yield, this model poses real empirical challenges. The assumptions for the error terms are quite different than those used by Berck and Helfand and Paris.

A common approach would be to linearize the function about an estimate of $\boldsymbol{\varphi}$, $\hat{\boldsymbol{\varphi}}$, and $E(v) = 0$ (Beal and Sheiner) or about $\hat{\boldsymbol{\varphi}}$ and $\hat{v}$, where $\hat{v}$ is the posterior mode equivalent to the best linear unbiased predictor (BLUP) in the linear case (Lindstrom and Bates). Vonesh and Carter propose an estimated generalized least squares estimation on the linearized model. These first- and second-order linearizations based on a finite number of moments (usually no more than two) may represent a symmetric distribution well but will not be adequate for most other situations. According to Davidian and Gallant, features such as multimodality and asymmetry will not be detected from first and second moments. Recent developments involve estimating both fixed and random effects jointly by maximizing the log of an approximate marginal likelihood function of y.

The marginal probability density function, on which the marginal likelihood function is based, is formed by integrating the joint density function with respect to all the random effects variables. Let $f_y(y_{jt} \mid \mathbf{x}_{jt}, \boldsymbol{\varphi}, \sigma_\varepsilon^2, v_t)$ and $f_v(v_t \mid \sigma_v^2)$ denote the conditional pdf of y and pdf of v, respectively, where $\mathbf{x}_{jt} = (\mathbf{x}_{1jt}, \ldots, \mathbf{x}_{njt})$ is an $n(k + 1)$ x 1 vector of all factors. Then the joint probability density function is

$$f(y_{jt}, v_t \mid \mathbf{x}_{jt}, \boldsymbol{\varphi}, \sigma_\varepsilon^2, \sigma_v^2) = f_y(y_{jt} \mid \mathbf{x}_{jt}, \boldsymbol{\varphi}, \sigma_\varepsilon^2, v_t) f_v(v_t \mid \sigma_v^2) \tag{29}$$

where $\boldsymbol{\varphi}$, σ_ε^2, and σ_v^2 are the unknown parameters, and v_t is assumed to have mean zero and variance σ_v^2. The marginal likelihood function of y is obtained by integrating equation (29) with respect to v_t and taking the product over t and j

$$(30) \qquad l(\boldsymbol{\varphi},\sigma_{\varepsilon}^{2},\sigma_{v}^{2}\,|\,\mathbf{x},y)=\prod_{t=1}^{T}\prod_{j=1}^{J_t}\int_a^b f_y(y_{jt}\,|\,\mathbf{x}_{jt},\boldsymbol{\varphi},\sigma_{\varepsilon}^{2},v_t)f_v(v_t\,|\,\sigma_v^2)dv_t$$

where *a* and *b* are the limits of integration for the distribution of v_t, *t* is the number of years under consideration, and J_t is the number of cross-sectional observations in year *t*.

Theoretically, (30) is the function whose logarithmic transformation is supposed to be maximized in a maximum likelihood estimation. However, because v_t enters nonlinearly, (30) has no closed form and can only be approximated numerically. Several approaches are available for approximating such non-tractable integrals, including Monte Carlo integration and Gaussian quadrature. Of all the numerical integration techniques, Gaussian quadrature is believed to offer the highest degree of accuracy (Liu; Ghomi and Hashemin; Stiegert and Hertel).

For nonlinear mixed effects models, such as the one in equation (30), the SAS NLMIXED procedure uses a special type of Gaussian quadrature called adaptive Gaussian quadrature. Adaptive Gaussian quadrature uses the Gauss-Hermite abscissas and probability weights in the approximation while centering the integral over the empirical Bayes estimate of v_t (SAS Institute Inc., 2000). Pinheiro and Bates argue that adaptive Gaussian quadrature gives one of best approximations to the marginal likelihood function.

If the year random effect needs to be estimated, then equation (30) extends to

$$(31) \qquad l(\boldsymbol{\theta}\,|\,\mathbf{x},y)=\prod_{t=1}^{T}\prod_{j=1}^{J_t}\int_c^d\int_a^b f_y(y_{jt}\,|\,\mathbf{x}_{jt},\boldsymbol{\theta},u_t,v_t)f_v(v_t\,|\,\sigma_v^2)f_u(u_t\,|\,\sigma_u^2)\;dv_t du_t\,,$$

where *c* and *d* are the limits of the distribution of *u*, $f_u(\cdot)$ is the pdf of *u*, and $f_y(\cdot)$ is conditioned on both *v* and *u*. The symbol $\boldsymbol{\theta}$ represents a vector of the unknown parameters, defined as $\boldsymbol{\theta}=(\boldsymbol{\varphi},\sigma_{\varepsilon*}^{2},\sigma_{u}^{2},\sigma_{v}^{2})'$.

Data and Empirical Procedures

Data were obtained from a long-term experiment conducted at the North Central Oklahoma research station near Lahoma. The study was established in 1970 to investigate winter wheat grain yield response to fertilizer application, using a randomized complete block design (Raun et al.; Westerman et al.). The treatments include a control (no nitrogen) and five levels of nitrogen (20, 40, 60, 80, and 100), in pounds per acre per year. Each treatment was replicated four times. Data from 28 years (1971-1998) were used for estimation.

Both a stochastic and nonstochastic LRP are estimated with the wheat yield response data. In both cases, the year effect is assumed to be random. Both the model parameters and the variance components were estimated jointly with the SAS NLMIXED procedure, using the adaptive Gaussian quadrature approximation of equations (30) and (31), respectively. The random effects are assumed to be normally distributed. As with any nonlinear optimization, convergence is not assured. The problem had to be carefully scaled to obtain convergence.

Results

A summary of the estimation results for the stochastic and nonstochastic response functions is reported in table 1. All parameters and variance components are significant at the one percent level. The restriction of a nonstochastic plateau is soundly rejected. The expected plateau wheat grain yield is about 41 bushels per acre for both models. The threshold level of nitrogen is 64 pounds per acre for the LRP, but only 50 pounds per acre for the linear response stochastic plateau function. One key difference is that the estimated marginal productivity of nitrogen is higher with the stochastic model.

The optimum level of nitrogen when the plateau is nonstochastic is either zero or 64 pounds per acre. With wheat price assumed to be \$3.00/bu, the VMP of nitrogen is \$0.70/lb. The optimal choice of nitrogen remains at 64 pounds per acre as long as the price of nitrogen is above zero and is less than the VMP of \$0.70/lb.

For the linear response stochastic plateau function, the optimal level of nitrogen changes with the price of nitrogen. Figure 2 contains the optimal level of nitrogen for three price ratios for both the linear response stochastic plateau and the conventional LRP (nitrogen prices of \$0.01, \$0.10, and \$0.60 per pound and a wheat price of \$3.00 per bushel). The optimal level of nitrogen at these three prices is 67, 58, and 47 pounds per acre.

Notice that when r = \$0.10/lb, which is close to current prices, the optimal level of nitrogen is less under the linear response stochastic plateau than it is under the conventional LRP. The major reason for this difference is the greater marginal productivity of nitrogen with the linear response stochastic plateau. As Figure 2 demonstrates, fertilizer recommendations with the linear response stochastic plateau can be either less than or greater than recommendations with the nonstochastic plateau depending on relative prices. The seemingly contradictory empirical observations, with some authors arguing that farmers applied less nitrogen than recommended (de Janvry; Ryan and Perrin) and others arguing otherwise (Babcock), are perfectly consistent with expected profit maximization and may not have much to do with risk aversion. Also, Babcock's model is derived from assumptions, and it does not have the stochastic plateau with a higher marginal productivity of nitrogen.

Table 2 shows expected profits for each of the cases depicted in figure 2. Again, profits will vary according to the value of the output/input price ratio. The losses from using a nonoptimal level of nitrogen are small. Thus, it should not be a surprise to observe successful farmers using a range of nitrogen levels.

Current recommendations from OSU's agronomy department are to apply two pounds of nitrogen for each bushel of yield goal. With a yield goal of 41 bu./acre, the advice would be to apply 82 lbs./acre. Thus, current recommendations are higher than those with either plateau model.

Conclusions

Determination of optimal fertilizer levels has been studied for many decades. A number of researchers have found that crop-response-to-nitrogen functions that include a yield plateau are more appropriate than functions that do not include a plateau. In prior work, the plateau has usually been assumed nonrandom. However, the determinants of the plateau are stochastic. BHP used the switching regression technique to estimate an alternative form of a stochastic plateau. The linear response stochastic plateau function developed here is not nested with the BHP model. The linear response stochastic plateau model is estimated with data obtained from a long-term wheat grain yield response to nitrogen experiment. With current prices, the optimal level of nitrogen was lower with the stochastic plateau than with the nonstochastic one.

The use of the LRP with stochastic plateau provides insight into why farmers may apply more or less nitrogen than would appear optimal. The optimum level of nitrogen for a linear response stochastic plateau can be lower or higher than that of an LRP depending on output/input price ratio. Thus, the seemingly contradictory empirical observations, with some authors arguing that farmers applied less nitrogen than recommended (de Janvry; Ryan and Perrin) and others arguing otherwise (Babcock). Also, the expected profit function is relatively flat with current prices and so the optimal level is likely difficult for farmers to determine.

References

Ackello-Ogutu, C., Q. Paris, and W.A. Williams. "Testing a von Liebig Crop Response Function against Polynomial Specifications." *Amer. J. Agr. Econ.* 67(1985):873-80.

Babcock, B.A. "The Effects of Uncertainty on Optimal Nitrogen Applications." *Review Agr. Econ.* 14(July 1992):271-80.

Beal, S.L. and L.B. Sheiner. "Heteroscedastic Nonlinear Regression." *Technometrics* 30(1988):327-38.

Berck, P. and G. Helfand. "Reconciling the von Liebig and Differentiable Crop Production Functions." *Amer. J. Agr. Econ.* 72(1990):985-96.

Bock, B.R. and F.J. Sikora. "Modified-Quadratic/Plateau Model for Describing Plant Response to Fertilizer." *Soil Sci. Soc. Amer. J.* 54(1990):1784-89.

Bullock, D.G., and D.S. Bullock. "Quadratic and Quadratic-Plus-Plateau Models for Predicting Optimal Nitrogen Rate of Corn: A Comparison." *Agron. J.* 86(1994):191-95.

Cerrato, M.E., and A.M. Blackmer, "Comparison of Models for Describing Corn Yield Response to Nitrogen Fertilizer." *Agron. J.* 82(1990):138-43.

Chambers, R.G., and E. Lichtenberg. "A Nonparametric Approach to the von Liebig-Paris Technology." *Amer. J. Agr. Econ.* 78(1996):373-386.

Cox, F.R. "Economic Phosphorus Fertilization Using a Linear Response and Plateau Function." *Comm. Soil Sci. Plant Anal.* 27(1996):531-43.

___. "Range in Soil Phosphorus Critical Levels with Time." *Soil Sci. Soc. Amer. J.* 56(1992):1504-09.

Davidian, M. and A.R. Gallant. "The Nonlinear Mixed Effects Model with a Smooth Random Effects Density." *Biometrika* 80(1993):475-88.

de Janvry, Alain. "Optimal Levels of Fertilization Under Risk: The Potential for Corn and Wheat Fertilization Under Alternative Price Policies in Argentina." *Amer. J. Agr. Econ.* 54(1972):1-10.

Frank, M.D., B.R. Beattie, and M.E. Embleton. "A Comparison of Alternative Crop Response Models." *Amer. J. Agr. Econ.* 72(1990):597-603.

Ghomi, S.M.T.F., and S.S. Hashemin. "A New Analytical Algorithm and Generation of Gaussian Quadrature Formula for Stochastic Network." *European J. Oper. Res.* 114(1999):610-25.

Grimm, S.S., Q. Paris, and W.A. Williams. "A von Liebig Model for Water and Nitrogen Crop Response." *West. J. Agr. Econ.* 12(1987):182-92.

Heady, E.O. and J. Dillon. *Agricultural Production Functions*. Ames: Iowa State University Press, 1961.

Heady, E.O. and J. Pesek. "A Fertilizer Production Surface with Specification of Economic Optima for Corn Grown on Calcareous Ida Silt Loam." *J. Farm Econ.* 36(1954):466-82.

Heady, E.O., J. Pesek, and W.G. Brown. "Crop Response Surfaces and Economic Optima in Fertilizer Use." Iowa State University Agr. Exp. Sta. Res. Bull. No. 424, March 1955.

Johnson, P.R. "Alternative Functions for Analyzing a Fertilizer-Yield Relationship." *J. Farm Econ.* 35(1953):519-29.

Judge, G.G., R.C. Hill, W.E. Griffiths, H. Lutkepohl, and T.C. Lee. *Introduction to the Theory and Practice of Econometrics.* New York: John Wiley & Sons, Inc., 1988.

Kennedy, P. *A Guide to Econometrics, Third Edition,* Cambridge, Massachusetts: The MIT Press, 1996.

Khuri, A. I. *Advanced Calculus with Applications in Statistics*. New York: John Wiley & Sons, Inc., 1993.

Lanzer, E.A., and Q. Paris. "A New Analytical Framework for the Fertilization Problem." *Amer. J. Agr. Econ.* 63(1981):93-103.

Lindstrom, M.J. and D.M. Bates. "Nonlinear Mixed Effects Models for Repeated Measures Data."" *Biometrics* 46(September 1990):673-87.

Littell, R.C., G.A. Milliken, W.W. Stroup, and R.D. Wolfinger. *SAS System for Mixed Models*, Cary, NC: SAS Institute Inc., 1996.

Liu, S. "Gaussian Quadrature and its Applications (Numerical Integration, Sensitivity Analysis, Design Centering, Maximum Entropy)." Ph.D. Dissertation, Purdue University, 1997.

Llewelyn, R.V. and A.M. Featherstone. "A Comparison of Crop Production Functions Using Simulated Data for Irrigated Corn in Western Kansas." *Agr. Systems* 54(1997):521-38.

Maddala, G.S., and F.D. Nelson. "Maximum Likelihood Methods of Markets in Disequilibrium." *Econometrica* 42(1974): 1013-30.

Nicol, K.J., E.O. Heady, and H.C. Madsen. "Models of Soil Loss, Land and Water Use, Spatial Agricultural Structure, and the Environment." Iowa State University CARD Rep. 49T, July 1974.

Paris, Q. "The von Liebig Hypothesis." *Amer. J. Agr. Econ.* 74(1992):1019-28.

Paris, Q., and K. Knapp. "Estimation of von Liebig Response Functions." *Amer. J. Agr. Econ.* 71(1989): 178-86.

Pautsch, G.R., B.A. Babcock, and F.J. Breidt. "Optimal Information Acquisition under a Geostatistical Model." *J. Agricultural and Resource Economics* 24 (1999): 342-366.

Perrin, R.K. "The Value of Information and the Value of Theoretical Models in Crop Response Research." *Amer. J. Agr. Econ.* 58(1976):54-61.

Pinheiro, J.C. and D.M. Bates. "Approximations to the Log-Likelihood Function in the Nonlinear Mixed-Effects Model." *J. Computational and Graphical Stat.* 4(1995): 12-35.

Raun, W.R., G.V. Johnson, S.B. Phillips and R.L. Westerman. "Effect of Long-Term N Fertilization on Soil Organic C and Total N in Continuous Wheat under Conventional Tillage in Oklahoma." *Soil & Tillage Res.* 47(1998):323-30.

Ryan, J.G., and R.K. Perrin. "Fertilizer Response Information and Income Gains: The Case of Potatoes in Peru." *Amer. J. Agr. Econ.* 56(1974):337-43.

SAS Institute Inc. *SAS/STAT User's Guide, Volume 2: GLM-VARCOMP, Version 6 Fourth Edition.* Cary, NC: SAS Institute Inc., 1990.

__. *The NLMIXED Procedure (Draft)*, Available at http://www.sas.com/rnd/app/da/new/expdoc/chap3/, May 1, 2000.

Sheiner, L.B. and S.L. Beal. "Evaluation of Methods for Estimating Population Pharmacokinetic Parameters. I. Michaelis-Menten Model: Routine Clinical Pharmacokinetic Data." *J. Pharmacokinetics and Biopharmarceutics* 8(1980):553-71.

Spillman, W.J. *Use of the Exponential Yield Curve in Fertilizer Experiments*. Washington DC: U.S. Department of Agriculture, Tech. Bull. No. 348, April 1933.

Stiegert, K.W. and T.W. Hertel. "Optimal Capacity in the Anhydrous Ammonia Industry." *Amer. J. Agr. Econ.* 79(1997):1096-1107.

Sumelius, J. "A Response Analysis of Wheat and Barley to Nitrogen in Finland." *Agr. Sci. Finland* 2(1993):465-79.

Vonesh, E.F. and R.L. Carter. "Mixed-Effects Nonlinear Regression for Unbalanced Repeated Measures." *Biometrics* 48(1992):1-17.

Wackerly, D.D., W. Mendenhall, and R.L. Scheaffer. *Mathematical Statistics with Applications.* Boston: Duxbury Press, 1996.

Waugh, D.L., R.B. Cate and L.A. Nelson. *Discontinuous Models for Rapid Correlation, Interpretation and Utilization of Soil Analysis and Fertilizer Response Data*. Technical Bulletin No. 7, Soil Fertility Evaluation and Improvement Program, North Carolina State University, August 1973.

Westerman, R.L., R.K. Boman, W.R. Raun, and G.V. Johnson. "Ammonium and Nitrate Nitrogen in Soil Profiles of Long-Term Winter Wheat Fertilization Experiments." *Agron. J.* 86(1994):94-99.

Wolfinger, R.D. "Laplace's Approximation for Nonlinear Mixed Models." *Biometrika* 80(1993):791-795.

Wolfinger, R.D. "Fitting Nonlinear Models with the New NLMIXED Procedure." *SUGI Proceedings*, SAS Institute Inc., Cary, North Carolina, 1999.

Table 1. Summary of regression results for wheat yield response functions

		Estimates and standard errors by type of response function[a]	
Statistic	Symbol	Stochastic linear plateau	Linear plateau
Intercept	β_0	25.3621	26.1954
		(1.2926)	(1.3590)
Level of nitrogen (lbs)	β_1	0.3075	0.2322
		(0.0172)	(0.0142)
Expected plateau yield (bu)	μ_m	40.5938	41.1260
		(1.3241)	(1.3290)
Nitrogen at expected plateau (lbs)	x_m	49.5340	64.3126
		(7.5096)	(3.1638)
Variance of plateau yield	σ_v^2	5.3324	
		(0.8491)	
Variance of year random effect	σ_u^2	39.3126	42.2476
		(11.1531)	(11.9921)
Variance of error term	$\sigma_{\varepsilon*}^2$	29.9353	43.5248
		(2.1326)	(2.4701)
Log likelihood[b]		-2122.8	-2185.1

[a]Standard errors are in parentheses.

[b]The null hypothesis that the nonstochastic plateau is the correct model (ie. $H_0 : \sigma_v^2 = 0$) is rejected at any conventional level of significance based on a likelihood ratio test. The calculated value of the likelihood ratio statistic is 124.6 which is considerably above the $\chi^2_{1,.01}$ critical value of 6.63.

Table 2: Expected Profit by Price of Nitrogen, Assuming the Linear Response Stochastic Plateau Is the Correct Model and p=$3.00/bu.

Function	Profit by price of nitrogen (*r*)		
	$0.01/lb	$0.10/lb	$0.60/lb
LRSP[a]	121.09	115.54	89.50
LRP[b]	121.07	115.29	83.13
Difference (SLRP-LRP)	0.02	0.25	6.37

[a] For a linear response stochastic plateau, the optimal quantity of nitrogen is 66.77lb/acre, 58.81lb/acre and 46.63lb/acre when *r* is equal to $0.01/lb, $0.10/lb and $0.60, respectively. This translates into an expected yield, E(y|x), of 40.58 bu, 40.47 bu and 39.16 bu, respectively.
[b] For an LRP, because MVP>MFC at all three prices, $x = x_m$= 64.31 lb/acre, which translates into $E(y|x)$ = 40.57.

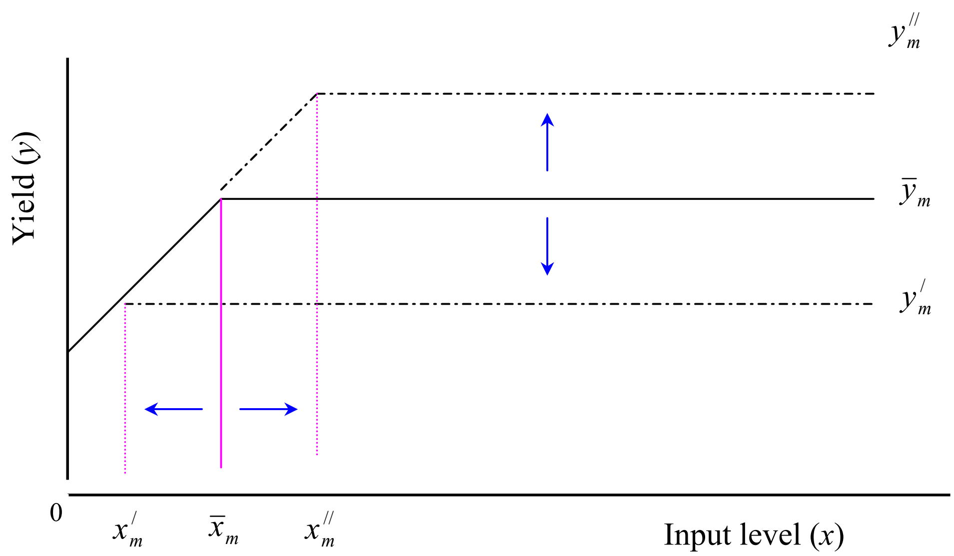

Figure 1. Univariate stochastic linear plateau response function

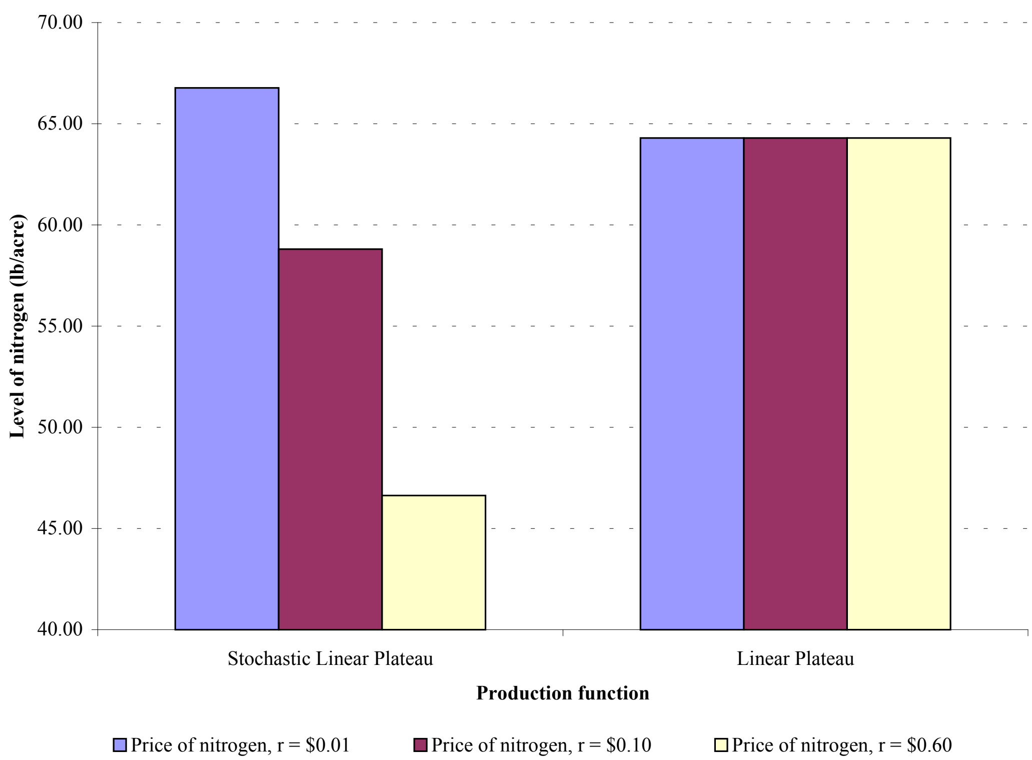

Figure 2. Expected profit maximizing levels of nitrogen derived from the stochastic linear plateau and linear plateau functions for varying price ratios (price of wheat constant at $3 per bushel).