Returns to Scale and Size in Agricultural Economics

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Differences between the concepts of returns to size and returns to scale are systematically reexamined in this paper. Specifically, the relationship between returns to scale and size are examined through the use of the envelope theorem. A major conclusion of the paper is that the level of abstraction in applying a cost function derived from a homothetic technology within a relevant range of the expansion path may not be severe when compared to the theoretical, estimative, and computational advantages of these technologies.

Key words: elasticity of scale, envelope theorem, returns to size.

Agricultural economists investigating long-run relationships among levels of inputs, outputs, and costs generally use concepts of returns to scale and size. Econometric studies which utilize production or profit functions commonly are concerned with returns to scale (de Janvry; Lau and Yotopoulos), while synthetic studies of relationships between output and cost utilize returns to size (Hall and LaVeen; Richardson and Condra). However, as noted by Bassett, the word scale is sometimes also used to mean size (of a plant). Thus, increasing (decreasing) returns to scale are used to refer to economies (diseconomies) of size, and vice versa. For example, Raup uses scale and size interchangeably in discussing economies and diseconomies of "large-scale" agricultural production. Feder, and Gardner and Pope also invoke the scale concept, but the bulk of their discussions center on the farm size issue. Such use of these concepts presents an interpretation problem for researchers and students in agricultural economics.

Previous research addressing this problem generally has not been definitive. McElroy establishes the relationship between returns to scale, Euler's theorem, and the form of production functions. However, the direct link between returns to size and scale is not developed. Hanoch investigates the elasticity of scale and size in terms of variation with output and illustrates that only at the cost-minimizing input combinations are the two concepts equivalent. Hanoch further provides a proposition stating that Frisch's "Regular Ulta-Passum Law" is neither necessary nor sufficient for a production technology to be associated with U-shaped average cost curves. Unfortunately, Hanoch's article does not consider specific production technologies such as homothetic or ray-homogenous technologies and does not explicitly establish the relationships between returns to scale and size. In an effort to clarify the concepts of size and scale economies, Chambers, in a 1984 proceedings paper on economies-of-size studies, presents the two concepts and discusses their interrelationships. Chambers presents a sound discussion of the concepts blending the works of previous efforts within this area. However, despite Chambers' and other past researchers' efforts to delineate the similarities and differences between returns to scale and size, confusion still exists in the profession. As noted in a recently published textbook by Beattie and Taylor, the distinction between returns to scale and size is not always clear, and the terminology is at times, inappropriately, employed interchangeably.

The purpose of this paper is to reexamine

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systematically the differences between the concepts of returns to scale and size. Specifically, the relation between returns to scale and size is examined through the use of the envelope theorem. In the first section, the production and cost relationships between size and scale are investigated, and based on these results, an envelope relation between the two concepts is developed. The final section concludes with some methodological views on the distinctions between these concepts.

Production and Cost Relationships between Size and Scale

The function coefficient ϵ is the most common means of discriminating scale economies. This coefficient is known under various names, such as elasticity of scale, local returns to scale, elasticity of production, and passus coefficient. Specifically, ϵ is defined as the proportional change in output resulting from a unit proportional change in all inputs. Mathematically, as noted by McElroy,

(1) $$\epsilon(y, X/|X|) = (\nabla f \cdot X)/y,$$

where ∇f is derived from $y = f(X)$, a production function which is assumed to be regular, monotone, and convex (Varian). Finally, as noted by Chambers (1985), $|X|$ is the Euclidean norm of the original input vector X and $X/|X|$ characterizes a ray from the origin in Euclidean N space.[1]

A general representation of variations in output associated with a proportional increase in inputs (k) is

(2) $$f(kX) = G[k, X/|X|, f(X)].$$

Equation (2) corresponds to a class of production functions with expansion paths which are linear from the origin. This class of functions may be called ray production functions and can be simplified as follows (McElroy):

(3a) $$f(kX) = k^r \cdot f(X),$$

a homogenous technology with r indicating the degree of homogeneity;

(3b) $$f(kX) = F[k^r \cdot h(X)],$$

a homothetic technology, where $h(X) = F^{-1}[f(X)]$ and F is a monotonic transform of the technology;

(3c) $$f(kX) = k^{H(X/|X|)} \cdot f(X),$$

a ray-homogenous technology, where $H(X/|X|)$ is a strictly positive and bounded function; and

(3d) $$f(kX) = F[k^{H(X/|X|)} \cdot h(X)],$$

a ray-homothetic technology (Chambers 1985).

Returns to size, alternatively called returns to outlay, can be defined as the proportional change in output associated with a proportional change in cost or outlay. Specifically, elasticity of size is denoted

$$\partial \ln y / \partial \ln c = \eta^{-1},$$

where η is defined as the elasticity of cost (Varian, Chambers 1984). Thus, based on the properties of elasticity, elasticity of size is equal to the ratio of average cost to marginal cost and is directly associated with the average cost curve.

As implied by McElroy, ray production functions process the following relation between elasticity of scale and size

(4) $$\eta^{-1} = \epsilon.$$

The conditions establishing (4) differ for the four technologies represented in (3). From (3a) it can be demonstrated that $\epsilon = r$, and thus (4) does not depend on the input or output levels. The cost function associated with (3a) is multiplicatively separable in output y and input price vector w, $c = y^{1/r}A(w)$. Thus, $\eta^{-1} = r$. For a homogenous technology, ϵ and η^{-1} are constant in the isoquant space and exhibit constant proportional returns. Equation (4) then holds at every point within the isoquant space. Every point in the isoquant space is a sufficient but not necessary condition for (4) to hold.

For a homothetic technology (3b), scale elasticity is defined as $\epsilon = (\nabla f \cdot X)/f(X) = h(y)/[h'(y)y]$, where $h(y) = F^{-1}(y)$ and $h'(y) = dh(y)/dy$ (Chambers 1985). In this case ϵ is a function of output and exhibits variable proportional returns. Denoting marginal cost as λ, size and scale elasticity for a homothetic technology may be related with the aid of the following proposition.

PROPOSITION 1. $c = \lambda[h(y)/h'(y)]$, *if and only if $f(X)$ is homothetic.*

Proof. Consider the cost function $c = w \cdot X(w,$

[1] The norm in Euclidean N space, R^n, is defined as $|X| = (X \cdot X)^{1/2}$ and is interpreted as the distance in R^n from X to the origin. $X/|X|$ is a vector of trigonometric functions defining an angle associated with a ray through the origin R^n. In R^2, $X/|X| = [\sin\theta, \cos\theta]$, where θ is an angle between 0 and $\pi/2$. Taking the inner product of $[\sin\theta, \cos\theta]$ with itself yields $\sin^2\theta + \cos^2\theta = 1$. In this manner, all points can be defined on the unit circle and the level of output is dependent on the angle θ.

y). All X on the expansion path are characterized by

$$w = F'(h)\nabla f^*, \tag{5}$$

where $h(y) = f^*(X)$ is a linear homogenous function. Substitute (5) into the cost function and using Euler's theorem, $c = \lambda F'(h)h(y)$. Noting that F is monotonic,

$$c = \lambda[h(y)/h'(y)]. \tag{6}$$ Q.E.D.

Dividing both sides of (6) by y and λ establishes the relation between size and scale elasticity

$$\epsilon = [h(y)/h'(y)y] = c/\lambda y = \eta^{-1}.$$

For a homothetic technology, (4) is true when y associated with ϵ corresponds to the cost-minimizing level of output. A necessary condition for (4) to be true is for ϵ to be associated with all points on the isoquant corresponding to the cost-minimizing output level. A sufficient condition is a point on the expansion path. Although (4) still holds at every point in the isoquant space, the value of the equality changes with output.

For a ray-homogenous technology, (3c), equation (4) no longer holds for all points in the isoquant space. For this technology, a point on the expansion path is a necessary and sufficient condition for (4) to be true. Size and scale elasticities are defined as $\epsilon = k^{H(X/|X|)}$ and $\eta^{-1} = k^{H(X/|X|)}$. Only when input ratios are the same for ϵ and η will (4) hold. Thus, when all inputs are increased by a fixed proportion, being on the expansion path is a necessary and sufficient condition for (4) to be true. The same correspondence holds when the production function is generalized to a ray-homothetic technology and to nonray production function classes.

Size as the Envelope of Scale

Hanoch illustrates for nonray production functions that elasticity of scale and size are equivalent only at points on the expansion path. He concludes that the behavior of the elasticity of scale does not yield conclusive results in determining the shape of average costs. Further investigation of this issue reveals an important property between the two elasticities.

PROPOSITION 2. *Elasticity of size is the envelope of elasticity of scale, or the long-run average cost curve is the envelope of average cost curves associated with output along a ray from the origin called scale average cost curves.*

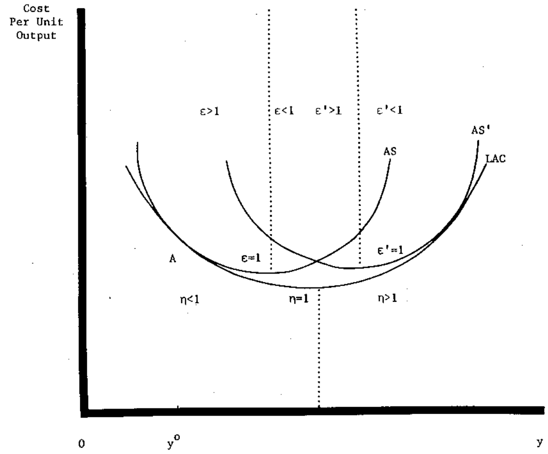

Figure 1. Average cost and scale average cost curves

Proof. Consider the following scale-cost function, $c(y, X(y))$, where $X(y)$ is the vector of cost-minimizing inputs on a scale expansion path. Let y^* be the level of output at which the actual cost function, $c(y)$ is equivalent to the scale-cost function,

$$c(y^*) = c(y^*, X(y^*)).$$

Let $X^* = X(y^*)$ be the associated cost-minimizing inputs for output level y^*, such that $c(y^*) = p \cdot X^*$. At any output level, the scale cost function must be at least as great as the ordinary cost function. Thus, the scale cost curve and ordinary cost curve must be tangent at y^*, which is a geometrical statement of the envelope theorem (Varian). The slope of the cost curves at y^* is

$$dc(y^*)/dy = \partial c(y^*, X(y^*))/\partial y + [\partial c(y^*, X(y^*))/\partial X](\partial X/\partial y),$$

but

$$\partial c[y^*, X(y^*)]/\partial X = 0.$$ Q.E.D.

Proposition 2 ties two important relationships of elasticities of scale and size together. First, Hanoch demonstrated that the change in the two elasticities with respect to a change in y are not equivalent unless a ray production technology exists. This result is apparent from Proposition 2 and is illustrated in figure 1. Elasticity of size and scale are equal at point A associated with scale average cost curve AS and output y^o. However, for alternative output levels other than y^o, η^{-1} and ϵ differ. Second, Frisch's "Regular Ulta-Passum Law" states that for an outward movement along an arbitrary curve in an input space, the function coefficient will decrease from a value in excess of one to values less than zero. This law establishes the U-shaped scale average cost curves in figure 1. Thus, Hanoch's concept and Frisch's law are directly related through proposition 2. Further, Hanoch's proposition 3, stating that Frisch's law is neither necessary nor sufficient for "classical" U-shaped average cost curves, follows directly from proposition 2. Elasticity of size, as the envelope of scale and Frisch's Law, implies that ϵ may range from $-\infty$ to ∞, within the range of $\eta > 1$, figure 1.

Conclusion

Basic concepts such as economies of scale and size can be very confusing when a clear distinction is not presented. Economies of scale is solely related to the particular production technology and does not require economic efficiency as one of its determinants. Economies of size requires the firm to be operating on its input efficiency locus and thus is an economic efficiency concept. In the cost space, this corresponds to long-run average cost. Declining long-run average cost indicates increasing returns to size. Similarly, rising average cost implies decreasing returns to size. A rigorous definition of the size and scale relationship reveals that elasticity of size is the envelope of elasticity of scale. Within the neighborhood about any point on the expansion path where $\epsilon = \eta^{-1}$, elasticity of scale can approximate elasticity of size and allows for the direct derivation of a cost function based on an associated ray production technology, such as the homogenous technology. Thus, the level of abstraction in applying a cost function derived from a homogenous production function within a relevant range of the expansion path may not be severe when compared to the theoretical estimative and computational advantages of homogenous functions. For example, Hoch reported Cobb-Douglas functions were superior to quadratic functions; however, returns to scale varied among regions which he attributed to different sized production units.

McElroy concluded that in view of the convenient properties of ray production functions, it is useful in many general economic applications to assume this class of functions. This assumption also does not appear to be damaging to agricultural economics. The widespread use of enterprise budgets to estimate costs of production, the common practice of including only one production activity for each enterprise in mathematical programming models, and the estimation of homogenous production functions all, at least implicitly, assume a ray production technology. This view is at odds with the conventional beliefs of many agricultural economists. For example, Madden recently noted that economies of scale is an "empty box" which should be discarded as unrealistic. The continued parallel use of scale with size concepts in agricultural economics suggests that this recommendation is not likely to be followed. In large part, the persistence of scale is related to its theoretical value in interrelating cost functions and production technology. At this point in the development of agricultural economics, the standard that theoretical concepts must be completely realistic surely is not taken seriously. Most widely used economic concepts are unrealistic and could also be considered empty boxes. Utility functions and the perfectly competitive model are two examples. Furthermore, Boggess recently reminded us that the production function is another such abstraction. This paper suggests only that the same methodological standards be applied in the size versus scale issue as in other theoretical issues in agricultural economics.

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