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Bruce R. Beattie and Jeffrey T. LaFrance

DEPARTMENT OF AGRICULTURAL AND RESOURCE ECONOMICS
DIVISION OF AGRICULTURE AND NATURAL RESOURCES
UNIVERSITY OF CALIFORNIA AT BERKELEY

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Abstract

Diminishing marginal utility is neither necessary nor sufficient for downward sloping demand. Yet upper-division undergraduate and beginning graduate students often presume otherwise. This paper provides two simple counter examples that can be used to help students understand that the Law of Demand does not depend on diminishing marginal utility. The examples are accompanied with the geometry and basic mathematics of the utility functions and the implied ordinary/Marshallian demands.

Key Words: Convex preferences, Diminishing marginal utility, Downward sloping demand

JEL Classification: A22

- Bruce R. Beattie is a professor in the Department of Agricultural and Resource Economics at The University of Arizona.
- Jeffrey T. LaFrance is a professor in the Department of Agricultural and Resource Economics and a member of the Giannini Foundation of Agricultural Economics at the University of California, Berkeley.

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THE LAW OF DEMAND VERSUS DIMINISHING MARGINAL UTILITY

In a combined total of more than a half century of university teaching experience, many students in our advanced undergraduate, master's, and beginning PhD level courses have come to us convinced that the principle of diminishing marginal utility (DMU) is a primary explanation for and cause of downward sloping demand (DSD) in the theory of consumer behavior. It has been generally accepted since the beginning of the 20^{th} Century that the Law of Demand does not require cardinal utility and the strong assumption of DMU (Samuelson, p. 93). Yet, clearly explaining why this is so continues to be a challenge in teaching consumption theory.

This paper presents two valid utility-function/applied-demand models that can be used by teachers of upper-division and beginning graduate courses to convince students that DMU is neither necessary nor sufficient for DSD nor necessary for convex preferences (downward-sloping-convex indifference curves, CIC). We have found the counter examples, and, in particular, the graphics provided in the paper to be most helpful in teaching the fundamentals of consumer behavior and demand. Generally speaking, it is at the upper-division undergraduate and beginning graduate level when most agricultural economics courses seriously tackle the theory of consumer behavior as foundation for applied demand and price analysis.

The paper is organized as follows. First, we show that DMU is *not necessary* for CIC. Although not essential for DSD, convex preferences are commonly presumed in the theory of consumer behavior. Next we establish that DMU is *not necessary* for DSD. Lastly, we show that DMU is not *sufficient* for DSD. Our approach is to construct simple counter examples for each case. We use two valid utility functions to develop the counter examples.[1] The utility functions are valid in that both satisfy the usual and accepted preference axioms of consumer theory (Varian; Henderson and Quandt; Silberberg and Suen; Mas-Colell, Whinston and Green). The essential mathematical results (utility function specifications, marginal utility equations, marginal utility slope equations, indifference curve equations, indifference curve slope and curvature equations, ordinary Marshallian demand equations and demand slope equations) and geometric interpretation are presented. The final section of the paper summarizes and concludes.

DMU, CIC, and DSD

Result 1. *DMU is* ***not necessary*** *for negatively-sloped-convex indifference curves*.

Assume a simple two-good (q_1, q_2) Stone-Geary utility (u) function:[2]

(1) $$u(q_1, q_2) = q_1^2 q_2^2 .$$

The marginal utility for good one, MU_1, for example, is given by

(2) $$\frac{\partial u}{\partial q_1} = 2q_1 q_2^2 .$$

The slope of (2) is

(3) $$\frac{\partial^2 u}{\partial q_1^2} = 2q_2^2 ,$$

which is strictly positive for all q_1, $q_2 > 0$; i.e., MU_1 is everywhere increasing**.** However, the indifference curve equation obtained by rearrangement of (1) is

(4) $$q_2 = \sqrt{u} / q_1 ,$$

a rectangular hyperbola in q_1. The slope and curvature of (4) is given by (5) and (6), respectively:

(5) $$\left.\frac{dq_2}{dq_1}\right|_u = -\frac{\sqrt{u}}{q_1^2} = -\frac{q_2}{q_1};$$

(6) $$\left.\frac{d^2q_2}{dq_1^2}\right|_u = \frac{2\sqrt{u}}{q_1^3} = \frac{2q_2}{q_1^2}.$$

Clearly (5) is strictly negative and (6) is strictly positive for all q_1, $q_2 > 0$.

Computer-generated, three-dimensional and two-dimensional graphs of the utility function (1) and indifference-curve map (4) are presented in figures 1a and 1b, respectively. It is readily seen in figure 1a that marginal utility (MU) increases everywhere on the utility surface for both q_1 and q_2. And in figure 1b one sees clearly that the indifference curves become more dense as one moves across the graph parallel to the q_1 axis, increasing q_1 while holding q_2 constant, or vice versa — again, reflecting increasing MU. Yet we observe in both figures that the indifference curves are everywhere negatively sloped per equation (5) and convex to the origin per equation (6).

Upshot: Despite the fact that the marginal utilities for both goods are everywhere increasing, the indifference curves are seen to be everywhere negatively sloped and convex to the origin. This simple utility function clearly shows that DMU is *not necessary* for CIC. More generally, we know that the convexity of the level curves for a two-variable model depends on an expression involving all first and second partial derivatives of the function. In the words of Silberberg and Suen, "…convexity of the indifference curves in no way implies, or is implied by, 'diminishing marginal utility,' . . . diminishing marginal utility and convexity of indifference curves are two entirely independent concepts. And that is how it must be: Convexity of an indifference curve relates to how marginal evaluation change *holding utility* (the dependent variable) *constant.* The concept of diminishing marginal utility refers to changes in total utilities, i.e., movements from one indifference level to another" (pp. 52-53).

Result 2. *DMU is* ***not necessary*** *for downward sloping demand.*

To establish that DMU is not necessary for downward sloping demand we continue with the Stone-Geary example used in establishing "Result 1" (depicted in figures 1a and 1b). The equation of the demand function for q_1, obtained from the solution of the first-order necessary conditions of the budget-constrained maximization of (1), is given by

$$q_1 = m/2p_1 , \tag{7}$$

where m is income and p_1 is the price of q_1.[3] The slope of (7) is

$$\frac{\partial q_1}{\partial p_1} = -\frac{m}{2p_1^2} . \tag{8}$$

Clearly, the own-price effect from (8) is strictly negative.

Upshot: The important feature depicted in this example is that the indifference curves are negatively sloped and convex as q_1 increases given u fixed. In consumer theory the behavior of the marginal utility relationship is immaterial. It is the convexity of the indifference curves, not DMU, that is crucial for DSD in this case.[4] Suffice it to say DMU *is not necessary* for DSD.

Results 1 and 2 follow from the fact that a utility function is unique only up to a monotonic transformation (Varian). An implication of non-uniqueness, in the context of our example Stone-Geary utility function, is that the implied demand functions for_the two goods are the same irrespective of the exponents on q_1 and q_2 in equation (1), so long as they are positive. This of course is another way of saying that whether marginal utility is decreasing (exponents <1), constant (exponents =1), or increasing (exponents >1) does not matter.

Result 3. *DMU is* ***not sufficient*** *for downward sloping demand.*

To establish that DMU is not sufficient for DSD we use as our example a utility function that gives rise to a linear upward-sloping demand for q_1 and exhibits DMU for both q_1 and q_2 (LaFrance)[5], viz.,

$$u(q_1, q_2) = (1 - q_1)\exp\left\{\frac{q_2 - 100}{1 - q_1}\right\}. \tag{9}$$

The demand function for q_1 implied by (9) is

$$q_1 = 101 + \frac{p_1}{p_2} - \frac{m}{p_2} \tag{10}$$

where m and p_1 are as defined previously and p_2 is the price of q_2. The slope of (10) with respect to p_1 is

$$\frac{\partial q_1}{\partial p_1} = \frac{1}{p_2}, \tag{11}$$

which is strictly positive rather than negative. Complete mathematical derivations, including establishment of DMU, are presented in the appendix for a generalized version of equation (9).

The essential geometry of this case is presented in figures 2a and 2b. Like the previous figures, figure 2a is the three-dimensional representation of the utility function and figure 2b is the two-space indifference map. In figure 2a we see that we have DMU for q_1 and for q_2. The curvature of the utility function in both the q_1 and q_2 direction is concave to the q_1q_2 plane. And in figure 2b we observe that the indifference curves become less dense as q_1 increases given q_2 and vice versa. Also we see clearly that the indifference curves are negatively sloped and convex to the origin in the regular region.

Upshot: DMU, in addition to being unnecessary, is not sufficient for DSD as sometimes alleged.[6] In this example we have a perfectly acceptable (well behaved) utility function giving rise to an upward sloping demand function even when the marginal utility of that good is diminishing—the long-known *Giffen good* case (Spiegel).

Conclusion

This paper provides examples of how to convince students of something that often must be unlearned, namely the idea that diminishing marginal utility is the principal rationale for convex indifference curves and downward sloping demand. The paper presents two simple utility functions to demonstrate the algebra and geometry of why:

- Diminishing marginal utility is *not necessary* for convex indifference curves.
- Diminishing marginal utility is *neither necessary nor sufficient* for downward sloping demand.

Downward sloping demand can be motivated by appealing to students' common sense. When asked, students will confess that when the price of a good rises, other things constant, they typically reduce their purchases of that good. The instructor can then proceed to explain that they do, what it is they know they do, for two reasons: First, they seek and find relatively less expensive substitutes. And, second, an increased price reduces effective purchasing power for all goods. For most (normal) goods the income effect reinforces the negative substitution effect, contributing further to reduced consumption of the subject good (Stigler, pp. 60-61; Slutsky). The result is unambiguous

DSD. There is no need to burden students with something (DMU) that is unnecessary and insufficient for making the case.

When students are ready for a formal treatment of consumer choice (typically at the intermediate level), the why and why not of DSD should be taught and learned drawing upon the assumptions and framework of ordinal utility maximization and the ideas of substitution and income effects. To dissuade students still wanting to believe that the underlying motivation for DSD is the cardinal utility idea of DMU, we have found the two utility functions and associated graphs in this paper most helpful.

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Appendix: Insufficiency of DMU for DSD

The following utility function gives rise to a positively-sloped linear demand function for q_1 despite DMU for both q_1 and q_2:

(A1)
$$u(q_1,q_2)=\frac{(\beta+\gamma q_1)}{\gamma^2}\exp\left\{\frac{\gamma(\alpha+\gamma q_2-q_1)}{\beta+\gamma q_1}\right\}$$

assuming $\alpha > m/p_1 > 0$, $\beta > 0$, and $\gamma < 0$. The first-order partial derivatives of (A1) are

(A2)
$$\frac{\partial u}{\partial q_1}=-\left(\frac{\alpha+\gamma q_2-q_1}{\beta+\gamma q_1}\right)\exp\left\{\frac{\gamma(\alpha+\gamma q_2-q_1)}{\beta+\gamma q_1}\right\}>0\ \forall\ \alpha+\gamma q_2>q_1>-\beta/\gamma\ .$$

(A3)
$$\frac{\partial u}{\partial q_2}=\exp\left\{\frac{\gamma(\alpha+\gamma q_2-q_1)}{\beta+\gamma q_1}\right\}>0\ \forall\ q_1\neq-\beta/\gamma\ .$$

The condition $q_1 \neq -\beta/\gamma$ is necessary for the utility function to be well-defined, while the conditions $\alpha+\gamma q_2 > q_1 > -\beta/\gamma$ are necessary for the utility function to be increasing in both goods. The ratio $-\beta/\gamma > 0$ may be arbitrarily small, but is not necessarily so. The utility function has a pole (can equal any real number) at the point $(q_1,q_2)=\left(-\beta/\gamma\,,\,-(\alpha\gamma+\beta)/\gamma^2\right)$.

Since $\dfrac{\partial u/\partial q_1}{\partial u/\partial q_2}=-\left(\dfrac{\alpha+\gamma q_2-q_1}{\beta+\gamma q_1}\right)=\dfrac{p_1}{p_2}$ and $q_2=\dfrac{m}{p_2}-\dfrac{p_1}{p_2}q_1$ at an interior solution for the demand equations, we obtain the demand for q_1 as

(A4)
$$q_1=\alpha+\beta\frac{p_1}{p_2}+\gamma\frac{m}{p_2}\ .$$

Note that the demand for good one is upward sloping with respect to its own price and downward sloping with respect to income, the classic case of a Giffen good. This property holds for all values of (p_1, p_2, m) that lead to an interior utility maximizing solution.

We now show that in the range where both goods are purchased in positive quantities and where preferences are strictly increasing in both goods, this utility function (A1) exhibits diminishing marginal utility in q_1 and in q_2. The second-order partial derivatives of (A1) are

(A5) $$\frac{\partial^2 u}{\partial q_1^2} = \frac{\left[\beta + \gamma\left(\alpha + \gamma q_2\right)\right]^2}{\left(\beta + \gamma q_1\right)^3} \exp\left\{\frac{\gamma\left(\alpha + \gamma q_2 - q_1\right)}{\beta + \gamma q_1}\right\} \le 0 \;\forall\; q_1 > -\beta/\gamma \;.$$

(A6) $$\frac{\partial^2 u}{\partial q_1^2} = \frac{\left[\beta + \gamma\left(\alpha + \gamma q_2\right)\right]^2}{\left(\beta + \gamma q_1\right)^3} \exp\left\{\frac{\gamma\left(\alpha + \gamma q_2 - q_1\right)}{\beta + \gamma q_1}\right\} \le 0 \;\forall\; q_1 > -\beta/\gamma \;.$$

Both (A5) and (A6) are strictly negative throughout the region of strict monotonicity, $\alpha + \gamma q_2 > q_1 > -\beta/\gamma$. In fact, *u* is concave and a simple transformation of this particular normalization (in particular, $-u^2$) is jointly strongly concave in (q_1, q_2), i.e.

(A7) $$\frac{\partial^2 u}{\partial q_1 \partial q_2} = \frac{-\gamma\left[\beta + \gamma\left(\alpha + \gamma q_2\right)\right]}{\left(\beta + \gamma q_1\right)^2} \exp\left\{\frac{\gamma\left(\alpha + \gamma q_2 - q_1\right)}{\beta + \gamma q_1}\right\}$$

(A8) $$|\mathrm{H}| = \left(\frac{\partial^2 u}{\partial q_1^2}\right)\left(\frac{\partial^2 u}{\partial q_2^2}\right) - \left(\frac{\partial^2 u}{\partial q_1 \partial q_2}\right)^2 \equiv 0 \;\forall\; (q_1, q_2).$$

The reason for condition (A8) is as follows. At any point (q_1, q_2) in the two-dimensional plane, define the constant $\gamma\left(\alpha + \gamma q_2 - q_1\right)/\left(\beta + \gamma q_1\right) = c$. Then the utility function is linear in q_1 (equivalently, linear in q_2, or jointly linear in q_1 and q_2) on the line defined by

$$q_2 = \left[-\alpha\gamma + \beta c + \gamma q_1(1 + c)\right]/\gamma^2 \;.$$

Note that this line passes through the point $\left(-\beta/\gamma, (\alpha\gamma+\beta)/\gamma\right)$, the pole of the utility function. Even so, the preference function is jointly concave in (q_1, q_2), and it is easy enough to show that the monotonic transformation $-u^2$ is a strictly concave function of the original u, which is strongly concave (has a strictly negative Hessian) throughout the region of regularity for u.

Thus, this utility function (or a simple transformation of it) possesses the property of diminishing marginal utility in both goods, yet generates a demand for one of the goods that violates the Law of Demand.

Endnotes

[1]The advantage of counter examples is, of course, that the *general* validity of a proposition can be refuted with a single counter example.

[2]The Stone-Geary utility function first appeared in the literature in the late 1940s, after its production economics counterpart – the Cobb-Douglas production function. Owing to its simplicity and tractability numerous textbook authors have used the Stone-Geary functional form to provide a concrete demonstration of convex indifference curves and the derivation of consumer demand functions. See, for example, Silberberg and Suen, Henderson and Quandt, Mas-Colell, Whinston and Green, and Varian.

[3]While unnecessary for our purpose here, a more general version of the Stone-Geary utility function, $u(q_1,q_2)=(q_1-\alpha_1)^{\beta}(q_2-\alpha_2)^{1-\beta}$, yields demands that are functions of both product prices, $q_i=\alpha_i+\beta_i(m-\alpha_1 p_1-\alpha_2 p_2)/p_i$, $i=1,2$, where $\beta_2=1-\beta_1$.

[4]We see when we get to "Result 3" that even convexity of the indifference curves does not guarantee DSD.

[5]This form of utility function generates a single linear demand equation (Hausman). This type of utility model is commonplace among applied researchers wanting to estimate systems of linear demands (Burt and Brewer; Cicchetti, Fisher, and Smith; LaFrance; LaFrance and deGorter; von Haefen).

[6]The untidy suggestion that DMU gives rise to (is sufficient for) DSD may trace to Friedman. In his influential *Price Theory* (1962, p.39), Friedman unfortunately stated, "Diminishing marginal utility will provide a negative slope for the demand curve. …"

Figure 1a. Surface plot for $u(q_1, q_2) = q_1^2 q_2^2$.

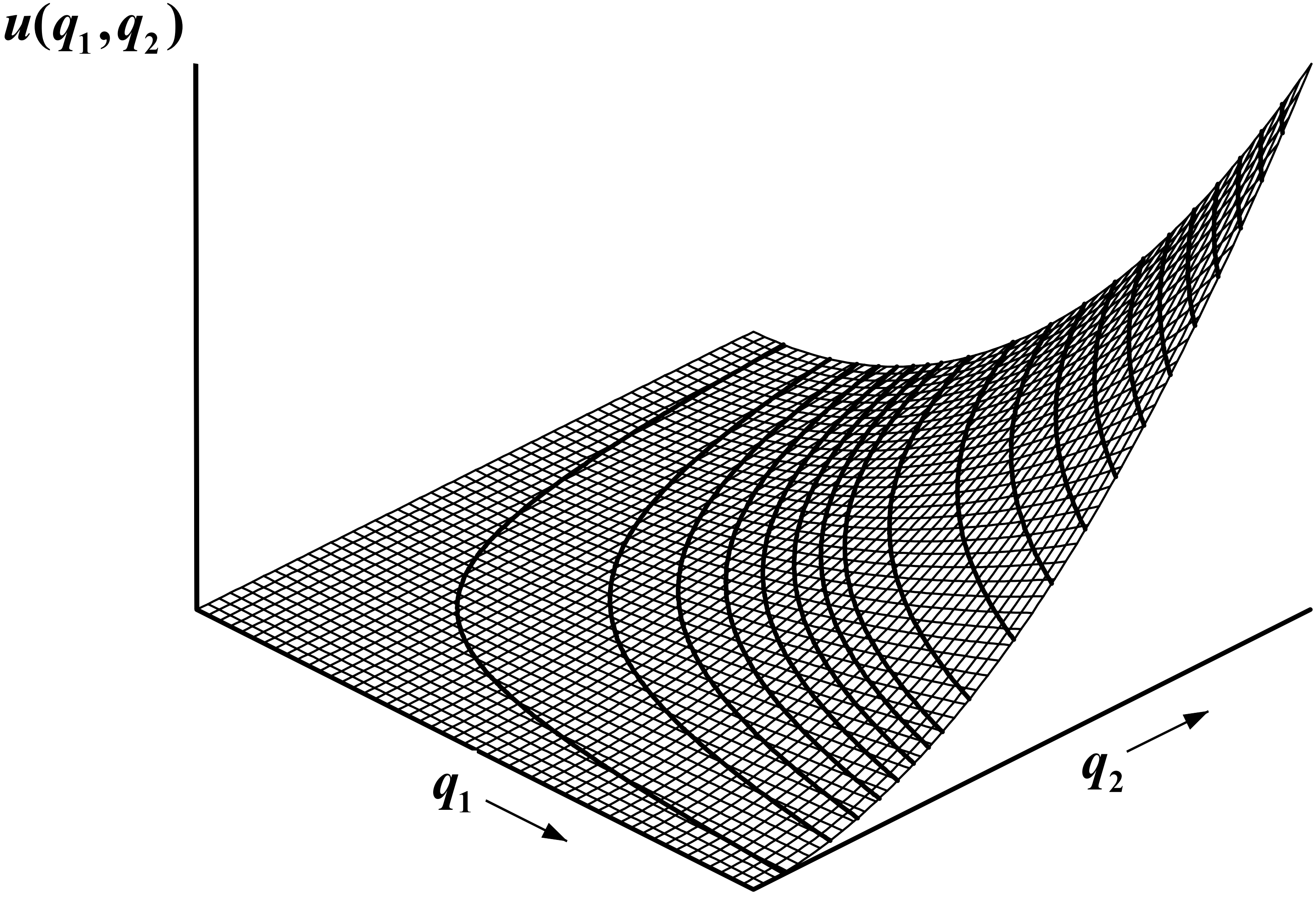

Figure 1b. Contour plot for $u(q_1, q_2) = q_1^2 q_2^2$.

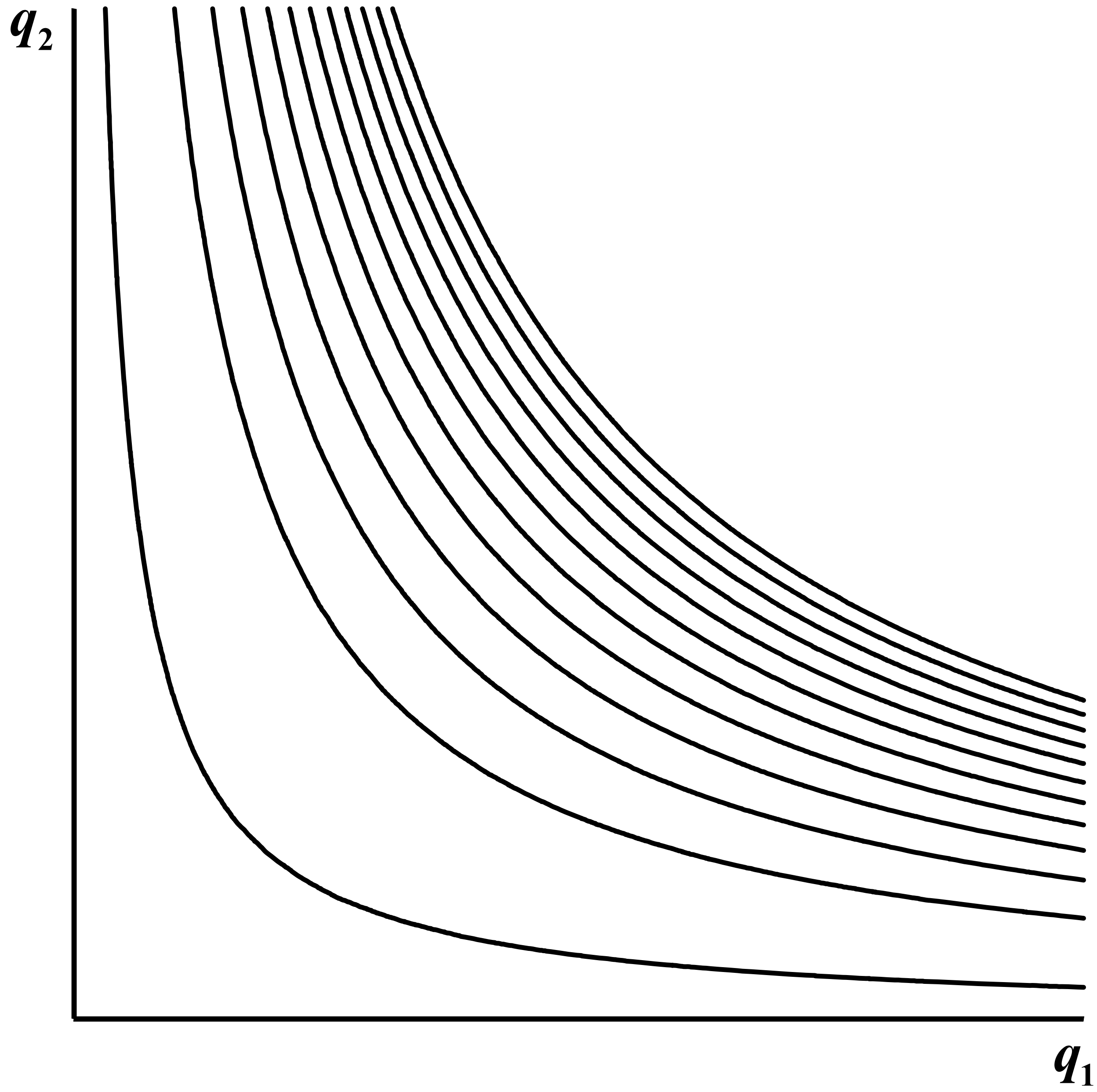

Figure 2a. Surface plot for $u(q_1, q_2) = (1 - q_1)\exp\left\{\dfrac{q_2 - 100}{1 - q_1}\right\}$.

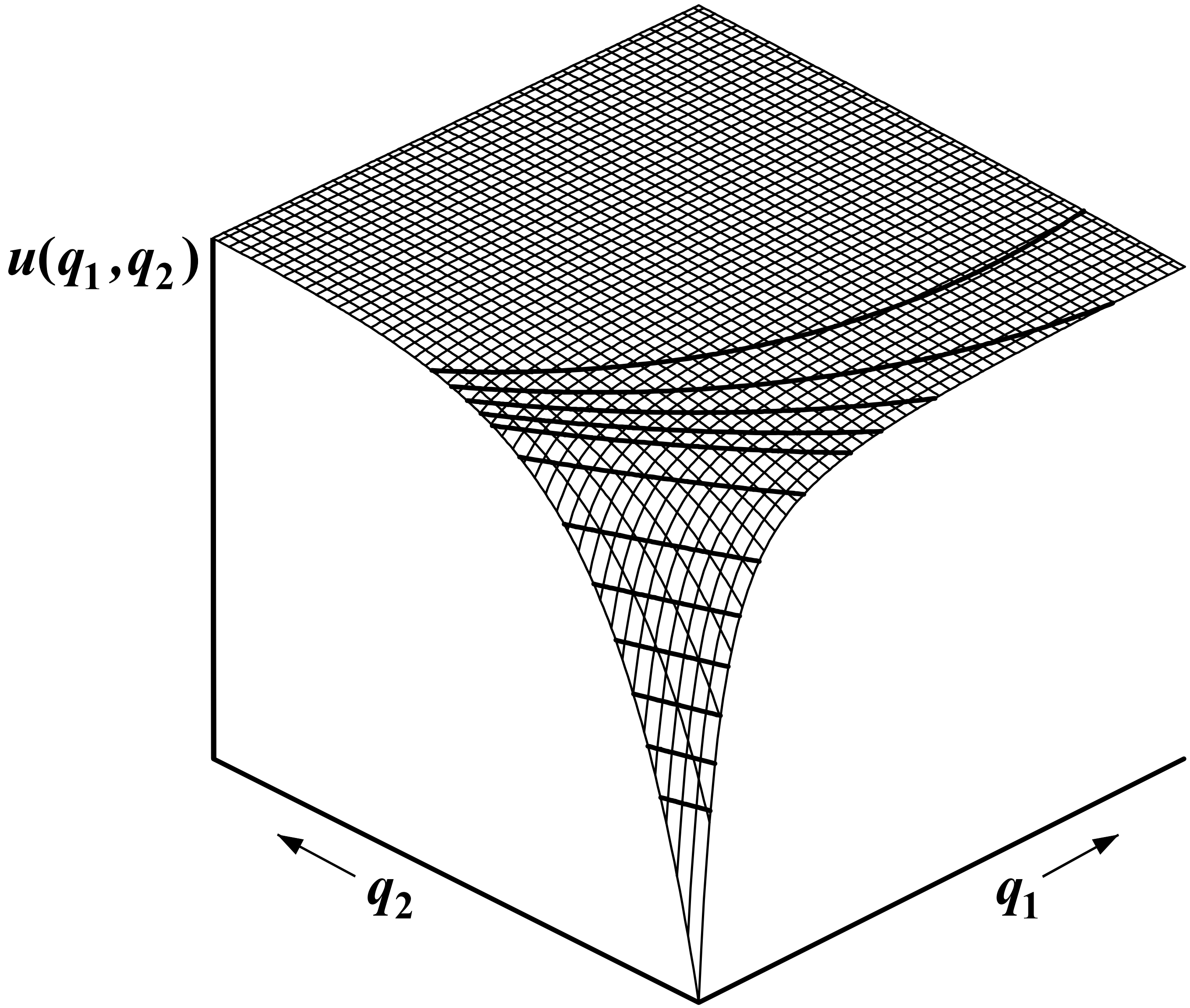

Figure 2b. Contour plot for $u(q_1, q_2) = (1 - q_1)\exp\left\{\frac{q_2 - 100}{1 - q_1}\right\}$.