Destabilising Stabilisation Policy in a Dynamic Menu Cost Model.

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Abstract

This paper analyses how systematic stabilisation policy by monetary authorities may change individual firms' price adjustment decision. The model is a stochastic dynamic menu cost model that results in $(S,s)$-price rules where the price is fixed inside a band. The resulting price rigidity causes output to fluctuate, and hence there is room for stabilisation policy. This paper shows that such a policy might actually be destabilising in the sense that the zone of fixed prices widens, leading to larger output fluctuations. In fact, output can be completely stabilised by a policy that amplifies shocks.
1 Introduction

In recent years there has been a growing literature on stochastic dynamic menu cost models, featuring the so-called \((S,s)\) price rules where the price is fixed inside a band (Sheshinski and Weiss (1983), Danziger (1983, 1984) and more recently Caplin and Leahy (1991) and Dixit (1991) among others). Since prices are not changed inside this band, quantities have to adjust to equilibrate markets, leaving aside the issue of rationing. Thus, price rigidity caused by menu costs leads to output fluctuations. This suggests a case for government intervention in order to attempt to stabilise the economy. Nevertheless, only a few papers within this literature consider systematic stabilisation policy and none - with C.T. Hansen (1998) as the only exception - that stabilisation policy may change the firm’s price setting incentives.

This paper shows that stabilisation policy might actually prove destabilising in the sense that the zone of unchanged prices widens, leading to even larger output fluctuations. The intuition is straightforward; the firm knows that (monetary) authorities attempt to systematically stabilise demand. Thus, if it observes an extreme realisation of demand that without stabilisation would have called for a price change, it knows that the authorities to a certain extent conduct stabilisation. Therefore, there is no need to change the price. Hence, the firm allows even larger shocks until it itself adjusts the price. In fact, the monetary authorities can stabilise the economy completely by pursuing a policy of enhancing demand shocks to an extreme degree.

The model I use is described in detail in the following section, and it is an extension of Dixit (1991). In addition to Dixit (1991), I include a demand function and a nominal scale variable (money) in order to study the effect of monetary policy on output, and how the policy feeds back to the individual price adjustment decision. The model considers a monopoly whose demand is continuously exposed to shocks. At the same time the monetary authorities are capable of carrying out instantaneous stabilisation by offsetting demand shocks to some degree. It is demonstrated that systematic stabilisation policy may be destabilising. Starting from a state where there is no offsetting of shocks at all and then increasing the degree to which the monetary authorities offset the shocks, implies that the firm allows larger shocks to demand before it adjusts its price. Hence output fluctuates more. But this is only true up to a certain degree of stabilisation. The reason is that a very strong offsetting of shocks basically means that the firm faces a demand that is more or less constant (the volatility is low). Hence, the option value of waiting for new information at later periods is very low, and the firm does better by paying its menu costs, and thereby obtaining the profit maximising price.

Finally, it should be noted that the discussion of whether or not a systematic policy is stabilising has involved the state space. That is, the variation in output levels. It has not been concerned with the time dimension. Actually, it turns out that even though attempts to stabilise output may lead to more volatility, the time until these more extreme output realisations are observed, is longer. So in that sense systematic stabilisation policy is always stabilising.

There are many papers that examine the welfare consequences of menu costs in case of demand shocks in models of either monopoly or monopolistic competition (see for instance Mankiw (1985), Blanchard and Kiyotaki (1987) or Ball and Romer (1990)). Also, several
papers study the optimal rate of inflation (e.g. Danziger (1988), Benabou (1992)). But as mentioned above, the literature is very scarce on how a systematic stabilisation policy feeds back into the individual price decision problem. However, the basic idea that a systematic policy that ex ante was thought to be stabilising, may prove to be ex post destabilising because the private incentives have changed, is common between the present paper and C.T. Hansen (1998). He has a lemma showing that increasing the degree of stabilisation leads to larger output fluctuations in a dynamic menu cost model. But the two papers differ significantly in their approach to specifying the models and they differ in some of the results. The latter being accounted for exactly because of the different models.

While my paper considers a continuous time framework similar to the option pricing models in finance, and thus is directly comparable to much of the recent research within this area such as Dixit (1991), Caplin and Leahy (1991) and Cabellero and Engel (1993), C.T. Hansen considers a simplified version of the yeoman-farmer model from Ball and Romer (1990). One of the most important differences between these two approaches is that the time structure of C.T. Hansen's model is somewhat restrictive (but required for tractability). When a shock hits the economy, the firms decide whether to adjust or not. The central bank can only respond with some time lag, and during the period until it responds no new shocks arrive. Hence my model can be seen as providing a framework for studying continuous arrival of shocks, and thus as complementary to C.T. Hansen (1998).

As mentioned above, I show that when the degree of stabilisation is sufficiently strong, output, as expected, fluctuates less. In C.T. Hansen (1998) this feature does not arise. Instead the relationship between the degree of stabilisation and output fluctuations is monotone (and positive). The reason is the difference in the shock process. In the present paper stabilisation policy alters the entire process and, as argued above, in effect the variance of demand shocks. In C.T. Hansen a new shock is drawn each period to which the central bank reacts (with a time lag) within the period. But the next period is distinct from the previous, and therefore the variance of the process is not affected as such, and there are no opposing effects of stabilisation. This explains C.T. Hansen's monotone relationship.

The remaining of the paper is organised as follows. Section 2 presents the model and discusses the specific way stabilisation is measured. Section 3 considers the benchmark case of no stabilisation policy, while section 4 deals with stabilisation policy and presents and discusses the results. Optimal monetary policy and the concept of stabilisation are investigated in section 5. Finally I offer a few concluding remarks in section 6. The formal solution of the model is derived in the appendix.

\footnote{P.S. Hansen (1996) also analyses stabilisation policy. He shows that by offsetting all negative shocks and by leaving the economy to itself in case of positive shocks, output can be permanently increased. But by pursuing such a stabilisation policy, the monetary authorities does not change the firm's incentives to adjust the price, exactly because the price inaction band is symmetric prior to monetary policy. What monetary policy does is P.S.Hansen (1996) is essentially to eliminate the negative part of the band. But the positive part is the same. Hence it does not inflict on the incentives to adjust the price in case of positive shocks.}
2 The Model

Consider a monopolist that produces with constant real marginal costs. Its demand function can be represented by (1) and the optimal price by (2)

\[ y = m \cdot \ln p \] (1)

\[ p^* = m \] (2)

where lowercase letters denote logarithms, and \( y \) is output, \( m \) is a nominal scale variable (money) and \( p \) the price of output.\(^2\)

Assume that the nominal scale variable is stochastic, but controlled such that it reverts towards its mean. In particular, it follows what is known as an Ornstein-Uhlenbeck process:

\[ dm = \mu m dt + \sigma dW \]
\[ dW \sim N(0, dt) \]

where \( dW \) is the increment of a standard Brownian motion, also known as a Wiener process. This specification can be interpreted as an attempt by the monetary authorities to conduct stabilisation policy, if the nominal scale variable is thought of as money supply. At every instant of time there is an offsetting effect to the evolution of money supply. It is a kind of policy which can be characterised as "leaning against the wind", and it is known as such in the exchange rate economics literature. But stabilisation is less than complete. The monetary authorities cannot neutralise any shock entirely. This can be rationalised by assuming that money supply consists of a controllable and an uncontrollable part. The latter being for instance the velocity of money. Thus, the above specification of money supply can be seen as a short-cut for specifying a separate process for the controllable and uncontrollable part. This simplification does not restrict the conclusions in a qualitative way. Modelling a process for the controllable part where complete stabilisation is possible, and a process for the uncontrollable that is purely stochastic, amounts to a common process where stabilisation is incomplete. In (3) stabilisation is stronger the larger is \( \mu \) since money supply is dragged more fiercely towards its mean value, which is zero. If \( \mu \) is zero there is no stabilisation at all, and if \( \mu \) becomes negative, the monetary authorities amplify shocks to the money supply.

If a nominal shock hits the economy, clearly it is optimal for the rm to adjust its price according to (2) and keep output constant. Introducing price adjustment costs, the so-called menu costs, it is however not an optimal strategy to change the price each time the economy is subject to a shock. Instead the rm keeps its price constant until the profit loss of not adjusting becomes too big. Beside this standard effect, the stochastic process governing \( m \) by itself affects the pricing rule. In case of a shock there is a probability that a new shock in the opposite direction will hit the economy in the future. This makes it worthwhile for the rm to wait and see before it alters its price. In addition the mean reversion of the process enhances the incentive to wait and see, because the rm knows that the shock to

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\(^2\) The specific form of the demand function is inessential. What matters is that it is homogenous of degree zero in nominal variables. It can be derived from say, an appropriately defined constant elasticity demand function, where unimportant constants have been suppressed.
a certain extent will be offset, reducing the necessity of a price adjustment. This feature is the new element compared to existing studies and the driving force behind the results. The wait-and-see-attitude together with the profit loss argument, creates a zone of inertia where the price is not changed. Only when a certain threshold is passed where the accumulated velocity shocks are big enough will the price be changed. So the solution will look like the very familiar \((S,s)\) rules known from the previously mentioned studies.

Given the above arguments, the objective of the monopolist is to minimise the costs of being out of equilibrium and the costs of adjusting the price. The former consists of the flow costs associated with a change in \(p^\mu\) when the price is kept constant, the latter consist of the menu costs \(z\) which have to be paid each time the firm resets its price. The minimisation problem results in two optimal barriers \((a,b)\) at which the price is changed, defining the band of inaction.

There are at least two methods by which this model can be solved. One is to follow Dixit’s (1991) approach. He solves the model in the case where \(\mu = 0\) by setting up the minimisation problem for the entire time horizon from time zero to infinity, and applies techniques from dynamic programming. He imposes the so-called value matching and smooth pasting conditions in order to solve for the optimal stopping time, i.e. defining the barriers \(a\) and \(b\). This cannot be accomplished analytically. Instead he utilises approximations to find the barriers. P.S. Hansen (1999) offers another approach which, without relying on smooth pasting and value matching conditions, enables him to solve the problem explicitly (again with \(\mu = 0\)). It should be noted though, that matters are complicated tremendously when \(\mu \neq 0\) and analytical solutions are impossible to obtain, regardless of the method used. Approximations as well as numerical solutions are indispensable. Given that with \(\mu = 0\) the method from P.S. Hansen (1999) yields exact results, I choose to follow his method in this paper. The specific way the model is solved is described in the appendix. Here I will just sketch the arguments.

Begin by calculating the total flow costs accrued up to the first time the process reaches either barrier. Then calculate the expected time until either barrier is reached the first time. This defines the cycle. The problem is then simply to minimise the average long run costs, which consist of the expected flow costs over the cycle, plus the cost of resetting \(z\), divided by the expected length of a cycle. The optimal boundaries are the ones that minimise these costs. Since the problem is symmetric (positive and negative shocks occur with equal probability), the upper and lower barrier is going to be (numerically) the same. This simplifies the problem enormously, but approximations as well as numerical solutions are needed, unless there is no monetary stabilisation policy in which case we are back to P.S. Hansen (1999). This will be demonstrated in the following section.

3 The Case Without Monetary Policy \((\mu = 0)\)

If \(\mu = 0\) then the average costs over the cycle \(Q(\phi)\) can be found in P.S. Hansen (1999) (see also appendix):

\[
\lim_{\mu \to 0} Q(\phi x = 0) = \frac{kb^2}{6} + \frac{z\sigma^2}{\mu^2}
\]
where $b$ is the barrier, $k$ a measure for the flow costs, $z$ menu costs, and $\sigma^2$ uncertainty. The optimal boundary can be found by differentiating this expression with respect to $b$, and it yields:

$$b = \frac{\mu 6z2\sigma^2 \frac{1}{2}}{k}$$

which is identical to the boundary found in P.S. Hansen (1999) and Dixit (1991).

The most striking result of (4), first accounted for by Dixit (1991), is that we can have fourth order small menu cost to generate first order effects. If $z \gg \varepsilon^4$ then $b \gg \varepsilon$. That is, fourth order small menu costs generate a first order zone of price inertia. Compared to the static case where Mankiw (1985) showed that it was second order small menu cost that generated first order effects, the zone of inertia is two orders of magnitude larger in the dynamic case. This is caused by the inclusion of uncertainty in the dynamic model. It is the wait-and-see attitude as described previously that accounts for this result.

From (4) some static comparative results can be obtained. First, what happens if the menu costs increase? It is easy to see that this implies that the range of inaction widens. This conforms with intuition. If it is more expensive to adjust the price, the firm needs it desirable to postpone the decision, hence the wider zone. Secondly, if the flow costs rise it means that the potential loss of being out of equilibrium is greater and therefore, the costs of the wait-and-see attitude have risen and the zone becomes more narrow. Finally, consider an increase in uncertainty. This widens the band and there is a larger range in which prices are unchanged and thus potentially larger fluctuations in demand.4

4Stabilisation Policy

As should be clear by now the existence of menu costs, causing an inaction band with price rigidity, lead to fluctuations in quantities since this model leaves aside the issues of rationing and quantity adjustment costs. The precise output process is easily obtained and it has a particular convenient form. The output process is identical to the price difference $p_t^m - p_t$ by (1) and (2). Hence, output will fluctuate stochastically between the two barriers. This opens for the possibility of monetary authorities to attempt to stabilise output. There are several ways of doing this. P.S. Hansen (1996) considers asymmetric monetary policy in the meaning that any negative shock is fully offset while positive shocks are left to itself. This policy results in output being permanently increased, but due to the way the problem is specified it does not lead to changes in the firm's incentives to adjust its price (see footnote 2). The present paper explicitly analyses this feedback question with another kind of stabilisation policy, however. Monetary authorities pursue a symmetric policy of what could be called "leaning against the wind". They partially offset shocks such that output is dragged towards its mean value (which is zero). The degree of stabilisation is measured by

Note that an increase in uncertainty that widens the band, does not necessarily imply less frequent price changes. The reason being that increased uncertainty has a countervailing effect, namely that it becomes more likely to observe extreme realisations. Hence the band may be reached more often even though it has widened. This is exactly what P.S. Hansen (1999) shows to be the case in the long run.
the mean reversion parameter, $\mu$. In the following, numerical results of the optimal barrier’s dependence on $\mu$ and the other parameters are obtained.

The parameter values for figure 1 which is a sort of base case scenario, are chosen in accordance with Dixit (1991). These specific numbers are in no way essential for the results as will become evident.

Figure 1 shows the optimal barrier’s dependence on the degree of stabilisation $\mu$ for particular values of the other parameters.

![Figure 1](image)

When $\mu = 0$, money is not controlled and the conclusions of the last section hold. As $\mu$ increases up to a certain threshold, the barrier increases as well. Hence, output becomes more volatile when stabilisation is stronger. Thus, we have a situation of destabilising stabilisation policy. The reason is that the systematic and credible monetary policy feeds back to the firm’s price decision. In an uncertain economy where it is costly to adjust the price, the firm is reluctant to move unless the accumulated demand shocks create too big a deviation between the optimal frictionless price and the actual. But once the firm knows for certain that the monetary authorities instantaneously reduces the impact of shocks, there is no need to adjust to shocks, that in the absence of stabilisation policy would have required price changes. Hence, the optimal barrier increases for the firm. However, as $\mu$ rises above the threshold, a strong degree of stabilisation causes output to be less variable and $b$ falls. The intuition is that when $\mu$ is high, the variance on demand is in fact relatively low. In the figure, the turning point is around $\mu = 1$, indicating that the degree of mean reversion is 100 times larger than the variance. This implies that the option value of waiting for the firm is reduced, because the firm faces an almost certain demand. Therefore, the optimal barrier decreases and systematic stabilisation policy proves stabilising.

Figure 2 depicts what happens when $\mu$ is decreased below zero. It is thus a situation where monetary policy in itself is not stabilising but destabilising, since any shock is amplified. It is seen that decreasing $\mu$ leads to a narrower band, and therefore reduced output fluctuations. Hence, a policy that from the outset is thought to be destabilising turns out to be stabilising. In fact, by reinforcing the shocks output fluctuations can be completely avoided. The explanation is straightforward. When shocks are reinforced there are no stabilising mean reverting forces, but the opposite. This makes demand more erratic and large
shocks more likely. Hence, the firm allows smaller accumulated shocks before it adjusts its price. The cost of not having the optimal price is too large.

Figure 2

\[ \sigma = 0.1 \quad k = 1.0 \quad z = 0.2 \]

Note, that the effect on the barrier of decreasing \( \mu \) when \( \mu < 0 \) is much stronger than the effect of increasing \( \mu \) when \( \mu > 0 \). The reason is, as explained above, that when \( \mu \) increases and \( \mu > 0 \) there are two forces that have an opposing effect on the optimal barrier. One that causes \( b \) to rise because of a higher degree of stabilisation, and one that leads to a fall in \( b \) because a higher degree of stabilisation renders demand more stable. When \( \mu \) decreases and \( \mu < 0 \), both forces work in the same direction, thereby leading to the stronger effect on the barrier.

The next few figures illustrate that the comparative statics results from the situation without monetary policy described in section three, also hold true when stabilisation is conducted. Let me first consider what happens when menu costs increase. This is shown in Figure 3 for a given degree of stabilisation (\( \mu > 0 \)).

Figure 3

\[ \sigma = 0.1 \quad k = 1.0 \quad \mu = 0.2 \]

In the limit with no menu costs (\( z = 0 \)) there is no boundary since it is not associated with any costs to change the price. Hence, the optimal frictionless price equals the actual
price at every instant of time. As menu costs increase, so does the boundary. Just as in the case without monetary policy at all, an increase in menu costs makes price adjustments more expensive to the rm. Therefore, the band is widened accordingly.

The same arguments apply when \( \mu \) is fixed at any negative level and the figure is not included.

**Figure 4**

\[
\mu = 0.2 \quad k = 1.0 \quad z = 0.2
\]

Figure 4 illustrates how the boundary depends on the standard deviation of demand for a given \( \mu > 0 \). As expected, the optimal barrier increases with uncertainty. This is a standard result from option theory; that the value of waiting increases with uncertainty. The reason is that when conditions always are uncertain a shock now may be reversed soon. But then it may not be optimal to pay the menu costs right away in order to change the price; it might be better to wait. Hence, the wider zone.

Similar conclusions can be reached when \( \mu < 0 \). It does not alter the concept of value of waiting whether \( \mu \) is positive or not.

## 5 Optimal Monetary Policy and the Concept of Stabilisation

Even though I have demonstrated that it is possible to stabilise output completely by reinforcing shocks sufficiently, it may not be an optimal policy from a welfare point of view. It depends as usual on the welfare function to be optimised. If welfare is defined to mean a situation where output does not fluctuate, then the optimality question is simple, though surprising: amplify shocks, don't reduce them. If on the other hand, welfare include considerations of the rm's profit, then this kind of policy cannot be optimal since the rm would have to pay price adjustment costs quite often, to be more precise - instantaneously. If these costs are not ignored, then the optimal policy is to offset any positive or negative shock entirely. That is, letting \( \mu \) ! 1. Though, the limit of \( b \) when \( \mu \) approaches infinity is bounded, it does not imply that output fluctuates, since stabilisation is total. The rm never reaches the boundary and it never pays the menu costs.
A natural question arises though. Namely whether this above mentioned optimal policy of complete stabilisation is viable. At the beginning I assumed that money consisted of a controllable and an uncontrollable part. This seems to accord with the view most economists have on the possibility of controlling money supply. The uncontrollable part will always be there. But if that is the case, then as this model demonstrates, attempts to stabilise may actually be destabilising.

Another issue in the discussion of an optimal policy concerns the concept of stabilisation. Implicitly, I have interpreted stabilisation as the band width and hence the levels output can attain. That is, an interpretation considering the state space. But what about the time domain? When the degree of stabilisation becomes stronger as $\mu$ increases, it also implies that output is pushed towards its mean. Even though the band becomes wider, the time it takes until the band is visited the first time may in fact increase. This is what actually turns out to be true. The time until the boundary is reached the first time (the first passage time) increases. An expression for the first passage time exists, as a matter of fact, namely equation (A4) from the appendix. It cannot be solved analytically but numerical solutions yield the following relationship:

$\mu$

Figure 5

\[
\sigma = 0.1 \quad k = 1.0 \quad z = 0.2
\]

From the figure it is evident that the expected waiting time ($H(x)$) until the barrier is reached the first time goes up. The more the authorities stabilise (higher $\mu$), the longer the waiting time. Thus, it may be true that as $\mu$ increases more extreme realisations of output are possible, but they are not probable. In terms of the time domain, stabilisation policy is stabilising.

Now the big question is; how does the optimal policy look like? I do not believe that there is an easy answer to this question. It seems difficult to combine a trade-off between a state space and a time domain. One has to compare higher output volatility that results in output realisations that are more extreme, with the period within which each state is visited. This task is beyond the scope of this paper.
6 Conclusion

This paper’s main conclusion is that a systematic stabilisation policy by monetary authorities may prove to be destabilising in the sense that it leads to more output volatility. The reason is that if the firm observes an extreme observation of demand that without stabilisation would have called for a price change, it knows that the authorities to a certain extent conduct stabilisation. Therefore, there is no need to change the price. Hence, the firm allows even larger shocks until it itself adjusts the price. This mechanism I believe, is quite general for the individual firm, but of course one should be careful in drawing conclusions on the macro level. What is true for one firm may not be true in aggregate, as the paper by Caplin and Spulber (1987) is a prominent example of. But on the other hand, we also know that the results in Caplin and Spulber are in no way robust. They were accounted for by very special and restrictive assumptions. Papers like Caplin and Leahy (1991), Caballero and Engel (1993) and Danziger (1999) suggest that the most basic properties of single firm partial equilibrium models are not washed away when aggregation occur, though some effects may be dampened. The aim of this paper is to demonstrate a simple effect that has been strangely absent from most of the menu cost models, and it would be highly surprising to me if the effect would disappear in a fully general equilibrium model. Nevertheless, it goes without saying that it would be very interesting and important to actually check this and not rely on my suspicions. A general equilibrium model would further have the advantage that welfare considerations are more easily analysed when consumers are explicitly allowed in the model. But both these extension are beyond the scope of this paper.

Furthermore, it has implicitly been assumed that it is costless for the monetary authorities to regulate money supply, but costly for the firm to adjust its price. This may be a restrictive assumption depending on the way these costs were to be specified. If they were of a lump-sum nature money would only be changed discretely, just as prices. The monetary authorities would have an inaction band just like the firm. If this band width is larger than the firm’s, the firm would have to adjust to shocks that it would not have adjusted to, had there been no costs of changing money supply. In this case stabilisation policy would have no effect for small shocks, only large. For large shocks, the effect would be as in this paper. If on the other hand the costs of changing money supply were convex, I suspect that nothing in a qualitative way would change. The authorities would still stabilise continuously, though not as strongly due to the increasingly higher costs of doing so. But this is simply a question of the level of $\mu$, and hence it does not conflict with the main conclusion of this paper.

The model is like any other model highly stylized and maybe to simplistic, but the question with which this paper deals is quite important and it has been ignored in much of the macroeconomics literature. It is definitely worth further studies.

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4They show that even though there is price rigidity at the individual level, money is neutral in aggregate.
7 References

References


8 Appendix

The objective of the monopolist is to minimise the costs of being out of equilibrium and the costs of adjusting the price. The former consists of the flow costs associated with a change in  \( p^\ast \) when the price is kept constant, and can by a second order Taylor approximation be written as  \( k(p^\ast - p)^2 \) where  \( k = \frac{1}{2} \frac{d^2 p}{dp^2} \). The latter consist of the menu costs  \( z \) which have to be paid each time the firm resets its price. Denote that time by  \( T^\ast \).  \( T^\ast \) is of course in itself stochastic. Define  \( p^\ast - p \leq x \). It follows an Ornstein-Uhlenbeck process similar to (3). The resetting time is defined as:

\[
T^\ast = \min\{t \geq 0 : x_t \leq a \text{ or } b \geq x_t\}
\]

where  \( a \) and  \( b \) are the lower and upper threshold, respectively, at which the firm changes its price.

8.1 Solution of the Model

8.1.1 The General Case

Assume that the state space of  \( f(x(t) : t \geq 0) \) is an interval  \( I = ]a, b[ \) where  \( a < l < r < b \), and that  \( x(t) \) is regular in the interior of  \( I \) (all states can be reached with positive probability). Assume also, that  \( f(x) \) is bounded and continuous. Then it is possible to write down the following integral which is well defined given the above assumptions, and given that a Brownian motion has a continuous sample path.

\[
G(x) = E[g(x)] = E \int_0^{T^\ast} f(x(s)) ds : x(0) = 0, \quad a < x < b \tag{A1}
\]

where  \( T^\ast \) is the first time the process reaches either  \( a \) or  \( b \). If  \( f(x) = 1 \) for all  \( x \) then  \( G(x) = T^\ast \), the expected time to reaching either  \( a \) or  \( b \). Hence, as written before, finding  \( E[T^\ast] \) is merely a special case of (A1).

To compute the solution of (A1) the integral can be split into two parts. The first consists of a small time interval  \( \zeta \) in which the probability of reaching either  \( a \) or  \( b \) is negligible. The remaining part consists of the integral from to  \( T^\ast \). Therefore we obtain an expression similar to a dynamic programming recursion which solution is quite standard. Hence, I state the solution without any further calculations and note that (A1) has to fulfill the following second order differential equation with obvious boundary conditions:\footnote{The solution is derived for a general diffusion process. In our case  \( \mu(x) = \frac{1}{2} x \) and  \( \sigma^2(x) = \sigma^2 \).}

\[
\mu(x) \frac{dG}{dx} + \frac{\sigma^2(x)}{2} \frac{d^2G}{dx^2} = f(x), \quad a < x < b, \quad G(a) = G(b) = 0 \tag{A2}
\]

A solution to this differential equation can be found by following Karatzas and Shreve (1991 p. 339) with a slight modification allowing for a function  \( f(x) \) instead of simply a constant.
Define the scale function $S(x)$ as:

$$S(x) = \frac{Z_x}{\exp i 2 \frac{\mu(\xi)}{\sigma^2(\xi)} d\xi} \cdot \frac{Z_\tau}{d\tau}$$

Define also a speed measure $M$ with speed density $\frac{dM}{dx} = m(x)$, where we will write the speed measure as $m(dx)$.

$$m(dx) = \frac{2}{S^q_x \sigma^2(x)} dx$$

Having defined these two functions the solution to (A2) can be written as:

$$G(x) = \int_a^x (S(x) - S(\xi)) f(\xi) m(d\xi) + \frac{S(x)}{S(b) - S(a)} \int_a^b (S(b) - S(\xi)) f(\xi) m(d\xi) \quad (A3)$$

With $f(\xi) \neq 1$ we get the general expression for the expected first passage time to either $a$ or $b$ which we denote by $H(x)$.

$$H(x) = \int_a^x (S(x) - S(\xi)) m(d\xi) + \frac{S(x)}{S(b) - S(a)} \int_a^b (S(b) - S(\xi)) m(d\xi) \quad (A4)$$

The way the model is specified implies that the boundaries are symmetric and that the resetting point is zero. This follows because according to the stochastic process governing $x$, equal sized positive and negative shocks to the money supply are offset to the same extent. Thus, the process is symmetric around zero, causing the lower boundary to equal minus the upper. Hence, $a = -b$. The resetting point is zero precisely because the problem is symmetric and because flow costs are minimised at zero. This simplifies the problem as only one barrier has to be found.

Finding the optimal barrier amounts to, as written previously, finding the expected costs over a cycle adding the menu costs and divide by the expected length of a cycle. The optimal boundary is the one that minimises these average costs denoted by $Q(x)$.

$$Q(x, b, \mu, \sigma, z, k) = \frac{G(x, b, \mu, \sigma, z, k) + z}{H(x, b, \mu, \sigma, z, k)} \quad (A5)$$

Assuming that the state is initially in equilibrium, that is $x = 0$, the optimal boundary at which the price is changed, can be found by minimising these costs with respect to $b$. This gives (A6).

$$\frac{\partial Q}{\partial b} = 0 \quad b = b(\mu, \sigma, z, k) \quad (A6)$$

The scale function is a function that changes the probabilities of arriving at different states and can thus be used to, for instance, change the drift of a process.
8.1.2 The Specific Case

In our case \( \mu(x) = \mu x \) and \( \sigma^2(x) = \sigma^2 \). Inserting this into the scale function we get:

\[
S(x) = \int_{x}^{\infty} \frac{e^{-\frac{1}{2} \sigma^2 \tau^2}}{\sigma^2 \tau^2} \, d\tau
\]

This integral can be solved, though it does not give a neat solution. But this is only part of the entire solution to, say, \( G(x) \). \( S(\xi) \) has to be multiplied by \( f(\xi) \) (which in our case is equal to \( kx^3 \)) and divided by \( m(d\xi) \), and this again has to be integrated. All in all, there is no analytical solution to this. Therefore I choose to make a Taylor approximation to \( e^{-\frac{1}{2} \sigma^2 \xi^2} \). Unfortunately, the order of approximation has to be small in order to obtain a solution to \( G(x) \). In fact, it has to be a first order Taylor approximation. But still, in some neighbourhood, however small, it comes close to the exact solution. Hence we can write \( S(x) \) as:

\[
S(x) = \int_{x}^{\infty} (1 + \frac{\mu}{\sigma^2} \xi^2 + \cdots) \, d\tau
\]

and \( m(dx) \) as:

\[
m(dx) = \frac{2}{1 + \frac{\mu}{\sigma^2} x^2} \, dx
\]

Substitute (A7) and (A8) into (A1) and into (A4) to obtain:

\[
G(x) = \int_{x}^{\infty} \frac{e^{-\frac{1}{2} \sigma^2 \xi^2}}{\sigma^2 \xi^2} \, d\xi
\]

\[
H(x) = \int_{x}^{\infty} \frac{2}{1 + \frac{\mu}{\sigma^2} x^2} \, dx
\]

The optimal barrier is found by substituting \( G(x) \) and \( H(x) \) into (A5) and solved according to (A6). The resulting expressions are in no way neat and tidy and are omitted, but they allow in principle a solution to the problem. However, \( b \) cannot be found in a closed form solution. Therefore, I solve the model numerically.

8.2 Solution when \( \mu = 0 \)

To obtain a solution in the limit when \( \mu = 0 \) substitute (A9) and (A10) into (A5) and take limits as \( \mu \to 0 \). This yields the equation in the text. Note, that in the limit when \( \mu = 0 \) the approximations reproduces the exact result of the barrier, as obtained in P.S. Hansen (1999) and as in Dixit (1991) with analytical approximations, indicating that the approximations are probably not too bad.