AGGREGATION OF MICRO-FUNCTIONS TO OBTAIN A WHOLE-FARM PRODUCTION FUNCTION*

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In this paper, a re-assessment of the aggregation problem in farm production functions is attempted. Aggregation starts from measurable farm management entities such as the milking cow, breeding ewe etc., and appropriate whole farm production functions are derived that are consistent with these biological units of farming, or as they will be called, the micro-functions of agriculture. The aggregate function, as derived, is based on the geometric means of micro outputs and inputs and could appropriately be said to 'represent' all the micro-functions. It is thus called, later in the paper, the representative production function for the farm firm.

For this paper, the micro-functions of agriculture are based on the application of variable inputs to the dairy cow, the breeding ewe and the crop acre. These are basic technological or biological units which are usually collected up to firm size by farming entrepreneurs. Each of these units represents some kind of durable factor to which variable factors can be added. Feed and milking services for the dairy cow; hay and shearing services for the breeding ewe; fertilizer and seed for the crop acre. These variable factors are likely to be affected by scale of operation (such as the organization of milking services for dairy cows) and any aggregation of such micro-functions must take this into account. Supporting livestock are set aside for the moment although they clearly stand in some joint relation to each productive unit of stock. Fattening stock can be regarded as basic productive units like breeding stock. Consideration of factors which are common to micro-functions within firms, and joint relationships between them require special treatment. The view taken of the whole farm is that it represents an aggregation of the above micro-functions or biological units.

The biological units of production tend to have fixed periods of production, e.g. the cow lactation, the growth period of the lamb, and the maturing period of a crop. Thus the durable factor is usually fixed in both time and space, but variable factors are subject to managerial judgement and skill and can be varied considerably. Day-to-day feeding and other husbandry factors are aggregated in time and their effect is measured in terms of the growth period of the product with which they are associated. Feeding and other tasks are daily requirements, but seed and fertilizer are usually applied once in the growth period of a crop.

Changes in marginal productivity of variable factors may result from varying output rates per time period, or from differences in aggregate feed and service patterns and other decisions subject to farmer's skill.

The basic objective of analysis in this area is to obtain meaningful

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estimates of micro-elasticities of production and marginal products so that good policy recommendations can be made. The policy-maker should be provided with estimates of marginal products to the different factors at the micro-level and the macro-level so that correct (realistic) allocation decisions can be made. Entrepreneurs, on the other hand, are more likely to be interested in the adjustments possible at the micro-level alone.

The problem in the interpretation of production function analysis is that economists tend to move from level to level of aggregation without realizing the consequences. In agricultural economics, particularly, the comparative wealth of information on the micro-units of production has led to a loose application of the appropriate principles at the whole firm level, possibly due to the availability of annual data at that level.

This paper is concerned with setting out the steps of aggregating from micro-units of production to whole firm units explicitly so as to clarify the exact productive concept appropriate to each stage. The principles of aggregation are not new, as Plaxico's 1955 article [10] demonstrates, but little work appears to have been done in spelling out the exact implications of aggregation in terms of the micro-functions involved.[1]

The paper sets out the theory of aggregation as it applies to such simple micro-functions and then analyses the properties of an aggregate production function which is consistent with the basic micro-functions. This aggregate function is later called a representative production function.

Aggregation of Micro-functions

In the above discussion, the view taken of agricultural micro-functions has been defined in some detail as the level of reference to the basic production function is crucial to the discussion. For the purpose of further discussion, consider the Cobb-Douglas form of such an agricultural micro-function,

$$Y_{it} = K_{it} N_{it}^{a_{it}} M_{it}^{b_{it}} \tag{1}$$

where

Y_{it} = output of the i-th unit in production period t,
K_{it} = a constant,
N_{it}, M_{it} = inputs of specific factors during period t,
a_{it}, b_{it} = Cobb-Douglas elasticities of production,

and where the equation is separable and the function is homogeneous of degree 1. Further analysis of different production periods is not considered hereafter but could be developed where production periods for different units differ or functions are constrained by time effects like rotations.

Next consider the problem of aggregating like micro-production functions. Clearly, all the biological units defined by equation (1) can be aggregated because each unit is identical to the aggregate. But if it is assumed that the elasticities and the input levels are different, then a set of rules must be followed for correct aggregation.

[1] A recent renewal of interest is apparent in the literature. See recent articles by Sadan [11] and El-Issawy [3].

In theory, the aggregate function should reproduce the true micro-function but there will be errors in the aggregate owing to small or large variations in response rates, i.e. the production elasticities. Furthermore, it is not clear how a sample of such units, expressed in Cobb-Douglas form can be added together. The whole relationship embodied in (1) is multiplicative and hence must be aggregated as an entity. As Domar [2] points out, the right hand side of the equation is a geometric index number representing output, and the individual inputs cannot be aggregated arthmetically.

The aggregation problem in the Cobb-Douglas case can be overcome by adding in the logarithmic form of the micro-units,[2] (lower case letters in what follows). The equations representing the two unlike micro-units in (2) below are then said to be additively separable, e.g. dropping the time subscripts,

$$(2) \qquad \begin{aligned} y_1 &= k_1 + a_1\, n_1 + b_1\, m_1 \qquad a_1 \neq a_2,\ b_1 \neq b_2 \\ y_2 &= k_2 + a_2\, n_2 + b_2\, m_2 \end{aligned}$$

To obtain the aggregate function we must now add the two micro-functions in (2). The *i* subscript falls away and denoting the aggregate exponents as a and b, we define them as:

$a = \frac{a_1 + a_2}{2}$, $b = \frac{b_1 + b_2}{2}$, and can write the aggregate function as:

$$(3) \qquad y = k + a\, n + b\, m$$

where

$$\begin{aligned} y &= y_1 + y_2 \\ a\, n &= a_1\, n_1 + a_2\, n_2 \\ b\, m &= b_1\, m_1 + b_2\, m_2 \end{aligned}$$

Aggregation of unlike outputs requires price or other weights, thus y is some price-weighted index number measure of output.[3] Aggregate output for the firm is defined in terms of products and not sums, and aggregate inputs are weighted products according to their exponents. Thus for input N,

$$(4) \qquad N = N_1^{\frac{a_1}{a}}\, N_2^{\frac{a_2}{a}}$$

Thus the correct whole-farm inputs of labour and capital, say, if the firm consists of a number of diverse production activities, are the weighted products of individual micro-inputs. Such products preserve the micro-function relationships so that changes in inputs are consistently related to changes in aggregate output. These quantities will clearly bear little relation to whole farm observations of N and M in farm recording schemes, accounts etc. As Plaxico [10] points out, only perfect substitutes and complements can be aggregated arithmetically without violating the consistency rule.

Under the assumption of unlike micro-production elasticities, the whole firm elasticities must bear some defined relationship to the micro-elasticities if realistic policy parameters are to be estimated. The analyst has a choice at this point. He can assume arithmetic averaging of the micro-elasticities, and weight the inputs in multiplicative aggregation,

[2] The standard reference is Klein [7]; see also Simkin [12] and Walters [14].
[3] See Mundlak [9] for rules of consistent output aggregation.

thus representing the aggregate production elasticity as some average response rate for a given factor. Secondly, he can sum the products of inputs, without weighting, in which case, the algebra requires that a weighted average of micro-elasticities is used. The appropriate technique to the purpose in hand should obviously be chosen in each case.

In what follows, arithmetic averaging of micro-elasticities is assumed for purposes of exposition. Aggregates made up of products of terms are difficult to conceptualize in practice and it makes economic sense to express the appropriate aggregate production function for the whole firm as the average of all micro-functions.

Write this average production function for the farm as:

(5) $$\bar{Y} = \bar{K}\,\bar{N}^a\,\bar{M}^b$$

where,

$\bar{Y}$, $\bar{K}$, $\bar{N}$ and $\bar{M}$ are weighted averages of all micro-data and a and b are arithmetic averages of the micro-exponents.

Then, with output measured in constant prices, we can define the appropriate equations for the set of aggregate outputs and inputs for the average production function as:

(6) $$\bar{Y} = [Y_1\,Y_2]^{\frac{1}{2}}$$

$$\bar{N}^a = [N_1^{a_1}\,N_2^{a_2}]^{\frac{1}{2}}$$

or $$\bar{N} = N_1^{\frac{a_1}{2a}}\,N_2^{\frac{a_2}{2a}}$$

and similarly for $\bar{M}$.

The definition of aggregate input arrived at therefore resembles Solow's collapsible form as long as arithmetic averaging of exponents is considered desirable [13].

Summarizing, it can be seen that if whole farm *data* only is available for cross-section analysis, and if it is believed that micro-functions of the farm are appropriate to the problem, then the above weighting rules break down due to the lack of data. As Griliches [4] has observed, if we knew the micro-elasticities in (1) we would not be trying to estimate them.

Given that *whole* farm (firm) data from *farm records* represents the arithmetic aggregation of inputs and outputs in the micro-functions, then it must be assumed that up to this level of aggregation, inputs and outputs are perfect substitutes. If it cannot be assumed perfect substitution takes place, then it must be accepted that the whole farm function is *not* consistent with the micro-functions, but must be considered a useful representation in its own right. As farm data is aggregated to size groups, type groups, and regional totals in an *arithmetic* fashion, then production functions based on each level of aggregation can bear no consistent relationship to each other.[4]

A Working Model of the Farm Firm

A more detailed analysis of an aggregative model for the farm illustrates the points analysed above and also provides certain insights into some of the interpretation problems which might be encountered during

[4] I am indebted to T. W. Francis for numerous discussions on this point.

aggregation. Slightly more descriptive notation is employed to bring out the practical problems of defining an aggregate or representative production function for the farm.

Consider a farm which grows a number of crops, carries a number of different types of livestock, and has the usual collection of farm buildings, machinery and land. It will be useful to distinguish between intermediate products such as hay and feed crops, and final products for sale, e.g. wheat and milk. It is also useful, in what follows, to regard buildings and machines as durable capital goods which provide a flow of capital services to intermediate products and final products. Maintenance of capital assets is assumed. The exact details and relationships are specified below.

Each crop output can be regarded as a function of labour inputs (N), miscellaneous inputs such as seed, fertilizer, and sprays (M), machine services (MS) and acreage (A). The exponents, a_i, b_i, c_i, d_i, are the 'true' production elasticities respectively for the i^{th} crop ($i = 1, 2 \ldots n$). Write the micro-function as:

$$Y_i = N_i^{a_i} M_i^{b_i} MS_i^{c_i} A_i^{d_i} \tag{7}$$

Each type of livestock output can be regarded as a function of labour, miscellaneous inputs, machine services, building services (BS), feed services (FS) such as hay or feed crops, and capital in the form of livestock (LSC). The production elasticity for building services is denoted by e, that for feed services by g and livestock capital h. The micro-function for the s^{th} livestock type ($s = 1, 2, \ldots u$) is written:

$$Y_s = N_s^{a_s} M_s^{b_s} MS_s^{c_s} BS_s^{e_s} FS_s^{g_s} LSC_s^{h_s} \tag{8}$$

Machine services ($m = 1, 2 \ldots p$) are produced by a production function which includes as inputs, labour, miscellaneous inputs like fuel and oil, building services and capital in the form of machines (T):

$$MS_m = N_m^{a_m} M_m^{b_m} BS_m^{e_m} T_m^{h_m} \tag{9}$$

Building services ($q = 1, 2 \ldots r$) are produced by a production function which includes labour, miscellaneous inputs and capital in the form of buildings (BD):

$$BS_q = N_q^{a_q} M_q^{b_q} BD_q^{h_q} \tag{10}$$

Finally, feed services ($w = 1, 2 \ldots z$) involve land (A) and have essentially the same type of production function as crops:

$$FS_w = N_w^{a_w} M_w^{b_w} MS_w^{c_w} A_w^{d_w} \tag{11}$$

In our system, there are thus 2 types of function which produce final products and 3 types of function which produce intermediate services. Each type of function is based on a durable factor of production and may be expected to exhibit variable proportions as factor inputs are varied.

Each function can be expressed in terms of its durable input without changing the meaning of the remaining exponents, i.e. for (7),

$$\frac{Y_i}{A_i} = \left(\frac{N_i}{A_i}\right)^{a_i} \left(\frac{M_i}{A_i}\right)^{b_i} \left(\frac{MS_i}{A_i}\right)^{c_i} \tag{12}$$

Assuming constant returns to scale, the exponent for land for each crop can be calculated from the expression $1-a_i-b_i-c_i$ and the further inclusion of A_i as an additional variable in the above equation would provide an estimate of the returns to scale.[5]

Only the crop and feed service functions are divided by A_i; the livestock, machine services and building service functions would be divided through by the variables LS_s, T_m, BD_q, to obtain the equivalent expression to the model set out as equation (1) earlier. In theory every micro-unit which is different (e.g. different dairy cows) should be expressed as a separate function.

Following Klein [7], the aggregate variables are next defined as averages, but the analysis would be the same if carried out in aggregates.

Assuming suitable price weights for micro-outputs, the average function representing all appropriate micro-functions, can be written:

$$(13) \qquad \overline{Y} = \overline{N}^a\, \overline{M}^b\, \overline{MS}^c\, \overline{A}^d\, \overline{BS}^e\, \overline{FS}^g\, \overline{LSC}^h$$

Average output per function can be written in two ways, as products of individual outputs or as sums of logarithms of individual outputs:

$$(14) \qquad \overline{Y} = \left(\prod_{i=1}^{n} Y_i \prod_{s=1}^{u} Y_s\right)^{\frac{1}{n+u}}$$

$$= \text{anti-log}\left(\frac{1}{n+u}\left(\sum_{i=1}^{n} \log Y_i + \sum_{s=1}^{u} \log Y_s\right)\right)$$

For definitions of the respective aggregate inputs, we can express them as weighted products, and the reader can transform the equations for himself:

$$(15) \qquad \overline{N} = \left(\prod_{i=1}^{n} N_i^{a_i} \prod_{s=1}^{u} N_s^{a_s}\right)^{\frac{1}{(n+u)a}} \qquad \text{where, } a = \frac{\Sigma a_i + \Sigma a_s}{n+u}$$

(see equation 6)

$$\overline{M} = \left(\prod_{i=1}^{n} M_i^{b_i} \prod_{s=1}^{u} M_s^{b_s}\right)^{\frac{1}{(n+u)b}}$$

$$\overline{MS} = \left(\prod_{i=1}^{n} MS_i^{c_i} \prod_{s=1}^{u} MS_s^{c_s}\right)^{\frac{1}{(n+u)c}}$$

$$\overline{A} = \left(\prod_{i=1}^{n} A_i^{d_i}\right)^{\frac{1}{n.d}}$$

$$\overline{BS} = \left(\prod_{s=1}^{u} BS_s^{e_s}\right)^{\frac{1}{u.e}}$$

$$\overline{FS} = \left(\prod_{s=1}^{u} FS_s^{g_s}\right)^{\frac{1}{u.g}}$$

$$\overline{LSC} = \left(\prod_{s=1}^{u} LSC_s^{h_s}\right)^{\frac{1}{u.h}}$$

[5] This is adequately set out in Cozens [1] p. 202, for those interested.

It is most important to note that aggregate input, defined in this way, bears a consistent relationship to aggregate output. Any variation in micro-inputs in any part of the system is appropriately reflected in changes in aggregate output.

The interpretation of (13) is that it 'represents' all the component micro-relationships that go to make up the collective entity known as the farm firm. In practical terms, it is the production function for the 'average' cow on a dairy farm, for example, suitably weighted by *all* the inputs concerned in final production. Further interpretive details are discussed below.

The above equations complete the listing of inputs which appear in the production functions for final products. If we wish to aggregate the functions for intermediate products, then the mean labour input would be written, for example,

$$\bar{N}\text{ int.} = \left(\prod_{m=1}^{p} N_m^{a_m} \prod_{q=1}^{r} N_q^{a_q} \prod_{w=1}^{z} N_w^{a_w}\right)^{\frac{1}{(p+r+z)a}} \quad (16)$$

The important point to note here is that the aggregate production function for final products does not include all the conventional inputs but only those relevant to the production functions for the final products.[6] Specification of the micro-functions involves indentifying those inputs which contribute to intermediate products and those which contribute to final products. There is no point in aggregating the intermediate inputs incorporated in machine services for final products because they clearly belong to a different micro-function.[7] In fact, should labour expended in maintaining capital be included at all? Following Solow, those services that do enter the micro-functions for final products could be named 'capital services in general'.[8]

Up to this point all inputs are assumed to be measured in physical units. Within firms, constant prices for like inputs can be safely assumed, hence aggregation of micro-values of inputs does not present any difficulty. It follows that it is highly desirable to be able to assume constant input prices between firms if cross-section estimation is to be attempted. Differences in prices will require further weightings in the aggregation process and the derivation of estimation data starts to look particularly tedious and difficult, although not insuperable.

Returns to scale can be varied among the different micro-functions. If equations (7) to (11) are homogeneous of degree 1, as stated at the outset, then equation (13) is homogeneous of degree 1 as well. But adopting an idea expressed by Sadan [11], each micro-function could be left free to reflect its own micro-economies of scale. In this way, economy in the use of buildings, tractors, and other fixed equipment would be properly reflected in each micro-equation.

Following the general principle already set out, the aggregate sum of elasticities will be the weighted average of all micro-elasticities. In fact, under the particular assumptions of this paper, the sum of the

[6] The logic and justification for regarding production functions in two stages are set out in Solow [13] pp. 102-103.

[7] Some agricultural economists are interested in all the micro-functions and would want each type of function aggregated.

[8] See Solow [13] p. 104.

co-efficients in equation (13) will be the arithmetic average of all the final product micro-elasticities:

$$(17) \quad a + b + c + d + e + g + h$$

$$= \frac{\Sigma a_i + \Sigma a_s + \Sigma b_i + \Sigma b_s + \Sigma c_i + \Sigma c_s}{n + u} + \frac{\Sigma d_i + \Sigma e_s + \Sigma g_s + \Sigma h_s}{u}$$

Discussion

1. The author recognizes the tremendous data problems which the approach suggested above involves. Further it is clear that a circular argument is also involved, namely how do we aggregate correctly when the appropriate weights (elasticities) have not yet been estimated? Nevertheless, the author feels the approach has considerable heuristic value in drawing attention to the kinds of problem involved in formalizing production function relationships.

In addition, any theory which is better adjusted to the facts of an industry is surely worth consideration. In this respect, the above 'representative' function could well improve the basic understanding of production economics students, and has already been tried out successfully on one such class.

2. In principle, a complete cross-section sample of farms could be analysed according to the above principles and a cross-section production function estimated from the properly weighted aggregates. The concept of a representative function is obviously a fairly restrictive one in the farm management context. The sample should be confined to farms with a similar product mixture, if the estimated elasticities and marginal products are to have any *practical* meaning. In such cases, the analyst would perhaps be more satisfied with estimating the micro-functions individually.

On the other hand, policy work requires the aggregate results as well as the micro results, and the representative production function is the appropriate tool of analysis. All kinds of enterprise variety can be accommodated in such a model, and the aggregate level of factor productivity can be faithfully reproduced. In the process, the farm management meaning of the factor elasticities would tend to be lost in the averaging of exponents implicit in the method.

3. The chicken and egg problem of the micro-elasticities could be approached in two possible ways. First, assumptions could be made about the respective values of the elasticities (possibly on a factor share basis) and then tested by iterative procedures to find out if the assumptions were 'sound'. Secondly, a brick-building approach could be adopted where all the relevant information, i.e. farm management data, agronomy data, animal husbandry information and so on, is brought together to obtain likely production relationships which can be aggregated later.

4. Some care is needed in defining the appropriate micro-functions. For a given crop, all inputs are likely to be identical for each acre, hence aggregation is no problem, but the same crop in different fields may have treatment differences, soil differences and even rotation

differences. These would need to be set out separately. Livestock are more difficult. Can breeding ewes all be regarded as identical for aggregation purposes? Even if we are satisfied to do this, could a herd of milking cows be so regarded?

5. In the literature, the Massell-Johnson model [8] is the nearest approach to the kind of aggregate model recommended here. In their model, factor returns to 3 separate crops (in a smallholder economy) are estimated subject to various constraints on 'management' differences between farms. In a covariance sub-model, systematic differences between crops and between farms are isolated, and consistent estimates are obtained of the respective factor elasticities for each crop. In a second sub-model, crop functions are estimated individually, with outside information on management differences, to obtain the individual factor elasticities. The results of both sub-models are then interpreted more in a farm adjustment sense than a national policy sense.

In the light of the argument of this paper, a further stage of the analysis would have been to re-combine the results of the crop micro-functions into a single whole farm policy model.[9] Such a policy model has already been constructed by the present author in another context [6]. In this case, the estimates of the micro-elasticities obtained through the second sub-model referred to earlier, were utilized to derive appropriate crop producing activities in a linear programming model of a complete farming area. Aggregate factor elasticities were not derived, but marginal value productivities and shadow prices for land and labour were obtained which could be designated as enterprise-weighted aggregate policy parameters.

These models only isolate the individual *factor-product* elasticities, which nevertheless are the main problem in aggregation. For information on the product-product elasticities, the investigator must turn to the transcendental functions of Mundlak [9] and El-Issawy [3], which are consistent with the principles of the present paper, although they do not yield results useful for aggregation of the implied micro-functions described. For information on *partial* production functions, where the micro-functions are further disaggregated into biological and non-biological parts, the reader is referred to the recent article by Sadan [11].

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