CHASING ABSOLUTE COST AND PROFIT SAVINGS
IN A WORLD OF RELATIVE INEFFICIENCY

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Abstract

Econometric models to estimate allocative and technical inefficiency include stochastic shadow distance frontiers, shadow cost frontiers, and shadow profit frontiers. In these models, the cost savings from eliminating both sources of inefficiency is often reported in total and then decomposed into the contribution of each source. This paper shows that this calculation, as formalized in Kumbhakar (1997) and used extensively in empirical applications over the last 25 years, is non-unique without additional information that is typically not available. The same results are obtained for shadow profit and shadow distance systems. The decomposition of cost (profit) savings is underidentified, since it is conditional on the arbitrarily normalized value for one shadow input (input or output) price when the shadow cost (profit) system is estimated. When the normalized value is rescaled, estimates of shadow costs will change by the scale factor, but the new model is observationally equivalent to the original one. We provide theoretical proofs and empirical examples of the restrictions (additional information) necessary to identify cost savings. Several methods of satisfying this decomposition requirement are discussed.

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1. Introduction

A researcher can measure the degree of technical efficiency (TE) exhibited by a firm as the lost output for a given level of inputs (or equivalently, wasted inputs to achieve a given level of output) for specific firms or industries. This requires an assumption about the efficiency level of one firm; the standard assumption is that this index firm is 100 percent efficient. Conditional on this normalizing assumption, the efficiency losses in physical quantities can be converted into dollar losses. Different normalization assumptions will lead to different estimates of dollar losses.

If a firm is assumed to minimize shadow costs (maximize shadow profits) as a function of shadow or virtual prices, one must estimate a shadow cost (shadow profit) model, where shadow prices differ from actual prices. Reasons for these deviations include price floors, shortages, government regulation, and tax codes. Inefficiencies result when the shadow prices, which the firm employs in the optimization process, differ from market prices. If we assume that the firm minimizes shadow costs, allocative efficiency (AE) exists if the ratio of any pair of input marginal products equals the ratio of input shadow prices. Analogously, we can assume shadow cost minimization but estimate shadow quantities rather than shadow prices using a shadow distance system. If we assume that a single-output firm maximizes shadow profits, price inefficiency exists if the marginal value product for any input fails to equal its shadow input price normalized by shadow output price. For multiple-output firms we must add the requirement that their normalized output price equals their marginal cost of production. The calculation of inefficiency in all three models involves ratios of shadow prices, which implies that one shadow price must be set to some constant in order to achieve identification. For purposes of measuring AE and price efficiency, the choice of the numeraire value is arbitrary and typically is chosen as unity.
Frequently, however, the researcher wishes to convert estimated TE and AE measures into cost or profit savings that could be achieved if the firm were to operate efficiently and to decompose these savings into portions attributable to TE and AE. What is not well understood is that the choice of the value for the numeraire shadow price is now critically important. Different values of the numeraire produce different estimates of cost and profit savings. That is, the choice of the numeraire value, which was arbitrary for the purposes of estimating AE (price efficiency) in a shadow cost (shadow profit) system, is of utmost importance when measures of AE and price efficiency are translated into cost or profit savings. The reason is that relative measures of price efficiency are now translated into absolute measures of cost and profit savings. For the normalization to yield unique measures of cost and profit savings and hence a unique decomposition, a researcher must know the factor of proportionality between at least one shadow price and one market price. We term this the decomposition requirement. Perhaps the best chance to satisfy this requirement is to know that one input (e.g., labor) is bought and sold in a market that instantaneously adjusts to an equilibrium, so that its shadow price will equal its observed market price. However, we believe that the researcher will rarely satisfy this requirement.

To fully understand this problem, we must first consider the decomposition method proposed by Kopp and Diewert (1982), with its simplification by Zieschang (1983) and Mensah (1994). Henceforth, we refer to the Kopp-Diewert method, as modified by Zieschang, as the KDZ method. This method satisfies the decomposition requirement in the most restrictive manner possible since each shadow price is assumed to equal its actual price. Using this method, one can decompose cost efficiency (also called productive efficiency) into the product of scores representing AE and TE. Starting with the initial econometric estimation of the parameters of a cost equation, KDZ decompose economic
efficiency into its component parts for each data point. They do so by solving for a set of relative prices that satisfy a system of non-linear equations at the point where the firm is technically efficient, subject to maintaining the original input mix. However, as we show below, if actual prices differ from shadow prices, the KDZ decomposition method must satisfy the decomposition principle.

Interestingly, reported levels of allocative inefficiency are very small (less than 5% in Kopp-Diewert and less than 1% in Mensah), since AE can occur only through the error term attached to each factor demand equation. Alternatively, if we assume that firms minimize shadow costs as a function of shadow prices, where shadow prices can vary systematically from actual prices, the magnitude of estimated allocative inefficiency should be much greater.

As an alternative to the KDZ method, many researchers have allowed for this variation by econometrically estimating shadow cost, shadow distance, and shadow profit systems and then decomposing estimated cost or profit savings. However, none of these papers mentions the decomposition requirement. A partial list of papers published or forthcoming in highly-ranked journals include: Atkinson and Cornwell (1994) in the *International Economic Review*; Kumbhakar (1997), who formalized this procedure for the translog shadow cost function, Atkinson and Primont (2002), and Kumbhakar and Tsionas (2005a), all of which were published in the *Journal of Econometrics*; Berger and Humphrey (1991) in the *Journal of Monetary Economics*; Kumbhakar (1996) and Atkinson et al. (2003) in the *Southern Economic Journal*; Kumbhakar and Tsionas (2005b) in the *Journal of the American Statistical Association*; and two papers forthcoming in the *Journal of Econometrics* by Kumbhakar and Wang (forthcoming 2006 a,b). The highly cited book *Stochastic Frontier Analysis* by Kumbhakar and Lovell (2000) also advocates the use of this decomposition
for shadow cost and profit functions. A paper by Lovell and Sickles (1983) indicates that a profit function decomposition can be performed for a shadow profit system but do not carry out the decomposition.¹

Kumbhakar (1997) bases his theoretical decomposition of total costs savings on the normalization of one shadow price to 1. While this normalization does not affect the estimation of AE, his proposed decomposition is non-unique, unless the decomposition requirement is satisfied.

The remainder of this paper proceeds as follows. In section 2, we present the methodology to obtain parametric estimates of AE and TE using a shadow cost system. We prove the validity of the decomposition requirement which establishes the uniqueness of Kumbhakar’s shadow cost decomposition only if the relationship between at least one shadow price and one actual price is known. Section 3 establishes that the KDZ decomposition is unique when shadow prices are assumed to equal actual prices (since the decomposition requirement is satisfied for all inputs), but will be non-unique when shadow prices are allowed to differ from actual prices, unless the decomposition requirement is satisfied. The same requirement is established for the shadow cost decomposition estimated using a shadow distance system in Section 4 and for the shadow profit decomposition in Section 5. Empirical examples are provided in Section 6, possible solutions are presented in Section 7, and conclusions follow in Section 8.

¹ An earlier paper by Schmidt and Lovell (1979) successfully decomposes costs savings by estimating a Cobb-Douglas production function with the first-order conditions for cost minimization. They satisfy the decomposition requirement by assuming that each shadow price equals its actual price.
2. Econometric Estimation and Decomposition of A Shadow Cost Function

A number of researchers have estimated shadow cost functions, where estimable parameters scale actual input prices. One scale parameter must be normalized to an arbitrary value in order to identify the other scale parameters. Ratios of computed scale parameters provided estimates of AE. Additional firm-specific parameters were also introduced to capture TE. Although the measures of AE are invariant to the normalization chosen for one input, the decomposition of total cost savings from achieving AE and TE into the portions due to each is non-unique unless the decomposition requirement is satisfied, which we now define.

The Decomposition Requirement: TE and AE measures cannot be converted to estimates of cost savings and cost savings cannot be decomposed into portions due to TE and AE unless a researcher has additional information on the absolute level of TE for (at least) one firm and the (unnormalized) shadow price of (at least) one input. Without these additional pieces of information, the cost savings measures are underidentified, i.e., non-unique.

This requirement is obvious for measuring TE, due to the definition of TE as performance relative to the best-practice firm. Thus, our emphasis will be on the discussion and proof of this requirement for AE. Unless this requirement is satisfied, typically using information beyond what a researcher commonly possesses in empirical applications, shadow cost savings and their decomposition are non-unique. We assume that the decomposition requirement is typically difficult to satisfy, so that the decomposition will typically be non-unique.

What may seem strange to the reader at this point is that the decomposition requirement exists at all for AE, since the papers by KDZ and Mensah established methods by
which one can decompose cost efficiency into components due to TE and AE. In terms of Figure 1, TE is the ratio of costs at point B to costs at point A, while AE is the ratio of costs at point C to costs at point A. What we show in the next section is that KDZ and Mensah demonstrate these methods for the case where each shadow price equals its actual price, which more than satisfies the decomposition principle. That is, they decompose actual costs. However, if one attempts to decompose shadow costs which differ from actual costs using the KDZ and Mensah methods, then the decomposition principle must be satisfied. Since we argue that this is unlikely in practice, the decomposition is most likely non-unique. Before returning to the KDZ method in greater detail, we first examine the alternative method of estimating a shadow cost function, where actual prices are allowed to deviate from shadow prices, and performing a cost decomposition.

2.1. Estimating Technical and Allocative Efficiency Jointly in a Shadow Cost Function

For ease of exposition, we first consider estimating only TE and then both AE and TE using a shadow cost function.

2.1.1 Estimating Technical Efficiency Only

Technical inefficiency can be viewed in two different ways, both focusing on the impact of a firm’s mistakes on its productivity. Output technical inefficiency reflects the amount by which a firm can increase output while holding inputs constant. Input technical inefficiency reflects the amount by which a firm can lower input use while holding output(s) constant.

A firm is output technically efficient if it produces maximal output from a given quantity of measurable inputs. The Farrell measure of output technical inefficiency for firm \( i, \ i = 1, \ldots, F \), is

\[
y_i = a_i f(x_i), \quad 0 < a_i \leq 1,
\]  

(2.1)
where $a_i$ is a firm-specific parameter measuring the degree to which observed output, $y_i$, is less than frontier output, $f(x_i)$, and $x_i = (x_{1i}, \ldots, x_{Ni})$ is the $i^{th}$ firm's $(N \times 1)$ vector of inputs. Assume $f$ is a standard, neoclassical production function, common to all firms. Let the most technically efficient firm, say $i = E$, define the frontier so that $a_E = 1$. All other firms are technically inefficient if $a_i < 1$. Then relative output technical efficiency, measured as a percent of frontier efficiency, is defined as

$$O_i = a_i/a_E = a_i.$$  \hspace{1cm} (2.2)

The cost frontier dual to (2.1) is derived as

$$C_i(y_i/a_i, p_i) = \min_{x_i} [p_i'x_i \mid f(x_i) = y_i/a_i],$$  \hspace{1cm} (2.3)

where $p_i = (p_{1i}, \ldots, p_{Ni})$ is the $(N \times 1)$ vector of input prices for firm $i$. Note that the measure of relative output technical efficiency remains $O_i$.\(^3\)

The Farrell measure of input technical inefficiency is defined by writing observed output as

$$y_i = f(b_ix_i), \quad 0 < b_i \leq 1,$$  \hspace{1cm} (2.4)

where $b_i$ is a firm-specific, scalar parameter that scales input usage. As in (2.1), the implication of (2.4) is that the most efficient firm, $E$, has $b_E = 1$. Firms with $b_i < 1$ can produce a given amount of output and reduce input use, holding input mix constant. Relative input technical efficiency is commonly defined as

$$I_i = b_i/b_E = b_i.$$  \hspace{1cm} (2.5)

\(^2\) While this normalization provides an increasingly accurate ranking as $F \to \infty$, it ignores the fact that in many applications no firm may be fully technically efficient. This normalization, though often quite reasonable, should not be overlooked.

\(^3\) All cost, distance, and profit functions discussed in this paper are assumed to satisfy the standard regularity conditions.
Because a cost function is linearly homogeneous in $p_i$, the cost frontier for the $i^{th}$ firm corresponding to (2.4) is

$$C_i(y_i, p_i/b_i) = \min_{b_i x_i} [(p_i'/b_i)(b_i x_i) \mid f(b_i x_i) = y_i] = (1/b_i)C_i(y_i, p_i). \quad (2.6)$$

When panel data are available, input technical efficiency can be estimated simply as a coefficient of a firm-specific dummy variable, since the $b_i$'s appear only as firm-specific intercepts. This generally is not possible for the estimation of relative output technical efficiency. If and only if $C_i$ is linearly homogeneous in $y_i$, $C_i(y_i/a_i, p_i) = (1/a_i)C_i(y_i, p_i)$ and the $a_i$'s will appear only as firm-specific intercepts. With non-constant returns to scale, the $a_i$ does not factor out. Thus, only under constant returns to scale is $a_i = b_i$.

Together these results imply that even in models with only TE, cost savings estimates are a lower bound. That is because the econometrician can never recover the actual technology, only a bound of the production function or isoquants based on observed performance by the firms in the data set used. Thus, cost savings will either be exact (if one or more firms is truly 100% efficient) or underestimated (if all firms are at least somewhat technically inefficient).

### 2.1.2. Estimating Technical and Allocative Efficiency

Again following Atkinson and Cornwell (1994), we now temporarily suppress the technical efficiency terms in (2.6) and broaden the scope to include measurement of AE via a shadow cost function, $C^*$, which is obtained as

$$C^*_i(y_i, p_i^*) = \min_{x_i} [p_i'^* x_i | f(x_i) = y_i], \quad (2.7)$$
where \( \mathbf{p}_i^* = (p_{i1}^*, \ldots, p_{Ni}^*) \) is a \((N \times 1)\) vector of shadow prices. For the \( i^{th} \) firm, these prices are related parametrically to market prices, \( p_{ni} \), as

\[
p_{ni}^* = k_{ni} p_{ni}, \quad n = 1, \ldots, N. \tag{2.8}
\]

While the \( k_{ni} \) can also vary over time, for simplicity we assume that they are time invariant.

The first-order conditions corresponding to (2.7) are given by

\[
\frac{\partial f(x_i)}{\partial x_{ni}} = \frac{k_{ni} p_{ni}}{k_{li} p_{li}}, \quad n, l = 1, \ldots, N; n \neq l. \tag{2.9}
\]

Thus, we can identify only the ratios, \( k_{ni}/k_{li} \), for each firm. This is a crucial point which implies that the shadow cost function is identified only up to a factor of proportionality.

Additional information must be introduced through a normalization restriction on one \( k_{ni}, \ \forall \ i \), in order to identify the magnitude of shadow costs. Estimated shadow costs in the context of (2.8) are defined as

\[
C_i^*(\mathbf{p}_i^*, y_i) = \sum_n k_{ni} p_{ni} x_{ni}. \tag{2.10}
\]

Thus, unless the normalization restriction is correct, estimated shadow costs will be some multiple of the true value. A restriction on, say \( k_{Ni}, \ \forall \ i \), would allow identification of \( C_i^* \).

However, if the assumed value of \( k_{Ni} \) is, say, three times its true value, estimated shadow costs will be three times too large. This is equivalent to saying that the identification of the ratios \( k_{ni}/k_{li} \) is not sufficient to recover a unique estimate of shadow costs.

While the firm is assumed to minimize total shadow cost, we observe only actual cost and shares. The firm’s actual cost and shares are, respectively, \( C_i^A = \sum_n p_{ni} x_{ni} \) and \( s_{ni}^A = p_{ni} x_{ni}/C_i^A \). Shadow cost shares are defined as

\[
s_{ni}^* = k_{ni} p_{ni} x_{ni}/C_i^*, \tag{2.11}
\]
which implies
\[ \frac{\pi_n}{s_n^* C^*_n} = \frac{\pi_n}{k_n^p} \]
By substituting this expression for \( \pi_n \) into \( C^A_i \) and \( s_n^A \), we obtain expressions for actual cost and shares in terms of shadow cost and shares:
\[
C^A_i = C_i^* \sum_n \frac{s_n^* k_n^{-1}}{n}, \quad (2.12)
\]
and
\[
s_n^A = \frac{s_n^* k_n^{-1}}{\sum_n s_n^* k_n^{-1}}. \quad (2.13)
\]
We can now easily reintroduce the \( a_i \) parameters or the \( b_i \) parameters in order to estimate input or output-based TE in addition to AE.

2.2. Translog Shadow Cost Estimation and Decomposition

Assuming the availability of panel data, that is, \( T \) time-series observations on \( F \) firms, we follow Atkinson and Cornwell (1994) and write the translog version of the shadow cost frontier in (2.7) as
\[
\ln C^*_it = \gamma_0 + \gamma_y \ln y_{it} + \frac{1}{2} \gamma_{yy} (\ln y_{it})^2 + \sum_n \gamma_{ny} \ln y_{it} \ln (k_n p_{nit})
+ \sum_n \gamma_n \ln (k_n p_{nit}) + \frac{1}{2} \sum_n \sum_l \gamma_{nl} \ln (k_n p_{nit}) \ln (k_l p_{lit}), \quad (2.14)
\]
where \( \gamma_{nl} = \gamma_{ln}, \forall n, l, n \neq l \) and \( t = 1, \ldots, T \). The shadow share equations corresponding to (2.14) are
\[
\pi_n^* = \frac{\partial \ln C^*_it}{\partial \ln (k_n p_{nit})} = \gamma_n + \sum_{l} \gamma_{nl} \ln (k_l p_{lit}) + \gamma_{ny} \ln y_{it}. \quad (2.15)
\]
Then, from (2.7), (2.11), and (2.12), actual cost can be rewritten as:
\[
\ln C^A_{it} = \gamma_0 + \gamma_y \ln y_{it} + \frac{1}{2} \gamma_{yy} (\ln y_{it})^2 + \sum_n \gamma_{ny} \ln y_{it} \ln (k_n p_{nit})
+ \sum_n \gamma_n \ln (k_n p_{nit}) + \frac{1}{2} \sum_n \sum_l \gamma_{nl} \ln (k_n p_{nit}) \ln (k_l p_{lit})
+ \ln \left\{ \sum_n k_n^{-1} [\gamma_n + \sum_l \gamma_{nl} \ln (k_l p_{lit}) + \gamma_{ny} \ln y_{it}] \right\}. \quad (2.16)
\]
Finally, using the definition of $s_{ni}^\ast$ in (2.11), $s_{nit}^A$ in (2.13) is expressed as:

$$s_{nit}^A = \frac{[\gamma_n + \sum_l \gamma_{nl} \ln(k_{li}p_{lit}) + \gamma_{ny} \ln y_{it}]k_{ni}^{-1}}{\sum_n[\gamma_n + \sum_l \gamma_{nl} \ln(k_{li}p_{lit}) + \gamma_{ny} \ln y_{it}]k_{ni}^{-1}}.$$ (2.17)

In these equations, parameters capturing technical inefficiency are suppressed. If output technical inefficiency is specified, $y_i/a_i$ is substituted for $y_i$ in both the cost and share equations. However, if the input measure is specified, technical inefficiency enters the cost equation only in the form of a firm-specific intercept, $\ln(1/b_i)$.

Without restrictions on the $a_i$s ($b_i$s), the parameters $\gamma_y$, $\gamma_{yy}$ and $\gamma_{ny}$ are indeterminate. Thus, without loss of generality, we normalize $a_i$ ($b_i$) to 1 for the most efficient firm. While the normalization of the $a_i$ is necessary to compute TE, this measure is now relative to the most efficient firm, rather than absolute. If the designated “most-efficient” firm were to become more efficient, the computed TE for all the other firms would change.

Holding output constant, $C_{it}^\ast$ is linearly homogeneous in shadow prices. This implies the following restrictions on the cost function parameters:

$$\sum_n \gamma_n = 1$$ (2.18)

$$\sum_n \gamma_{ny} = 0$$ (2.19)

$$\sum_n \gamma_{nl} = \sum_n \sum_l \gamma_{nl} = 0.$$ (2.20)

We also impose symmetry on the structural parameters (all coefficients other than the $k_{ni}$).

Restrictions on the $k_{ni}$ also are required. From (2.9) minimization of $C_i$ subject to an output constraint requires that marginal rates of technical substitution equal ratios of actual input prices for all $n$ inputs and for all $i$ observations, which we refer to as relative price efficiency. Hence, relative price efficiency implies that $k_{ni} = k_{li}$, $\forall n, l, i$, $n \neq l$. Since
we can measure only relative efficiency using a cost function, the equations in (2.15) are homogeneous of degree zero in the \( k_{ni} \). Thus, we arbitrarily choose one input, say the \( N^{th} \), as the numeraire and normalize \( k_{Ni} \) to 1, \( \forall i \). We then jointly estimate (2.16) and (2.17) subject to (2.18)–(2.20) and symmetry of the structural parameters. Then, the estimated \( k_{ni} \) for all \( F \) firms are interpreted relative to \( k_{Ni} \). Choice of the value of the numeraire input has no impact on the estimated value of the log-likelihood. Hence, models with different values of the numeraire are observationally equivalent. One could rescale the \( k_{ni} \) by \( \eta \) and jointly reestimate (2.16) and (2.17) obtaining the same values for the structural coefficients. Although \( \eta \) in non-unique, all the estimated \( k_{ni} \) and their estimated standard errors will be scaled by \( \eta \). Further, since we are only interested in ratios of the \( k_{ni} \), we are still able to compute unique measures of AE.

However, the indeterminacy of \( \eta \) causes problems when one attempts to compute and decompose cost savings from achieving AE, using the estimated shadow cost model. Kumbhakar (1997) repeats Atkinson and Cornwell (1994) in deriving the above equations for the translog shadow cost system using slightly different notation. Using their notation, the Kumbhakar decomposition of \( \ln C^A_{it} \) is

\[
\ln C^A_{it} = \ln C^o + \ln C^{al} + v, 
\]

where

\[
\ln C^o = \gamma_0 + \sum_n \gamma_n \ln p_{nit} + \frac{1}{2} \sum_n \sum_l \gamma_{nl} \ln p_{nit} \ln p_{lit} \\
+ \sum_n \gamma_{ny} \ln y_{it} \ln p_{nit} + \gamma_y \ln y_{it} + \frac{1}{2} \gamma_{yy} (\ln y_{it})^2,
\]
\[
\ln C^{al} = \ln G + \sum_n \gamma_n \ln k_{ni} + \frac{1}{2} \sum_n \sum_l \gamma_{nl} \ln k_{ni} \ln k_{li} \\
+ \sum_n \gamma_{ny} \ln y_{it} \ln k_{ni} + \sum_n \sum_l \gamma_{nl} \ln p_{nit} \ln k_{li}, \tag{2.23}
\]

and

\[
G = \left\{ \sum_n k_{ni}^{-1} [\gamma_n + \sum_l \gamma_{nl} \ln (k_{li} p_{lit}) + \gamma_{ny} \ln y_{it}] \right\}. \tag{2.24}
\]

Kumbhakar (1997) is correct in asserting that if \( k_{ni} = 1, \forall n \), then \( \ln C^o \) is the log of allocatively efficient costs. Note that one would obtain \( \ln C^o \) in two steps. First, one would estimate (2.16) and (2.17) with \( k_{Ni} \) normalized to 1 (equivalently \( \xi_J = 0 \) in Kumbhakar). Using the estimated parameters, one would then compute \( \ln C^o \) from \( \ln C^* \) in (2.14) by holding the structural coefficient estimates constant and setting \( k_{ni} = 1, \forall n, n = 1, \ldots, N - 1 \) (equivalently setting \( \xi_j = 0, \forall j = 1, \ldots, J - 1 \) in Kumbhakar).\(^4\)

Notationally, we denote this process as \( \ln \hat{C}^*(k_{ni} = 1) \).

However, the following two lemmas show that this decomposition is generally invalid and only is correct when each shadow price equals its actual price, as in the KDZ decomposition method.

**Lemma 1.** Different values for the numeraire \( k_{Ni} = \eta \) and hence for shadow costs, \( C^* \), are observationally equivalent.

**Proof.** One can arbitrarily set \( k_{Ni} = \eta \) without changing (2.16) and hence the log likelihood. This can easily be seen by scaling each \( k_{ni} \) by \( \eta \) and applying the adding up conditions in (2.18)–(2.20). However, this scaling adds \( \ln \eta \) to \( \ln C^* \) in (2.14), which is seen by again applying (2.18)–(2.20) to (2.14). Hence, different values of \( \eta \) and \( \ln C^* \) are observationally equivalent.\(^\star\)

\(^4\) Clearly, one cannot estimate the equation for \( \ln C^*, \text{(2.14)}, \) by itself, since the left-hand side is unobserved.
Lemma 2. The decomposition in (2.21) is non-unique without knowledge of $\eta$. The interpretation of $\ln C^o$ as the log of allocatively efficient shadow cost is not valid in general. Instead, the log of allocatively efficient shadow cost for any positive numeraire value, $\eta$, is $\ln C^o_\eta = \ln C^o + \ln \eta$, where $\eta$ is the non-unique value of the numeraire $k_{Ni}$ and $\ln C^o$ is Kumbhakar’s equation (2.22).

Proof. Kumbhakar normalizes $k_{Ni} = 1$ obtaining $\ln C^*(k_{ni} = 1) = \ln C^o$ and thus $\ln C^o = \ln C^o + \ln C^al$, ignoring the error term. However, since Lemma 1 establishes that this normalization is non-unique, we generalize Kumbhakar’s formulation by normalizing $k_{Ni}$ to $\eta, \eta > 0$, rather than to 1. After obtaining estimates of the parameters of (2.14) with $k_{Ni} = \eta$ and the other $k_{ni}, n \neq N$, unrestricted, we set $k_{ni} = \eta, \forall n, n = 1, \ldots, N - 1$, in (2.14) to obtain $\ln \hat{C}^*(k_{ni} = \eta) = \ln \hat{C}^o_\eta$. The adding up restrictions in (2.18)–(2.20) imply that $\ln \hat{C}^*(k_{ni} = \eta) = \ln \hat{C}^o + \ln \eta$. Only when we employ the normalization $k_{Ni} = \eta = 1$ in estimation and we then set $k_{ni} = 1, \forall n$, do we obtain $\ln \hat{C}^*(k_{ni} = 1) = \ln \hat{C}^o$, since $\ln(1) = 0$.

The implication of Lemma 2 is that as the non-unique value of $\eta$ increases, estimated shadow cost subject to allocative efficiency, $\hat{C}^o_\eta = \hat{C}^*(k_{ni} = \eta)$, rises and cost savings, defined as $C^A - \hat{C}^*(k_{ni} = \eta)$, falls. If $\eta$ is large enough, estimated cost savings will be negative—something which occurs occasionally in the empirical literature.

Lemma 3. The decomposition of cost savings in percentage terms is non-unique, without knowledge of $\eta$. This applies to the calculation of percent of total cost savings due to AE and TE for a firm or for one firm relative to another firm.

Proof. Lemma 2 establishes the indeterminacy of total cost savings without knowledge of $\eta$. This indeterminacy is not eliminated by computing the percent of total cost savings.
saving due to achieving AE for a firm as

\[
\frac{[C_{it}^A - \hat{C}_{it}^*(k_{ni} = \eta)]}{C_{it}^A},
\]

since \(\eta\) cannot be factored out and cancelled from numerator and denominator. Further, this indeterminacy is not eliminated by computing the ratio of percent cost savings due to achieving AE for any two firms, since again \(\eta\) does not cancel everywhere in

\[
\frac{[C_{it}^A - \hat{C}_{it}^*(k_{ni} = \eta)]}{C_{it}^A}
\]

and

\[
\frac{[C_{lt}^A - \hat{C}_{lt}^*(k_{nl} = \eta)]}{C_{lt}^A}.
\]

The non-uniqueness of computed TE cost savings follows from its definition relative to the frontier firm.

**Theorem 1.** The translog cost function decomposition requirement: the researcher must have information on the absolute level of technical efficiency for (at least) one firm and the (unnormalized) shadow price of (at least) one input for a translog shadow cost function in order to uniquely estimate total cost savings, to uniquely decompose these savings into their component parts, and to uniquely compute relative cost savings within or between firms.

**Proof.** This result follows from Lemmas 1, 2, and 3 where we must know the value of \(\eta\), which implies that we must know the relationship between at least one shadow and one actual price in order to make the decomposition unique. Once this relationship is known, \(\eta\) can no longer be arbitrarily scaled and cost savings are uniquely identified.

2.3. Estimation and Decomposition of any Shadow Cost Function

The results of the previous section were for the translog shadow cost function. However, we can now generalize Theorem 1:
Theorem 2. The general cost function decomposition requirement: the researcher must have information on the absolute level of technical efficiency for (at least) one firm and the (unnormalized) shadow price of (at least) one input for any shadow cost function in order to uniquely estimate total cost savings, to uniquely decompose these savings into their component parts, and to uniquely compute relative cost savings within or between firms.

Proof. This result follows from substituting any functional form for (2.7) into (2.12) and (2.13). Since $C^\ast$ is linearly homogeneous in $p^\ast$, rescaling the $k_{ni}, \forall \ n$, by $\eta$ scales $C^\ast$ by $\eta$. However, from (2.12) and (2.13) one can easily see that the new and old models are observationally equivalent. Thus, total shadow costs and hence cost savings are non-unique without knowledge of the relationship between at least one shadow and one actual price. By retracing the steps of Lemmas 2 and 3 with any functional form for $C^\ast$, the percent cost savings for a given firm and relative cost savings between firms are also nonunique unless we know the relationship between at least one shadow and one actual price. Once this relationship is known, $\eta$ can no longer be arbitrarily rescaled.

To satisfy this decomposition requirement, which allows one to uniquely estimate and decompose cost savings, one must know at least one absolute shadow value, not just relative shadow prices. However, we need additional economic theory to support setting $\eta$ to any value, including 1. Without this information, the decomposition employed by Kumbhakar (1997), Berger and Humphrey (1991), Mensah (1994), Atkinson and Primont (2003), and Kumbhakar and Tsionas (2005a, 2005b) yields non-unique results. Since the required information is usually difficult to obtain, the decomposition will typically be non-unique. Empirical examples of these theoretical results are offered later.
3. The KDZ Decomposition Method

3.1. The KDZ Decomposition When Shadow Prices Equal Actual Prices

We now return to a more detailed examination of the KDZ method as applied to the decomposition of a previously estimated actual cost function and show that this method by definition satisfies the decomposition principle. Then we demonstrate that use of the KDZ method to decompose an estimated shadow cost function requires satisfying the decomposition principle and is therefore subject to the same problems as the econometric shadow cost estimation approach.

KDZ and Mensah were motivated to decompose economic efficiency into the portions due to AE and TE using an estimated cost equation based on actual (rather than shadow) prices. KDZ wished to avoid estimation of a production or distance function, which would allow comparison of ratios of marginal products to ratios of input prices. The KDZ solution was to estimate the cost function econometrically and then utilize Shephard’s lemma to derive input demand equations. They then evaluated these equations at actual prices and the technically efficient output level to obtain input usage at levels that possessed both AE and TE. Using non-linear optimization techniques, they next solve for a set of relative prices that satisfy a system of non-linear equations at the point where the firm achieves TE, subject to maintaining the original input mix.

Referring to Figure 1, the ratio $OA/OB$ measures TE, $OC/OB$ measures AE, and $OC/OA$ measures productive efficiency (PE) as the product of AE and TE. For more than two inputs,

\[
TE = \frac{||x^B||}{||x^A||}
\]

\[
AE = \frac{||x^C||}{||x^B||}
\]
where \(|\|\mathbf{x}\|| = (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}}\) and \(\mathbf{x} \cdot \mathbf{x}\) is the inner product of the vector \(\mathbf{x}\) with itself, we define each measure as the ratio of vector norms. One can also compute \(\text{TE}, \text{AE},\) and \(\text{PE}\) using ratios of total costs at the corresponding points.

We first compute total cost at point A as

\[
TC^A = \sum_{n=1}^{N} x_n^A p_n^A, \tag{3.1}
\]

where \(x_n^A\) is the actual quantity of input \(n\) and \(p_n^A\) is the actual price of input \(n\). Using Shephard’s lemma, we then compute \(x_n^E\) at the point of tangency between the isocost line defined by actual prices, \(\mathbf{p}^A = (p_1^A, \ldots, p_N^A)\), and the isoquant as

\[
x_n^E = \frac{\partial C(y^*, \hat{\mathbf{p}}^A, 1)}{\partial p_n}, \quad \forall \ n, n = 1 \ldots, N, \tag{3.2}
\]

where \(\hat{\mathbf{p}}^A = (p_1^A/p_N^A, \ldots, p_{N-1}^A/p_N^A)\) is a vector of relative prices defined subject to the numeraire input \(N\) and \(y^*\) is the frontier production level. We then compute total costs at point E as

\[
TC^E = \sum_{n=1}^{N} x_n^E p_n^A. \tag{3.3}
\]

The difficult problem is to solve for \(\mathbf{x}^B\) at point B in Figure 1. To do this we need to solve for the relative prices that support point B and lie on a radial contraction of point A where it intersects the isoquant. While we were unable to reproduce the results of the Kopp-Diewert decomposition in column 2 of their Table 1 using a number of different Matlab and Gauss algorithms that employed analytical as well as numerical first-order partial derivatives for their equation (14), we were able to solve the Zieschang simplification of the Kopp-Diewert approach for a simple demonstration data set using Matlab. The
Zieschang method is summarized clearly in Mensah (1994) as the solution for \( N - 1 \) price relatives for a system of \( N - 1 \) equations:

\[
\frac{\partial C(y^*, \hat{p}^B, 1)}{\partial \hat{p}_n} / \frac{\partial C(y^*, \hat{p}^B, 1)}{\partial \hat{p}_N} - x_{AN}/x_{AN} = 0, \ n = 1, \ldots, N - 1,
\]

where the relative prices \( \hat{p}^B = (p^B_1/p^B_N, \ldots, p^B_{N-1}/p^B_N) \) support point B. Absolute prices can be obtained from relative prices using formula (6a) from Mensah, which correctly replaces unit total costs used by Zieschang with \( C^A \).

Then we compute \( x_n^B \) as

\[
x_n^B = \frac{\partial C(y^*, \hat{p}^B, 1)}{\partial p_n}, \ \forall \ n = 1, \ldots, N,
\]

where total costs at point B are

\[
TC^B = \sum_{n=1}^{N} x_n^B p_n^A.
\]

3.2. The KDZ Decomposition When Shadow and Actual Prices Differ

What happens when we apply the KDZ decomposition to a previously estimated shadow cost function? We show that, as with the shadow cost decomposition problem just considered, unless the decomposition requirement is satisfied, the KDZ method yields a non-unique decomposition. Assuming that \( k_{ni} = k_n \) and that \( p_{ni}^* = p_n k_n \) for simplicity, the isocost line with slope \( p_1^*/p_2^* \) is the shadow isocost line which supports input use at point F in Figure 1. This isocost line corresponds to the isocost line in terms of actual prices \( p_1^A/p_2^A \), only if each shadow price equals its actual price. As already shown, we can arbitrarily rescale the numeraire \( k_{Ni} \) to any positive, finite value without changing the

---

5 Note also that the supporting price line at the equivalent of point B in Figure 4.1 of Kumbhakar and Lovell (2000) is not provided and that their formula following equation (4.2.31) for deriving absolute prices from relative prices is correct only in the case of unit total costs.
optimal values of $x_1$ or $x_2$. Thus, the values of the $k_n$ are non-unique. Unless we know that at least one actual price equals some factor times its shadow price, absolute values of shadow prices remain unidentified.

We now demonstrate that one can uniquely decompose cost saving when each shadow price equals its actual price (as assumed by KDZ) using the KDZ method, but that when shadow prices differ from actual prices, this decomposition is nonunique unless the decomposition requirement is satisfied. We utilize the parameters that Kopp and Diewert estimated for their three-input Cobb-Douglas production function:

$$y_i = 0.049 x_1^{0.25} x_2^{0.1} x_3^{0.7}. \quad (3.7)$$

They derive the dual cost function as

$$C(y, p) = 40.4 p_1^{0.238} p_2^{0.095} p_3^{0.666} y^{0.952}, \quad (3.8)$$

but could just as easily have estimated this cost function directly and obtained the same estimated coefficients, since the two functions are self-dual.

Using data for one observation on $x = (x_1, x_2, x_3)$ and $y$ as given in Table 2 of their paper, we find that a variety of techniques involving numerical as well as first-order analytical derivatives are unable to solve their equations (13a)-(13e) for $y, p^B, \text{ and } x^B = (x_1^B, x_2^B, x_3^B)$. Even using this data with (3.4), we were unable to achieve convergence. However, using the more uniformly scaled data $x = (5, 7, 8), p = (3, 2, 1), \text{ and } y = 6$, the FSOLVE minimizer in MATLAB solves (3.4) in 16 iterations, yielding the values of $\hat{p}_1 = p_1^B/p_3^B$ and $\hat{p}_2 = p_2^B/p_3^B$ given in column 2 of our Table 1. We then evaluate (3.5) at these price relatives to obtain estimates of $x^B$, which are given in column 2 of Table 1 along with the estimates of $AE = TC^F/TC^B, TE = TC^B/TC^A$, and total potential cost.
savings = $TC_A - TC_F$. obtained using (3.6), (3.3), and (3.1). Note that since each actual price equals its shadow price, points E and F are identical.

We now allow shadow prices to differ from actual prices, assuming that $k$ values have been estimated jointly with structural parameters in a previous step. First, we scale actual prices, $p_1$, $p_2$, and $p_3$, by the same factor, so that $k_1 = k_2 = k_3 = 2$, and resolve for relative prices at point B. Results are reported in column 3. Since the firm still satisfies the conditions for AE (all $k_n$ are equal), the results are identical to column 2, except for the ratio $TC_F/TC_B$, which has increased solely due to the increase in $TC_F$, and the total cost saving from attaining AE and TE, $TC^A - TC^F$, which has declined.\(^6\) We now simulate a change in AE by scaling actual prices, $p_1$, $p_2$, and $p_3$, differentially by $k_1 = 3$, $k_2 = 2$, and $k_3 = 1$, respectively, and resolve our model. In column 4, $x^F_1$, $x^F_2$, and $x^F_3$ have changed consistent with a different degree of AE, $TC^F/TC^B$ has decreased, and $TC^A - TC^F$ has increased relative to column 3. In column 5, we simulate the effect of doubling the numeraire $k_{Ni}$ value from column 4, but maintain the allocative inefficiency level from this column. This implies that we must double all $k_{ni}$ values from their column 4 levels. In the new solution, $x^F_1$, $x^F_2$, and $x^F_3$ remain unchanged since quantities are homogeneous of degree zero in shadow prices. However, $TC^F/TC^B$ has increased to 1.6977 and $TC^A - TC^F$ has fallen to -11.95, which implies the untenable result that achieving AE and TE relative to the firm’s initial position at point A results in an increase in costs. Negative estimates of this kind are reported by Burger and Humphrey (1991) and occur because shadow costs and actual costs are being added and subtracted as if they were in the same units, which they are not unless the decomposition principle is satisfied.

\(^6\) Any row values that differ from the baseline KDZ values in column 2 are italicized.
In short, we have demonstrated that quantities at point F are homogeneous of degree zero and change only when the ratios of the $k_n$ change. However, changing the normalized value of $k_3$, whether this leaves the ratios of the $k_n$ unchanged or alters them, will change the estimates of $TC^F / TC^B$ and the absolute value of cost savings, $TC^A - TC^F$. Clearly, if shadow and actual prices differ and we fail to satisfy the decomposition requirement, the results from a cost decomposition are non-unique.

4. Estimation and Decomposition Using a Shadow Distance System

Estimation of cost savings and cost decomposition from a shadow distance system has problems identical to those encountered with estimation of a shadow cost system. The input distance function is defined as

$$D(y, x) = \sup_{\lambda} \{ \lambda : (x/\lambda) \in L(y) \}. \quad (4.1)$$

We now reverse the roles of shadow prices and input quantities employed to derive the shadow cost function. Assuming cost minimization, we obtain the dual cost function as

$$C(y, p) = \min_x \left\{ p'x : D(y, x) \geq 1 \right\} \quad (4.2)$$

and let $x^* = [\kappa_1 x_1, \ldots, \kappa_N x_N]$ be the $(N \times 1)$ vector of shadow input quantities that solves the minimization problem in (4.2).

The first-order condition corresponding to (4.2) is

$$p_n = \mu \frac{\partial D(y, x^*)}{\partial x_n}, \quad n = 1, \ldots, N, \quad (4.3)$$

where $\mu$ is the Lagrangian multiplier and $\frac{\partial D(y, x^*)}{\partial x_n}$ indicates the partial derivative of $D(y, x)$ with respect to $x_n$, evaluated at $x_n^*$. Following Atkinson and Dorfman (2005), we express the $n^{th}$ equation in (4.3) as

$$p_n = \left( \sum_l p_l x_l^* \right) \frac{\partial D(y, x^*)}{\partial x_n}, \quad n = 1, \ldots, N. \quad (4.4)$$
We can write the stochastic input distance function as

\[ 1 = D(y_{it}, x^*_it) h(\epsilon_{it}). \]  

(4.5)

As a flexible approximation to the underlying true distance function, the stochastic translog shadow input distance frontier is written as

\[ 0 = \ln[D(y_{it}, x^*_it) h(\epsilon_{it})] = \ln D(y_{it}, x^*_it) + \ln h(\epsilon_{it}) \]

\[ = \beta_o + \sum_m \beta_m \ln y_{mit} + (1/2) \sum_m \sum_w \beta_{mw} \ln(y_{mit}) \ln(y_{wit}) \]

\[ + \sum_m \sum_n \beta_{mn} \ln y_{mit} \ln x^*_nit + \sum_n \beta_n \ln x^*_nit \]

\[ + (1/2) \sum_n \sum_l \beta_{nl} \ln x^*_nit \ln x^*_lit + \ln h(\epsilon_{it}), \]  

(4.6)

where

\[ h(\epsilon_{it}) = \exp(v_{it} - u_{it}), \]  

(4.7)

\( v_{it} \) is noise and \( u_{it} \) is a one-sided error.

The conditions for AE are given by (4.3), where shadow prices are replaced by actual prices and actual quantities are replaced by shadow quantities. For firm \( f \) at time \( t \), we can directly estimate relative over- and under-utilization of any pair of inputs, \( x_{nit} \) and \( x_{lit} \), in comparison to the cost-minimizing ratio, \( (\kappa_{nt}x_{nit})/(\kappa_{lt}x_{lit}) \), by computing \( \hat{\kappa}_{nt}/\hat{\kappa}_{lt} \).

The same problems with identifying total cost savings and decomposing them apply as when we estimated the shadow cost system. Since we can identify \( \kappa_{ni}, \forall n \), only up to a factor of proportionality, shadow costs, defined as

\[ C^*_i(p_i, y_{ni}) = \sum_n x^*_ni p_{ni} = \sum_n \kappa_{ni}x_{ni} p_{ni} \]  

(4.8)
are not unique. In estimation, the numeraire $\kappa_{N_i}, \forall i$, must be arbitrarily normalized to some chosen value, $\xi$. This choice does not affect the values of estimated structural parameters or their estimated asymptotic standard errors. The estimated intercept term of the distance function adjusts so that its value minus $\ln \xi$ is always the same, regardless of the value chosen for $\xi$. The estimated $\kappa_{ni}$ values and their estimated asymptotic standard errors are scaled by $\xi$. The result is that the log likelihood does not change with $\xi$. Thus, models based on different values of $\xi$ are observationally equivalent. However, estimated shadow costs, estimated potential cost savings, and the decomposition of cost savings change. Thus, the shadow distance system will produce non-unique estimates of cost savings and their decomposition unless the decomposition requirement is satisfied.

**Theorem 3.** *The general distance function decomposition requirement: the researcher must have information on the absolute level of technical efficiency for (at least) one firm and the (unnormalized) shadow quantity of (at least) one input for any shadow distance function in order to uniquely estimate total cost savings, to uniquely decompose these savings into their component parts, and to uniquely compute relative cost savings within or between firms.*

**Proof.** This result follows from retracing the steps of the proof of Theorem 2 using shadow quantities in place of shadow prices. ■

5. Estimation and Decomposition Using a Shadow Profit System

5.1. *The Shadow Profit System*

Estimation of profit savings and profit decomposition from a shadow profit system has identical problems. First, define actual profits as revenue less costs,

$$\bar{\Pi}_i = [s'_iy_i - p'_ix_i], \quad (5.1)$$

24
where $p_i = (p_{1i}, \ldots, p_{Ni})$ is a $(N \times 1)$ vector of input prices and $s_i = (s_{1i}, \ldots, s_{Mi})$ is a $(M \times 1)$ vector of output prices. The corresponding profit function is

$$\tilde{\Pi}(p_i, s_i) = \sup_{x_i, y_i} \{ s'_i y_i - p'_i x_i : (x_i, y_i) \in T \},$$  \hspace{1cm} (5.2)$$

where $T = \{(x_i, y_i) : x_i \in \mathbb{R}^N_+, y_i \in \mathbb{R}_+^M, x_i \text{ can produce } y_i \}$. The non-normalized profit function, $\tilde{\Pi}(p_i, s_i)$ is linearly homogeneous in non-normalized input and output prices.

From (5.1) we can derive normalized profits, defined as

$$\Pi_i = r'_i y_i - q'_i x_i,$$  \hspace{1cm} (5.3)$$

where $q_i = (p_{1i}/s_M, \ldots, p_{Ni}/s_M)$ and $r_i = (s_{1i}/s_M, \ldots, s_{Mi}/s_M)$ and we have arbitrarily normalized by the price of output $M$. The normalized profit function gives the maximized value of normalized profit as a function of $(q_i, r_i)$:

$$\Pi(q_i, r_i) = \sup_{x_i, y_i} \{ r'_i y_i - q'_i x_i : (x_i, y_i) \in T \}. $$  \hspace{1cm} (5.4)$$

Normalizing $\{p_i, s_i\}$ by an arbitrarily chosen price and estimating $\Pi(q_i, r_i)$ imposes the restriction that $\tilde{\Pi}(p_i, s_i)$ is linearly homogeneous in $\{p_i, s_i\}$.

We can now introduce technical efficiency parameters. Output technical inefficiency parameters, $a_i$, $a_i > 1$, measure the potential a firm has to increase profits by increasing output for a given level of inputs. The normalized profit frontier capturing output technical efficiency for the $i^{th}$ firm is

$$\Pi^O_i(q_i/a_i, r_i) = \sup_{x_i, y_i} \{ a_i r'_i y_i - q'_i x_i : (x_i, y_i) \in T \}$$

$$= a_i \sup_{x_i, y_i} \{ r'_i y_i - (q'_i/a_i) x_i : (x_i, y_i) \in T \}$$

$$= a_i G^O_i(q_i/a_i, r_i),$$  \hspace{1cm} (5.5)$$
where $\Pi^O_i$ is a normalized profit function that we assume satisfies the regularity conditions of a normalized profit function. Given the normalization of the best firm as the frontier firm, to reach the frontier firm $i$ must increase profits by

$$a_i G^O_i(q_i/a_i, r_i) - G^O_i(q_i, r_i). \quad (5.6)$$

Input technical inefficiency measures the potential each firm has to increase profits by reducing inputs to produce a given level of output. Using the input technical inefficiency parameter, $b_i$, $0 < b_i < 1$, a restricted normalized profit frontier reflecting input technical inefficiency has the general form

$$\Pi^T_i(q_i/b_i, r_i) = \sup_{b_i x_i, y_i} \{r_i y_i - (q_i'/b_i) b_i x_i : (x_i, y_i) \in T \} = G^T_i(q_i/b_i, r). \quad (5.7)$$

Again given the normalization of the best firm as the frontier firm, to reach the frontier, firm $i$ must increase profits by

$$G^T_i(q_i/b_i, r_i) - G^T_i(q_i, r_i). \quad (5.8)$$

We now introduce AE parameters by defining relative input shadow prices as $q^*_i = (q^*_{1i}, \ldots, q^*_{Ni})$ and relative output shadow prices as $r^*_i = (r^*_{1i}, \ldots, r^*_{M-1,i})$. For the $i^{th}$ firm, these prices are related parametrically to normalized market input and output prices, $q_{mi}$ and $r_{ni}$, as

$$q^*_{ni} = \varphi_{ni} q_{ni}, \quad n = 1, \ldots, N, \quad (5.9)$$

and

$$r^*_{mi} = \varphi_{mi} r_{mi}, \quad m = 1, \ldots, M - 1, \quad (5.10)$$
where $\varphi_{mi} = \hat{\varphi}_{mi}/\hat{\varphi}_{Mi}$ and $\varphi_{ni} = \hat{\varphi}_{ni}/\hat{\varphi}_{Mi}$. We define $\varphi_{mi}$ and $\varphi_{ni}$ as the positive relative efficiency parameters measuring the deviation of normalized shadow prices from normalized actual prices. The parameters $\hat{\varphi}_{mi}$, $\hat{\varphi}_{ni}$, and $\hat{\varphi}_{Mi}$ are the absolute efficiency parameters associated with the absolute output price $s_{mi}$, $\forall \ m, m \neq M$, the absolute input price $p_{ni}$, $\forall \ n$, and the numeraire output price $s_{Mi}$, respectively.

The corresponding normalized shadow profit functions in terms of normalized shadow prices of inputs and outputs are defined as $\Pi^O_i(q_i^*/a_i, r_i^*)$ and $\Pi^T_i(q_i^*/b_i, r_i^*)$, while the non-normalized shadow profit functions in terms of non-normalized shadow prices of inputs and outputs are defined as $\tilde{\Pi}^O_i(p_i^*/a_i, s_i^*)$ and $\tilde{\Pi}^T_i(p_i^*/b_i, s_i^*)$.

Using Shephard’s Lemma,

\[
\frac{\partial \Pi^O_i(q_i^*/a_i, r_i^*)}{\partial (q_{ni}/a_i)} = -a_i x_{ni}, \quad (5.11)
\]

\[
\frac{\partial \Pi^T_i(q_i^*/b_i, r_i^*)}{\partial (q_{ni}/b_i)} = -b_i x_{ni}, \quad (5.12)
\]

\[
\frac{\partial \Pi^O_i(q_i^*/a_i, r_i^*)}{\partial (r_{mi})} = a_i y_{mi}, \quad (5.13)
\]

and

\[
\frac{\partial \Pi^T_i(q_i^*/b_i, r_i^*)}{\partial (r_{mi})} = y_{mi}, \quad (5.14)
\]

where the notation indicates that the derivatives of $\Pi^O_i$ and $\Pi^O_i$ are taken with respect to $q_{ni}, r_{mi}$ and evaluated at $q_{ni}, r_{mi}$.

Normalized shadow profits can be written as

\[
\Pi^* = y_{Mi} + r_i^* y_i - q_i^* x_i, \quad (5.15)
\]

while normalized actual profits are

\[
\Pi^a = y_{Mi} + r_i^' y_i - q_i^' x_i. \quad (5.16)
\]
Solving (5.15) for \( y_M \), substituting into (5.16), and using (5.14) and (5.12), we can express actual profits in terms of shadow profits:

\[
\Pi_i^a = \Pi_i^{\ast I} + \sum_{m=1}^{M-1} r_m (1 - \varphi_m) \frac{\partial \Pi_i^{\ast I} (q_i^* / a_i, r_i^*)}{a_i \partial (r_m)} + \sum_{n=1}^{N} q_n (1 - \varphi_n) \frac{\partial \Pi_i^{\ast I} (q_i^* / a_i, r_i^*)}{a_i \partial (q_n / a_i)}.
\] (5.17)

5.2. Estimation and Decomposition Using the Translog Shadow Profit Function

For the purpose of estimation, we employ the translog flexible functional form for the profit frontiers defined in (5.5) and (5.7). The translog provides a convenient second-order approximation to an arbitrary continuously twice-differentiable profit function.

The translog specification of shadow profits for the \( I \)-model is

\[
\ln \Pi_i^{\ast I} = \delta_o + \sum_n \delta_n \ln (q_{ni}^* / a_i) + \sum_m \delta_m \ln (r_{mi}^*)
+ \frac{1}{2} \sum_n \sum_{n'} \delta_{nn'} \ln (q_{ni}^* / a_i) \ln (q_{n'i}^* / a_i)
+ \frac{1}{2} \sum_m \sum_{m'} \delta_{mm'} \ln (r_{mi}^*) \ln (r_{m'i}^*)
+ \sum_n \sum_m \delta_{mn} \ln (r_{mi}^*) \ln (q_{ni}^* / a_i),
\] (5.18)

where \( \delta_{nn'} = \delta_{n'n}, \forall n, n', n \neq n' \) and \( \delta_{mm'} = \delta_{m'm}, \forall m, m', m \neq m'. \) Taking antilogs, we obtain an expression for shadow profits as

\[
\Pi_i^{\ast I} = \exp \left[ \delta_o + \sum_n \delta_n \ln (q_{ni}^* / a_i) + \sum_m \delta_m \ln (r_{mi}^*)
+ \frac{1}{2} \sum_n \sum_{n'} \delta_{nn'} \ln (q_{ni}^* / a_i) \ln (q_{n'i}^* / a_i)
+ \frac{1}{2} \sum_m \sum_{m'} \delta_{mm'} \ln (r_{mi}^*) \ln (r_{m'i}^*)
+ \sum_n \sum_m \delta_{mn} \ln (r_{mi}^*) \ln (q_{ni}^* / a_i) \right].
\] (5.19)
Using (5.19) we can express our partial derivatives in (5.11) and (5.13) as

\[
\frac{\partial \Pi^*_I(q^*_i/a_i, r^*_i)}{\partial (q_{ni}/a_i)} = \left( \frac{\Pi^*_I}{q_{ni}^*/a_i} \right) \left( \delta_n + \sum_{n'} \delta_{nm'} \ln(q^*_{ni}/a_i) + \sum_m \delta_{mn} \ln(r^*_m) \right)
\]  

(5.20)

and

\[
\frac{\partial \Pi^*_I(q^*_i/a_i, r^*_i)}{\partial (r_{mi})} = \left( \frac{\Pi^*_I}{r_{mi}} \right) \left( \delta_m + \sum_{m'} \delta_{mm'} \ln(r^*_m) + \sum_n \delta_{mn} \ln(q^*_n) \right).
\]  

(5.21)

Substituting (5.20) and (5.21) into (5.17) we obtain:

\[
\Pi^a = \Pi^*_I \left\{ 1 + \sum_{m=1}^{M-1} \left[ \frac{(1 - \varphi_m)}{\varphi_m} \right] \left( \delta_m + \sum_{m'} \delta_{mm'} \ln(r^*_{m'}) \right) 
\right.
\]

\[
+ \sum_n \delta_{mn} \ln(q^*_{ni}/a_i) \right)
\]

(5.22)

Substituting for behavioral profits we obtain an estimable expression for normalized actual profits:

\[
\ln \Pi^a = \delta_o + \sum_n \delta_n \ln(q^*_{ni}/a_i) + \sum_m \delta_m \ln(r^*_m)
\]

\[
+ \frac{1}{2} \sum_n \sum_{n'} \delta_{nn'} \ln(q^*_{ni}/a_i) \ln(q^*_{n'i}/a_i)
\]

\[
+ \frac{1}{2} \sum_m \sum_{m'} \delta_{mm'} \ln(r^*_m) \ln(r^*_{m'})
\]

\[
+ \sum_n \sum_m \delta_{nm} \ln(r^*_m) \ln(q^*_n)
\]

\[
+ \ln \left\{ 1 + \sum_{m=1}^{M-1} \left[ \frac{(1 - \varphi_m)}{\varphi_m} \right] \left( \delta_m + \sum_{m'} \delta_{mm'} \ln(r^*_{m'}) + \sum_n \delta_{mn} \ln(q^*_{ni}/a_i) \right) 
\right.
\]

\[
+ \sum_{n=1}^N \left[ \frac{(1 - \varphi_n)}{\varphi_n} \right] \left( \delta_n + \sum_{n'} \delta_{nn'} \ln(q^*_{n'i}/a_i) + \sum_m \delta_{mn} \ln(r^*_{mi}) \right) \right\}.
\]  

(5.23)
The actual share equations for input quantities can be expressed in terms of shadow prices as:

\[
M_{ni}^a = \frac{q_{ni}x_{ni}}{\Pi_{ai}} = -\varphi_{ni}^{-1}\left\{1 + \sum_{m=1}^{M-1}\left[\frac{(1 - \varphi_m)}{\varphi_m}\right] \left(\delta_m + \sum_{m'} \delta_{mm'} \ln(r_{m'i}/a_i) + \sum_n \delta_{mn} \ln(q_{ni}^*/a_i)\right)\right. \\
+ \sum_{n=1}^{N}\left[\frac{(1 - \varphi_n)}{\varphi_n}\right] \left(\delta_n + \sum_{n'} \delta_{nn'} \ln(q_{n'i}/a_i) + \sum_m \delta_{mn} \ln(r_{mi}^*)\right)\right\}^{-1} \\
\left(\delta_n + \sum_{n'} \delta_{nn'} \ln(q_{n'i}/a_i) + \sum_m \delta_{mn} \ln(r_{mi}^*)\right) \quad n = 1, \ldots, N, \quad (5.24)
\]

and the actual share equations for output quantities can be expressed in terms of shadow prices as:

\[
M_{mi}^a = \frac{r_{ni}y_{ni}}{\Pi_{ai}} = \varphi_{mi}^{-1}\left\{1 + \sum_{m=1}^{M-1}\left[\frac{(1 - \varphi_m)}{\varphi_m}\right] \left(\delta_m + \sum_{m'} \delta_{mm'} \ln(r_{m'i}/a_i) + \sum_n \delta_{mn} \ln(q_{ni}^*/a_i)\right)\right. \\
+ \sum_{n=1}^{N}\left[\frac{(1 - \varphi_n)}{\varphi_n}\right] \left(\delta_n + \sum_{n'} \delta_{nn'} \ln(q_{n'i}/a_i) + \sum_m \delta_{mn} \ln(r_{mi}^*)\right)\right\}^{-1} \\
\left(\delta_m + \sum_{m'} \delta_{mm'} \ln(r_{m'i}/a_i) + \sum_n \delta_{mn} \ln(q_{ni}^*/a_i)\right), \quad m = 1, \ldots, M - 1. \quad (5.25)
\]

The set of estimated equations consists of (5.25), (5.24), and (5.23). Since we have normalized by one price, we have imposed linear homogeneity in input and output prices and there is no need to substitute a set of adding up restrictions. From (5.18), (5.23), (5.24), and (5.25), only relative shadow prices matter for estimation of relative price efficiency of \(M+N-1\) inputs and outputs. The choice of the numeraire value for \(\varphi_{Mi}, \chi\), is unimportant.

However, if we wish to compute the profit loss due to price inefficiency, we must be able to identify the absolute values of the efficiency parameters, since profits are based on absolute prices, not relative prices. Because the normalized profit function is homogeneous
of degree zero in relative prices, we can only estimate \( \varphi_{mi}, \forall \ m, m \neq M \) and \( \varphi_{ni}, \forall \ n, \) not \( \tilde{\varphi}_{mi}, \forall \ m, m \neq M \) and \( \tilde{\varphi}_{ni}, \forall \ n. \) In order to recover these absolute values, we must know the value of \( \chi, \) based on outside information. Without this information, we will be unable to identify the absolute value of the efficiency parameters for the remaining \((N + M - 1)\) prices.

Lovell and Sickles (1983) and Kumbhakar (1996) proposes that overall profit loss be computed as

\[
\ln \hat{\Pi}\{a_i = 1, \varphi_n = \varphi_m = 1; \forall \ n, m\} - \ln \hat{\Pi}^a. \tag{5.26}
\]

However, this measure of total profit loss due to AE and TE is not unique, since it is based on relative shadow prices rather than absolute shadow prices.

We now formally prove these points:

**Lemma 4.** Different values of \( \chi \) for the numeraire \( \tilde{\varphi}_{Mi} \) and hence for the unnormalized shadow profit functions \( \hat{\Pi}^{*T} \) and \( \hat{\Pi}^{*O} \) are observationally equivalent.

**Proof.** Since the normalized shadow profit function is homogeneous of degree zero in shadow prices, one can arbitrarily set \( \tilde{\varphi}_{Mi} = \chi \) for any positive value of \( \chi \) without changing normalized actual profits, \( \Pi^a, \) in (5.17). However, unnormalized shadow profits based on \( \chi, \tilde{\varphi}_{ni}, \) and \( \tilde{\varphi}_{mi} \) will vary proportionally with \( \chi. \) Thus, models with different values of \( \chi \) and unnormalized shadow profits are observationally equivalent.

**Theorem 4.** The general profit function decomposition requirement: the researcher must have information on the absolute level of technical efficiency for (at least) one firm and the (unnormalized) shadow price of (at least) one input or output for any shadow profit function in order to uniquely estimate total profit savings, to uniquely decompose these savings into their component parts, and to uniquely compute relative profit savings within or between firms.
Proof. Based on Lemma 4, in order to uniquely estimate profit savings, one must know the absolute shadow prices for each input and output. This requires that we know the value of $\chi$, which allows us to convert the estimated relative shadow prices, $\varphi_m$ and $\varphi_n$, into absolute efficiency parameters for $M + N - 1$ input and output prices. Following the proofs of Lemmas 2 and 3, without this information, any decomposition of profit savings and comparison of relative profit savings between firms will be non-unique.

The typical implicit assumption in the literature is that $\chi = 1$. In this case, $\varphi_m$ and $\varphi_n$ equal their respective absolute efficiency parameters and (5.26) yields unique estimates. However, if $\chi$ equals some other positive value, then $\tilde{\varphi}_m = \chi \varphi_m$ and $\tilde{\varphi}_n = \chi \varphi_n$.

There are clear parallels with the previous discussion regarding estimating and decomposing the total cost savings due to AE and TE using a shadow cost function. With both models, we must make an arbitrary normalization of one of the parameters. The result is that in both models we are only able to estimate ratios of efficiency parameters. In the shadow cost model, we estimated input price efficiency parameters which can be interpreted relative to a normalized input price efficiency parameter, while with the shadow profit model, we estimated ratios of input price and output price efficiency parameters relative to a normalized output price efficiency parameter. Without knowledge of the absolute value of these normalized parameters, we are unable to compute the total cost of inefficiency for the shadow cost model or the total lost profit for the shadow profit model, both of which must be computed in absolute (rather than relative) terms.

6. Empirical Examples of Econometric Estimation and Decomposition

In this section we demonstrate our theoretical results for shadow cost and distance systems. We omit a demonstration of the shadow profit system, since the problem is highly similar to these two systems.
To illustrate the problem of indeterminacy of the estimated cost savings computed using a shadow cost system, we estimate a shadow cost system using a panel of data for thirteen \((F = 13)\) U. S. airlines (accounting for over 90 percent of domestic air traffic) which are observed quarterly over the period 1970.1–1981.4 (so that \(T = 48\)). The airlines in our sample are: American (AA), Allegheny (AY), Braniff (BN), Continental (CO), Delta (DL), Eastern (EA), Frontier (FR), North Central (NC), Ozark (OZ), Piedmont (PI), Texas International (TI), United (UA), and Western (WA). Information on output and input prices and quantities was obtained from over 250 accounts from the CAB form-41. These accounts were aggregated to form a measure of output \((Q)\), capacity ton miles; four input measures, capital \((K)\), labor \((L)\), energy \((E)\), and materials \((M)\); and two output attributes, average stage length \((\text{asl})\) and quality of service \((\text{qua})\). For the details of the construction of the data set see Sickles, Good and Johnson (1986). The arguments of the cost function are input prices, output, the two output attributes, a set of seasonal dummies \((\text{WINTER} (w), \text{SPRING} (sp), \text{and SUMMER} (s))\), and time \((t)\).

Assuming that the explanatory variables in the cost system are exogenous, we estimate the shadow cost system using non-linear least squares and report the estimated coefficients and asymptotic standard errors in Table 2. For simplicity, we omit firm dummies. However, qualitative results are unchanged by including firm dummies and computing the instrumental variables estimator.

By comparing columns 3 and 6 of Table 2 with columns 4 and 7, it is clear that doubling the numeraire \(k_{Ni}\) from 1 to 2 results in a doubling of the estimated \(k_{ni}\) and their estimated asymptotic standard errors, but leaves all other estimated coefficients, their estimated asymptotic standard errors, and the log likelihood, \(\mathcal{L}\), unchanged.\(^7\) From (2.16)

\(^7\) Again, we italicize changes in estimates relative to the baseline.
it is clear that scaling all $k_{ni}$ by $\eta$ yields $-\ln \eta$ from the last term and $\ln \eta$ from the other terms in this equation, which exactly offset each other.

To see how these choices of numeraire affect estimated cost savings, examine Table 3 where we use the coefficients of Table 2 to compute estimated shadow costs and the percentage cost savings from achieving AE, defined in (2.25), for $\eta=0.5, 1,$ and $2$. A doubling of the value of the numeraire $k_{Ni}$ from 1 to 2 produces an approximately 20% increase in shadow costs and consequently causes all percent cost savings to become negative (Table 3, column 7). This nonsensical result obtains because the value chosen for the numeraire $k_{Ni}$ is arbitrary (and obviously incorrect, violating the Decomposition Requirement). Resetting the value of the numeraire $k_{Ni}$ to 0.5 makes all the estimated cost savings positive.

Moreover, cost savings for a given firm and ratios of percent cost savings for one firm relative to another depend on $\eta$ as shown in Tables 3 and 4. In fact, close examination of these tables reveals that rescaling of the $k_{ni}$ can even produce a different ordering of firms’ potential cost savings.

To illustrate the problem of indeterminacy of the estimated cost savings computed using a shadow distance system, we estimate a distance system using a panel of data for 43 privately-owned electric utilities operating in the U.S. over the 37 year period 1961-97, for a total of 1591 observations. Since technologies for nuclear, hydroelectric, and internal combustion differ from that of fossil fuel-based steam generation and because steam generation dominates total production by investor-owned utilities during the time period under investigation, we limit our analysis to this component of steam electric generation. The input variables are fuel (E), labor (L), and capital (K). We distinguish between residential (R) and industrial-commercial (I) output. See Atkinson and Primont (2002) for details.
Assuming that the explanatory variables in the distance system are exogenous, we estimate this system using non-linear least squares and report the estimated coefficients and their estimated asymptotic standard errors in Table 5. For simplicity, we omit the firm dummies and estimate the equations using non-linear least squares. Qualitative results are unchanged by including firm dummies and computing the instrumental variable estimator.

By comparing columns 2 and 3, one sees that doubling the value of the numeraire \( \kappa_{Ni} = \xi \) from 1 to 2 results in a doubling of the estimated \( \kappa_n \) but leaves all other coefficients except for the intercept unchanged. This occurs because the distance function is linearly homogeneous in shadow quantities, so that a doubling of the estimated \( \kappa_n \) means that the intercept is diminished by \( \ln(\xi) = \ln(2) \). That is, the estimated intercept plus \( \ln(\xi) \) is unchanged for any value of \( \xi \). Once again, the estimated log likelihood, \( L \) is unchanged.

From columns 3 and 4, one can see that the estimated asymptotic standard errors for the \( \kappa_n \) are doubled, while all other estimated standard errors are unchanged. Although not reported to conserve space, the estimated percentage changes in costs can be arbitrarily altered by changing the value of \( \xi \) and the ratios of percent cost savings for two firms varies with the specific normalized value of \( \xi \), and ultimately cost savings become negative when \( \xi \) is made large enough.

7. Are There Solutions?

A possible solution to this problem of underidentification is to add external information or constraints. One could normalize the value of the numeraire \( k_{Ni} \) to make the cost savings attributable to AE equal to zero for one firm and positive for all other firms, on the grounds that no firms could incur negative cost savings from achieving AE. This is analogous to normalizing the TE of the frontier firm to one and constraining TE measures.
to lie in the (0,1) interval for all other firms. The constrained estimates of cost savings would only be valid for relative comparisons among firms since they are clearly biased downward by the normalization.

Additional information for an input market might allow us to normalize the value of the numeraire $k_N$ to one if the market were fully “frictionless” and perfectly competitive. For example, a researcher could justify this assumption if he were willing to assume that the labor market had negligible adjustment costs, fully adjusted to market conditions rapidly, and was generally in equilibrium at all times. Then by assuming, for example, that the shadow price of labor equals its market price, one could identify cost savings from achieving AE. Although such assumptions may not be totally accurate, they may be good approximations in some cases.

Another approach, likely to be less common, is suitable to a distance function when one has strong prior information about the level of a shadow input quantity in relation to its actual quantity. One could normalize the numeraire $\kappa_N$ to $\xi$ such that the shadow level of input usage equals some multiple of the actual level. Perhaps this approach could be taken in certain settings where a labor contract ensures that labor is overutilized by a percentage that is known or easily estimated.

8. Conclusions

Many researchers have econometrically estimated AE and TE using stochastic shadow cost frontiers, shadow distance frontiers, and shadow profit frontiers. Frequently, the cost savings from eliminating both sources of inefficiency are reported in total and then decomposed. This paper has shown that the calculation and decomposition of total cost savings as formalized in Kumbhakar (1997) and used extensively in empirical applications over the last 25 years, is non-unique unless the decomposition requirement is satisfied. We
argue that the required additional information is typically not available. We also provide a similar result for the decomposition of profit savings obtained by estimating shadow profits.

It is well-known that technical efficiency is measured relative to the “best” firm; what is less well-understood is that the cost savings from achieving AE are measured using shadow costs which are identified only subject to a normalization of one shadow price or quantity. We further prove decomposition requirements which state that cost and profit decompositions are non-unique unless a proportional relationship between at least one shadow and actual price is known. Estimation of the shadow cost (profit) system is conditional on the normalized value for one shadow input (input or output) shadow price parameter. The same is true for decomposition of cost savings obtained from estimation of a shadow distance system where one shadow input quantity parameter must be normalized. However, when the normalized value is rescaled, estimates of shadow costs will change by the scale factor, but the new model is observationally equivalent to the original one. This implies that the total savings of efficient relative to actual cost or profit will change, as will their decompositions into portions due to achieving AE and TE.

For the shadow cost system, we provide empirical examples illustrating that large enough values of the numeraire shadow price parameter produce estimates of negative cost savings from achieving AE, clearly an impossible result. In fact, such results have been reported in the literature and ascribed to measurement error or other sources of sampling variation; however, they are, in fact, also consistent with an incorrect normalization of the numeraire inefficiency parameter.

Essentially the researcher has two options. First, he can establish a “best” firm as the benchmark against which relative cost savings (or profit gains) are measured. Second, he
can use additional, external information about the relationship between actual and shadow prices (for shadow cost and profit functions) and shadow quantities (for shadow distance functions) to satisfy the decomposition requirement. Regardless, performing a cost or profit decomposition based on the arbitrary choice of a numeraire value for a shadow price or quantity is not defensible. Choice of this numeraire affects the calculation of cost and profit savings and their decomposition. Believable prior information must be introduced to satisfy the decomposition requirement in order to obtain a unique decomposition that has policy relevance.
References


Table 1: Cost Decomposition using the Zieschang Method

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<td>$x_3^B$</td>
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Table 2: Cost System Estimated Coefficients

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Note: Asymptotic standard errors in parentheses. An asterisk denotes significance at the .05 level using a two-tailed test.
Table 3: Estimated Firm Percent Cost Savings From Allocative Efficiency

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Table 5: Distance System Estimated Coefficients

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<td>$\gamma_{E_t}$</td>
<td>-0.0521</td>
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<tr>
<td></td>
<td>(0.0126)*</td>
<td>(0.0126)*</td>
</tr>
<tr>
<td>$\gamma_{L_R}$</td>
<td>-0.0377</td>
<td>-0.0377</td>
</tr>
<tr>
<td></td>
<td>(0.0074)*</td>
<td>(0.0074)*</td>
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<tr>
<td>$\gamma_{E_R}$</td>
<td>0.0392</td>
<td>0.0392</td>
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<tr>
<td></td>
<td>(0.0131)*</td>
<td>(0.0131)*</td>
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<tr>
<td>$\gamma_{K_E}$</td>
<td>-0.0589</td>
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<td>(0.0092)*</td>
</tr>
<tr>
<td>$\gamma_{L_E}$</td>
<td>-0.0709</td>
<td>-0.0709</td>
</tr>
<tr>
<td></td>
<td>(0.0066)*</td>
<td>(0.0066)*</td>
</tr>
<tr>
<td>$\gamma_{K_L}$</td>
<td>-0.0184</td>
<td>-0.0184</td>
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<tr>
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<td>(0.0036)*</td>
<td>(0.0036)*</td>
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<tr>
<td>$\gamma_{L}$</td>
<td>0.1214</td>
<td>0.1214</td>
</tr>
<tr>
<td></td>
<td>(0.0075)*</td>
<td>(0.0075)*</td>
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<tr>
<td>$\gamma_{E}$</td>
<td>0.7070</td>
<td>0.7070</td>
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<tr>
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<td>(0.0172)*</td>
<td>(0.0172)*</td>
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<tr>
<td>$\gamma_{0}$</td>
<td>0.1030</td>
<td>-0.5901</td>
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<td>(0.1040)*</td>
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<tr>
<td>$\gamma_{R}$</td>
<td>-0.2704</td>
<td>-0.2704</td>
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<td>(0.1284)*</td>
<td>(0.1284)*</td>
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<tr>
<td>$\gamma_{R_2}$</td>
<td>-0.4398</td>
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<tr>
<td></td>
<td>(0.2030)*</td>
<td>(0.2030)*</td>
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<tr>
<td>$\gamma_{I}$</td>
<td>-0.6801</td>
<td>-0.6801</td>
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<tr>
<td></td>
<td>(0.0994)*</td>
<td>(0.0994)*</td>
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<tr>
<td>$\gamma_{I_2}$</td>
<td>-0.4046</td>
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</tr>
<tr>
<td></td>
<td>(0.1759)*</td>
<td>(0.1759)*</td>
</tr>
<tr>
<td>$\gamma_{R_I}$</td>
<td>0.4017</td>
<td>0.4017</td>
</tr>
<tr>
<td></td>
<td>(0.1853)*</td>
<td>(0.1853)*</td>
</tr>
<tr>
<td>$\gamma_{R_t}$</td>
<td>0.0037</td>
<td>0.0037</td>
</tr>
<tr>
<td></td>
<td>(0.0038)</td>
<td>(0.0038)</td>
</tr>
<tr>
<td>$\gamma_{t_t}$</td>
<td>-0.0028</td>
<td>-0.0028</td>
</tr>
<tr>
<td></td>
<td>(0.0032)</td>
<td>(0.0032)</td>
</tr>
<tr>
<td>$\gamma_{t}$</td>
<td>-0.0082</td>
<td>-0.0082</td>
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<tr>
<td></td>
<td>(0.0060)</td>
<td>(0.0060)</td>
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<tr>
<td>$\gamma_{t_t}$</td>
<td>0.0010</td>
<td>0.0010</td>
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<tr>
<td></td>
<td>(0.0003)*</td>
<td>(0.0003)*</td>
</tr>
<tr>
<td>$\ell$</td>
<td>471.6529</td>
<td>471.6529</td>
</tr>
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</table>

Note: Asymptotic standard errors in parentheses. An asterisk denotes significance at the 0.05 level using a two-tailed test.
Figure 1
Cost Decomposition

\[ L(y) \]

\[ A \text{ or } (x_1^A, x_2^A) \]

\[ B \text{ or } x = (x_1^B, x_2^B) \]

\[ C \]

\[ F \text{ or } x = (x_1^F, x_2^F) \]

\[ E \text{ or } (x_1^E, x_2^E) \]

\[ (p_1^B, p_2^B) \rightarrow (p_1^*, p_2^*) \rightarrow (p_1^A, p_2^A) \]