A General Approach to Valuing Commodity-Linked Bonds

Yufei Jin

Department of Agricultural and Applied Economics

Texas Tech University

and

Calum G. Turvey

Department of Agricultural, Food and Resource Economics

Rutgers University

Selected Paper

Presented at the Annual Meeting of the AAEA

Denver Colorado

July 29-Aug 2 2004

This paper is based on Yufei Jin's Ph.D. dissertation while he was a doctorate student in the Department of Agricultural Economics and Business, University of Guelph. Turvey is currently a professor in the Department of Agricultural, Food and Resource Economics, Rutgers University.

A General Approach to Valuing Commodity-Linked Bonds

ABSTRACT

The purpose of this paper is to develop a general approach to valuing commodity-linked bonds (CLBs) based on the Heath-Jarrow-Morton (HJM) framework. The model deals with four dimensions of uncertainty: prices of the underlying commodity, the value of firm that issues bonds, interest rates, and convenience yields. A mathematical formula for the price of a commodity-linked bond is derived. The previous results in Black and Scholes (1973), Merton (1973), Schwartz (1982), and Atta-Mensah (1992) can be obtained by specifying appropriate restrictions in the general model. Using similar assumptions, as found in Miura and Yamauchi (1998) and Carr (1987), more reasonable results can be obtained through the application of the present model.

A General Approach to Valuing Commodity-Linked Bonds

1. Introduction

The rapid expansion of derivative markets in the past 20 years has given rise to numerous new financial products that firms can use to mitigate business or financial risks. Among the class of new products are ones that link the payoff structure of various classes of debts to the value of an underlying commodity. Broadly defined, these Commodity-linked bonds (CLBs) are debt instruments with a payment structure that is contingent on the outcomes of one or more underlying commodities.

For businesses serving the food and agricultural industries, retained earnings and hence firm value is largely dependant upon volatile market forces in the underlying commodities purchased for further processing or sold as final product into the process market. In order to preserve ownership structure and to reduce ownership dilution corporate financial strategy often relies on debt markets to finance growth and investment. However, the debt carrying capacity of most firms is limited or constrained by associated business risks. If agricultural commodity markets largely govern business risks then in general the more volatility in the prices of the underlying commodity the greater will be the financial risks associated with increased use of debt. Commodity-linked bonds (or perhaps more generally debt) have the potential to mitigate such problems. A food-based firm that faces increasing downside risks as prices fall can secure debt repayment by issuing bonds or otherwise acquiring debt by directly linking to that debt a put option on the underlying commodity. Likewise, a firm that faces the risk of rising commodity prices can mitigate financial risk by linking to its debt a call option on the underlying commodity. In either case the payoff from the option is used to secure the debt.

Commodity-Linked Bonds are not new. O'Hara (1984), reports the use of commodity–linked bonds as far back as 1863, when the Confederate States of America (CSA) issued war bonds payable in bales of Cotton. More recently, there has been a resurgence of interest in commodity-linked bonds in various industrial sectors. In 1980, the Sunshine Mining Company issued bonds payable in Ounces of Silver. Both Mexico and the British Oil and Gas Corporation have issued petroleum bonds, and the Reagan administration had proposed issuing oil-indexed bonds to finance the strategic Oil Reserve. More examples can be found in Atta-Mensah's dissertation (1992) and an agricultural example can be found in Jin and Turvey (2002).

While our paper provides the conceptual basis for a wide range of commodity-linked financial products its focus is on a corporate entity of sufficient scale that it can issue bonds. The structural difference between a CLB and a conventional bond is that the nominal return of the conventional bond held to maturity is known with certainty, with an uncertain real return due to inflation, while both the nominal and real monetary returns are unknown for commodity-linked bonds (Atta-Mensah, 1992).

There are two types of commodity-linked bonds: the forward type and the option type. With commodity bonds of the forward type, the coupon and/or principal payment to the bondholder are linearly related to the price of a stated amount of the reference commodity. If, upon maturity, the option part is in-the-money, then the face value of the bond is reduced accordingly. These CLB's can either be issued at a discount and/or with increased coupon rates to compensate for the transference of business risk to the bondholder. With commodity bonds of the option type, the coupon payments are similar to that of a conventional bond, but, upon maturity, the bondholder receives the face value plus an option to buy or sell a predetermined quantity of the commodity at a specified price. If at maturation the option part is in the money then the bond holder will receive the full face value of the bond plus the intrinsic value of the linked option. These CLB's will likely be issued at a premium and/or with lower coupon rates to reflect the expectation of additional compensation from the option component.

How CLBs are priced to equilibrium is a significant problem. From the commodity side, risks arise from the diffusion and volatility of the underlying commodity price as well convenience yield risks that might arise if the commodity is storable. From the bond side, uncertainty arises from interest rate risk and the potential of default and/or bankruptcy risk. The purpose of this paper is to develop a general approach for pricing Commodity-Linked Bonds (CLB) in the presence of stochastic commodity prices, interest rates, convenience yields, and asset values (firm values). It is assumed that CLB holders receive the coupon payment at a constant instantaneous coupon rate of c plus the final payment. Assuming normality of continuously compounded forward interest rates and convenience yields, log-normality of the spot price of the underlying commodity, and log-normality of the firm value, we obtain the mathematical formula of the price of CLBs using the standard method, of, e.g., Harrison and Kreps (1979) and Harrison and Pliska (1981). This four-factor model is the main theoretical contribution of this paper, but the paper also makes a contribution in that the final model can be applied to a wide range of risk management strategies in corporate and agricultural finance. In fact, the generality of our model is revealed by showing that the results obtained by Schwartz (1982), Atta-

Mensah (1992), Carr (1987), Miura and Yamauchi (1998) and Jin and Turvey (2002) are special cases of this model.

This paper is organized as follows. Section 2 reviews the previous research works. To set up our model, the interest rate and the convenience yield are discussed in section 3. The general model is developed in section 4. Section 5 specializes our model to the Gaussian case and obtains the mathematical formula of the price of CLBs. In Section 6, various special cases of our model and related problems are discussed. Section 7 concludes the paper.

2. Literature Review

The standard model for pricing commodity-linked securities uses the option pricing framework as pioneered by Black and Scholes (1973) and extended by Merton (1973) and Cox and Ross (1976). Schwartz (1982) provides a general framework for valuing CLBs. He considers the underlying commodity price risk, default risk, and interest rate risk and provides a second–order partial differential equation in four variables that governs the value of CLBs at any point in time. Schwartz states that the solution to the general problem, subject to certain boundary conditions, is difficult even by numerical methods, and he only provides some closed form solutions in special cases. In his paper there is no discussion about convenience yields. Carr (1987) derives a closed form pricing formula for a commodity-linked bond where the bond prices follows a third order geometric Brownian motion (lognormal) without referring to the interest rate process. His formula encompasses Schwartz's solution as a special case. However he leaves out the convenience yield from his model.

Gibson and Schwartz (1990) consider stochastic convenience yields for the valuation of commodity derivatives. Assuming that the price of the underlying commodity has a lognormal stationary distribution and net marginal convenience yields follow the mean reverting pattern, they derive the partial differential equation for the price of commodity derivatives defined as functions of the underling commodity spot price and the net marginal convenience yields. They also estimate the parameters for the behavior of the net marginal convenience yield from market data, and then calculate numerically the futures price of the commodity. Schwartz (1997) extends this model by introducing a third stochastic factor, the instantaneous interest rates. Hilliard and Reis (1998) extend this three factors model by introducing jumps in the spot price of the commodity and by using the term structure of interest rate to eliminate the market price of interest rate risk in their fundamental price equation. Milterson and Schwartz (1998) develop a new arbitrage model that includes three factors: the spot price of the commodity, forward interest rates and convenience yields. They address issues about the forward interest rates and convenience yield based on the Heath-Jarrow-Morton (HJM) framework and obtain closed form solutions for options on commodity futures as well as commodity forwards in the Gaussian case. None of the above models actually considers the default of the issuing firms as the commodity contingent claims mature.

Atta-Mensah (1992) established a general model for pricing CLBs, which includes four factors: the spot price of the underlying commodity, firm value, interest rates, and convenience yields. He follows Schwartz (1982) to derive a second-order partial differential equation as well as the boundary conditions. Recently, Miura and Yamauchi (1998) assume that both the spot price of the commodity and firm value follows geometric Brownian motions and that the net marginal convenience yield and interest rate follow mean-reverting processes and then price CLBs. They derive the partial differential equation, which must be satisfied by the bond and obtain the mathematical formula for the price of CLBs. Evnine (1983) extends the Cox, Ross, and Rubinstein option pricing model to incorporate an option on two or more stocks. Rajan (1991) applies Evnine's model to CLBs, prices CLBs in the presence of default risk and commodity price risk, and then compares his results to Schwartz's results.

In this paper, a new general approach to value CLBs is proposed. Based on the HJM framework for compounded forward interest rates and convenience yields and using similar strategies and skills adopted by Miltersen and Schwartz (1998), we obtain the mathematical formula of the price of CLBs. In the Gaussian case, the closed form solution is derived and the relationship between our model and previous models is also discussed. Generally, our model can be considered as an extension of the results from Schwartz (1982) and Miltersen and Schwartz (1998).

3. Interest Rates and Convenience Yields

In recent years the no-arbitrage model has been used to deal with the interest rate when pricing financial products (Hull, 2000). Ho and Lee (1986) present a no-arbitrage model and Hull and White (1990) extend the Ho and Lee model. Heath, Jarrow, and Morton (HJM) (1992) develop a general no-arbitrage model based on several factors and derive the relationship between the drift and standard deviation of an instantaneous forward rate.

Kaldor (1939) first gives the definition for the convenience yield of a commodity in the economic literature, and then offers an explanation of the relationship between the spot and future prices of a commodity. Later many scholars such as Working (1948), Brennan (1958), and Frechette (1997) reached the conclusion that the valuation equation for derivatives that are indexed to a commodity must take account of the convenience yield of the commodity linked to it. Fama and French (1987) provide evidence that the marginal convenience yield varies seasonally for most agricultural and animal products, and furthermore they (1988) find evidence of a mean-reverting process for metals' convenience yield using futures data from the London Metals Exchanges (LME). Gibson and Schwartz (1990) find that a constant convenience yield assumption did not work well for pricing oil indexed bonds and argued that convenience yield needs to be explicitly considered in modeling any meaningful valuation model. For pricing commodity–linked bonds, Ingersoll (1982) points out that the convenience yield should be considered when pricing CLBs. Atta-Mensah (1992) assumes that the convenience yields follow the Brownian motion in his model. Miura and Yamauch (1998) postulate that the convenience yields follow a mean-reverting process to value CLBs.

Miltersen and Schwartz (1998) use the HJM approach for interest rates and convenience yields to value the options on commodity futures. In the current paper, the HJM approach for interest rates and convenience yields is also used to price CLBs.

4. A General Approach to Valuing Commodity-Linked Bonds

In general, the bond under consideration is a coupon bond, with a payoff linked to the value of some underlying storage or non-storage commodity and the value of the firm issuing these bonds. The value of the firm represents default risk which deals with the probability that the corporation or government issuing a bond will fail to either make interest payments or redeem the bond at parity. Share prices for publicly traded stocks can be used to represent these risks. In the current model, four factors for pricing CLBs are considered: 1) the value of the firm issuing bonds, 2) the spot price of the underlying commodity, 3) interest rates, and 4) convenience yields. The following notations are used:

$S(t)$: the spot price of the underlying commodity,

$V(t)$: the value of the firm issuing bonds,

$P(t,T)$: the zero-coupon bond price at time t for all maturities T,

$F(t,T)$: forward prices of the underlying commodity,

$G(t,T)$: futures prices of the underlying commodity, for all maturities, $T \geq t$,

$f(t,s)$: continuously compounded forward interest rates, $s \geq t$, for all $t \geq 0$,

$\varepsilon(t,s)$: continuously compounded future convenience yields, $s \geq t$, for all $t \geq 0$,

$\delta(t)$: the spot convenience yields, and

$r(t)$: the spot interest rates.

Here $\delta(t)$ follow in Miltersen and Schwartz's (1998) definition as the flow of services that accrues to the holder of the physical commodity, but not to the owner of a contract for future delivery (per unit time and per unit of the commodity). That is, the instantaneous spot convenience yield includes (minus) the instantaneous cost of carry.

Following Schwartz (1997) and Miltersen and Schwartz (1998), define a filtered probability space, $(\Omega, \Gamma, \{\Gamma_t\}_{t\geq 0}, P)$, and four adapted stochastic processes fulfilling sufficient integrability conditions, such that the expectations used in the analysis are well defined. The four processes are the spot price of the underlying commodity, S, the value of the firm issuing the bonds, V, the spot convenience yields, δ, and the spot interest rates, *r*. Let $E[.,|\Gamma_t]$ denote the conditional expectation under an equivalent martingale measure conditional on the information at date *t*, Γ_t. Using standard arguments from Cox, Ingersoll, and Ross (1981), we have

(1a) $P(t,T)=E[e^{-\int_t^T r(s)ds}|\Gamma_t]$,

(1b) $P(T,T)=1$,

(2) $S(t)=E[e^{-\int_t^T r(s)ds}\,e^{\int_t^T \delta(s)ds}\,S(T)|\Gamma_t]$,

(3) $G(t,T)=E[S(T)|\Gamma_t]$,

for any given date *t* and future date $T \geq t$.

Based on the HJM (1992) approach, Miltersen and Schwartz (1998) have provided definitions for forward interest rates, $f(t,s)$ and future convenience yields, $\varepsilon(t,s)$. They define the continuously compounded forward interest rates, $f(t,s)$, such that the zero-coupon bond prices are

(4) $P(t,T)=E[e^{-\int_t^T r(s)ds}|\Gamma_t]=e^{-\int_t^T f(t,s)ds}$.

Similarly, they define the continuously compounded future convenience yields, $\varepsilon(t,s)$, such that the futures prices are

(5) $G(t,T)=\dfrac{S(t)}{P(t,T)}e^{-\int_t^T \varepsilon(t,s)ds}=S(t)\,e^{\int_t^T (f(t,s)-\varepsilon(t,s))ds}$

Miltersen and Schwartz (1998) call $f(t,s)$ the term structure of forward interest rates and $\varepsilon(t,s)$ the term structure of future convenience yields. HJM (1992) establish the connection between the forward interest rates and the spot interest rates as follows,

(6) $f(t,t)=r(t)$,

for all t. A similar connection between the future convenience yield and the spot convenience yields as established by Miltersen and Schwartz (1998) is as follows,

(7) $\varepsilon(t,t) = \delta(t)$,

for all t. Since an integral with the same number, t, as lower and upper limits is zero,

$$\int_t^t f(t,s)ds = \int_t^t \varepsilon(t,s)ds = 0 ,$$

then

(8) $G(t,t) = S(t)$.

Which simply states that he expected futures price at t is equal to the expected spot price as t.

4.1 A General Approach

This section develops a model for pricing CLBs that is closely related to Miltersen and Schwartz's model (1998), for pricing options on commodity futures with stochastic term structures of convenience yields and interest rates.

The following assumptions are postulated:

(A1) Assets are traded in a frictionless or perfect market where there are no taxes, transaction costs or short sale restrictions, and all assets are perfectly divisible;

(A2) Trading of assets is done continuously; and

(A3) The value of the firm that issues the bond, the price of the referenced commodity, the continuously compounded forward interest rates and the continuously compounded future convenience yields follow a continuous time diffusion process.

The spot price of the reference commodity and the value of the firm issuing bonds are modeled explicitly as

(9) $S(t) = S(0) + \int_0^t S(u)\mu_S(u)du + \int_0^t S(u)\sigma_S(u) \bullet dW_u$

and

(10) $V(t) = V(0) + \int_0^t V(u)\mu_V(u)du + \int_0^t V(u)\sigma_V(u) \bullet dW_u$

respectively. Equations (9) and (10) are geometric, continuous time representations of a classical Brownian motion process with Wiener processes described by dW. The Wiener process describes a classical Markovian random walk and is also consistent with a martingale. The assumptions regarding the continuously compounded forward interest rates and future convenience yields are obtained from the HJM (1992) analysis and Miltersen and Schwartz's (1998) discussion, respectively. As it turns out, in the HJM analysis it is most convenient to model the price fluctuations of the zero-coupon price by explicitly writing up the stochastic differential equation (SDE) for the continuously compounded forward interest rates, *f*. That is,

(11) $f(t,s) = f(0,s) + \int_0^t \mu_f(u,s)du + \int_0^t \sigma_f(u,s) \bullet dW_u$.

This assertion is true for the price fluctuations of the futures prices of the commodity, as Miltersen and Schwartz point out (1998). Likewise, the SDE for the continuously compounded future convenience yields, ε is,

(12) $\varepsilon(t,s) = \varepsilon(0,s) + \int_0^t \mu_\varepsilon(u,s)du + \int_0^t \sigma_\varepsilon(u,s) \bullet dW_u$.

Where W is a standard four-dimensional Wiener process, "•" denotes the standard Euclidean inner product of R^4, and the corresponding norm is defined as $\|x\|^2 = x \bullet x$ for any $x \in R^4$.

The possible correlation among the four processes comes via the specification of the diffusion terms, because all four SDEs used the same vector Wiener process, *W*. By now, we have not discussed how to specify the drift terms and the diffusion terms, however they must satisfy certain technical conditions, such that strong solutions of the stated SDEs exist. Usually there are two ways to specify the drift terms. One way is to assume a risk neutral world as in Cox and Ross (1976). This means that in the risk-neutral world the return on any traded investment is simply *r(t)*. Hull (2000) discusses this topic in detail in his book. Wilmott (1998) derives the relationship between the forward rate drift and volatility for the one variable in the risk-neutral world. Hilliard and Reis (1998) point out that the drift of the commodity spot price process is simply the $r(t)-\delta(t)$ in the risk-neutral world. Another approach is to invoke standard no-arbitrage restrictions. This implies that the drift of the spot commodity price process is determined as

$$\begin{aligned}\mu_S(t) &= r(t)-\delta(t)\\ &= f(t,t)-\varepsilon(t,t)\end{aligned}$$

under an equivalent martingale measure (Miltersen and Schwartz, 1998). HJM (1992) use the no-arbitrage restriction to derive the drift of the forward interest rates process under an equivalent martingale measure. Using a similar analysis, Miltersen and Schwartz (1998) derive the drift of the future convenience yield process under an equivalent martingale measure. Accordingly the drift terms of the four processes are specified as follows

$$\text{(13)} \quad \mu_S(t)=r(t)-\delta(t),$$

$$\text{(14)} \quad \mu_V(t)=r(t),$$

(15) $\mu_f(t,s) = \sigma_f(t,s) \bullet (\int_t^s \sigma_f(t,v)dv)$,

$$\begin{aligned}(16)\quad &\mu_\varepsilon(t,s) = \sigma_f(t,s) \bullet (\int_t^s \sigma_\varepsilon(t,v)dv) \\ &+ (\sigma_f(t,s) - \sigma_\varepsilon(t,s)) \bullet (\sigma_S(t) + \int_t^s (\sigma_f(t,v) - \sigma_\varepsilon(t,v))dv)\end{aligned}$$

Note that the firm value (14) can be treated as a traded security and therefore the expected return from the traded investment will be the risk-free rate in the risk-neutral world. The drift terms are similar to the Miltersen and Schwartz (1998) specification.

4.2 The Payoff to Commodity-Linked Bonds

The payoff to CLBs of the option type is comprised of two parts[1]: the coupon rates and final payment. For the CLB with a linked call option, the promised payment on the bond at maturity is equivalent to the face value of the bond (*F*) for sure, plus an option to buy the reference commodity at a specified exercise price (*K*). In the case of a default bond, holders can costlessly take over the firm, so the final payment at maturity date,T, will be[2]:

(17) $B(.,T) = B(S,V,r,\delta,T) = \min[V(T), F + \max(0, S(T) - K)]$

Here, the coupon rates are assumed to be a constant, *c*.

[1] The CLB of the option type will generally be sold at a premium relative to a bond without an option rider. For bonds of the Forward Type in which the final payoff is related to the performance of an underlying commodity, the right hand term in (17) would be written as F-max(0, S(T)-K). Jin and Turvey (2002) using a simpler model than that derived in this paper illustrate the difference between CLBs (agricultural loans in their case) issued at premiums (the option type) or discounts (the forward type).
[2] The max term in (17) is for a call option with a strike price (K). This is the problem solved in this paper. However, the model can also be solved for a put option (the right to sell) by substituting max(0,K – S(T)) into equation (17). Also we have discussed payoffs in terms of units of the underlying commodity. Clearly, in the absence of transactions costs, a cash-settled option would be equivalent so that the final payoff would be the face value plus the intrinsic value of the option.

4.3 The Price of Commodity-Linked Bonds

Using standard methods as described in Harrison and Kreps (1979) and Harrison and Pliska (1981) the price of CLBs equals the discounted expected value of the payment at maturity and the cumulative payment from the coupons, that is

$$
(18)\quad \begin{aligned} B(.,t) &= E[\int_t^T e^{-\int_t^v f(s,s)ds} c dv \,|\, \Gamma_t] + E[e^{-\int_t^T f(s,s)ds} B(.,T) \,|\, \Gamma_t] \\ &= E[\int_t^T e^{-\int_t^v f(s,s)ds} c dv \,|\, \Gamma_t] + E[e^{-\int_t^T f(s,s)ds} \min[V(T), F + \max(0, S(T) - K)] \,|\, \Gamma_t] \end{aligned}
$$

Here $0 \le t \le T$.

Equation (18) is a very common approach to pricing commodity derivatives. For example, Chen (1996) uses this strategy to price an interest rate derivative which involved three factors; Miltersen and Schwartz (1998) use it to value the commodity options in the presence of the stochastic processes of the interest rates and convenience yields. More examples can be found in Duffie (1992) and Wilmott (1998).

Defines *TE* as the total value of the equity of the firm issuing bonds, the following relationship holds at any time t,

$$
(19)\quad V(.,t) = B(.,t) + TE(.,t)
$$

and the corresponding final value of the equity of the firm issuing bonds at maturity can be expressed as follows:

$$
(20)\quad TE(.,T) = \begin{cases} \max[0, V(T)] & \text{for } S(T) \le K \\ \max[0, V(T) - S(T) + K - F] & \text{for } S(T) > K \end{cases}
$$

Given the above relationship, we can price CLBs in some cases using the following formula:

$$
(21)\quad B(.,t) = V(.,t) - E[e^{-\int_t^T f(s,s)ds} TE(.,T) \,|\, \Gamma_t] .
$$

5. The Gaussian Case

In this section, we assume Gaussian continuously compounded forward interest rates,Gaussian continuously compounded future convenience yields, the log-Gaussian value of the firm issuing the bonds and log-Gaussian spot commodity prices. These assumptions imply that all drift terms (the *μ's*) and the diffusion terms (the *σ's*) of the four processes are deterministic functions of the time parameters. We show that the additional assumptions lead to a closed form mathematical pricing formula for CLBs.

In appendix A, we derive the price of CLBs from equation (18). The closed-form mathematical formula for the price of CLBs with constant coupon rates c at $t=0$ can be written as follows:

$$
\begin{aligned}
B(.,0) &= E[c\int_0^T A_1^* e^{-X_1^*} dv] + E[A_1 e^{-X_1} \min[A_2 e^{X_2}, F + \max(0, A_3 e^{X_3} - K]] \\
&= c\int_0^T [A_1^* \int_{-\infty}^{+\infty} e^{-x_1^*} f_v(x_1^*) dx_1^*] dv \qquad (22) \\
&\quad + \iiint [A_1 e^{-x_1} \min[A_2 e^{x_2}, F + \max(0, A_3 e^{x_3} - K)]] f(x_1, x_2, x_3) dx_1 dx_2 dx_3
\end{aligned}
$$

With all notations explained in Appendix A. Moreover the bond value can be segregated into the two components as follows. The first component in (22) represents the coupon rates with stochastic interest rates and second term in (22) is the discounted value of final payoff with the influence of default risk, the underlying commodity, interest rates, and convenience yields[3].

The formula in (22) is a general formula since it incorporates all of the factors discussed in this paper. From it a number of special cases can be derived to suit particular needs, for example when there are no coupons, or there is no risk of bankruptcy, or no

[3] Miura and Yamauchi (1998) examine a similar model under an equivalent martingale measure. We have derived their results under our assumptions and can make those results available on request. Under the

convenience yield. In the next section we examine some of these special cases. In particular we examine special cases from (22) that lead to identical or similar solutions to other models found in the literature.

6. Special Cases

The four-sector model provided in (22) is a general model for which many other models can be derived. This section, demonstrates that this model includes, as special cases, many of the previous results. In a few cases, results differ from previous studies and in these instances we provides the appropriate comparisons.

6.1 No default risk

For many firms the total debt relative to equity is so small that the probability of default is negligible[4]. Here, the investor acknowledges that the payoff will be made with certainty, but because of the option component the amount of the payoff is uncertain. This subsection investigates the case in which there is no default risk. This can simply be obtained by setting $V(T) = +\infty$, which implies that $A_2 e^{X_2}$ is infinite. Without this risk the bond value will rise. With this assumption the price of CLBs can be expressed at *t=0* as follows:

$$B(.,0) = c\int_0^T P(0,v)dv + FP(0,T)$$

(23)
$$+G^*(0,T)N\left(\frac{\ln\frac{G^*(0,T)}{P(0,T)K} + \frac{1}{2}\sigma_3^2}{\sigma_3}\right) - KP(0,T)N\left(\frac{\ln\frac{G^*(0,T)}{P(0,T)K} - \frac{1}{2}\sigma_3^2}{\sigma_3}\right)$$

where

Martingale measure the formula is comprised of four basic integrals representing the value from the coupon, the value from the firm, the face value of the bond, and stochastic commodity prices.

[4] Schwartz (1982) examines several variants of the default model. In particular he examines a model with no coupons or other payments to shareholders or bondholders and no convenience yield risk. We have also derived Schwartz's model under these assumptions from (17) and can provide the derivation on request. Equation (17) is also similar to Carr's (1987) results when there is no convenience yield.

$$
\begin{aligned}
G^*(0,T) &= E[e^{-\int_0^T f(s,s)ds} S(T)] \\
&= P(0,T)G(0,T)e^{-\sigma_{13}}
\end{aligned}
.
$$

Substituting $G^*(0,T)$ into (23) gives us a closed-form formula for CLBs, expressed as:

$$
\begin{aligned}
B(.,0) &= c\int_0^T P(0,v)dv + FP(0,T) \\
&+ P(0,T)[G(0,T)e^{-\sigma_{13}} N\left(\frac{\ln\frac{G(0,T)}{K} - \sigma_{13} + \frac{1}{2}\sigma_3^2}{\sigma_3}\right) \\
&- KN\left(\frac{\ln\frac{G(0,T)}{K} - \sigma_{13} - \frac{1}{2}\sigma_3^2}{\sigma_3}\right)]
\end{aligned}
\quad (24)
$$

The above formula provides a closed-form expression for the price of CLBs with constant coupon rate *c* and with exercise price *K* written on the commodity futures price with maturity *t=T* at time *t=0* under no default risk condition. Expression (24) simply says that the value of the commodity bond is equal to the discounted value of the future coupon payments and face value of the bond plus a similar "call option" to buy the reference commodity bundle at the predetermined exercise price *K*.

If more restrictions are put in our model, many previous results can be obtained from our model.

Case A: Uncertainty Commodity Price

Many nonstorable agricultural commodities such as livestock or pork do not have convenience yields that are of significant economic interest. In this case we consider a situation in which we have no convenience yields, and the interest and coupon payment rates are constant. The only source of uncertainty is therefore the forward commodity price. Incorporating these assumptions into (22) the price of CLB is

$$
(25)\qquad \begin{aligned} B(.,0) = {} & \frac{c}{r}(1-e^{-rT}) + Fe^{-rT} + S(0)N\left(\frac{\ln\frac{S(0)}{K} + rT + \frac{1}{2}\sigma_3^2}{\sigma_3}\right) \\ & - Ke^{-rT}N\left(\frac{\ln\frac{S(0)}{K} + rT - \frac{1}{2}\sigma_3^2}{\sigma_3}\right) \end{aligned}.
$$

This expression is exactly the same as the results obtained by Schwartz (1982) and Atta-Mensah (1992). This model is also similar to the one used in Jin and Turvey (2002). In their model, F is the principal on an agricultural loan with c=0. To reduce this further, note that if F = 0, c = 0 (no bond), and interest rate =r (constant), then P (0, T) = exp(-r*T) and (28) reduces to the standard Black (1976) model for pricing options on futures.

Case B: Uncertainty Commodity Price and Deterministic Convenience yield

In this case all assumptions in case A are maintained except that the convenience yield is constant (δ). The CLB price in this case is as follows:

$$
(26)\qquad \begin{aligned} B(.,0) = {} & \frac{c}{r}(1-e^{-rT}) + Fe^{-rT} + S(0)e^{-\delta T}N\left(\frac{\ln\frac{e^{-\delta T}S(0)}{K} + rT + \frac{1}{2}\sigma_3^2}{\sigma_3}\right) \\ & - Ke^{-rT}N\left(\frac{\ln\frac{e^{-\delta T}S(0)}{K} + rT - \frac{1}{2}\sigma_3^2}{\sigma_3}\right) \end{aligned}.
$$

Which is equivalent to that of Atta-Mensah (1992) after solving his partial differential equation. One can see immediately that if δ=0, the result in A is obtained.

Case C: Uncertainty in Commodity Prices with Stochastic Convenience yield

The assumptions in this case are that the interest rate is a constant and the convenience yield is a stochastic process as defined by G(0,T). With these assumptions, the price of the CLB is equal to

$$B(.,0) = \frac{c}{r}(1 - e^{-rT}) + Fe^{-rT} + S(0)G(0,T)N\left(\frac{\ln\frac{G(0,T)}{K} + \frac{1}{2}\sigma_3^2}{\sigma_3}\right) - Ke^{-rT}N\left(\frac{\ln\frac{G(0,T)}{K} - \frac{1}{2}\sigma_3^2}{\sigma_3}\right) \quad (27)$$

Atta-Mensah (1992) uses the futures price to construct the portfolio and then solve the partial differential equation to obtain the same results.

Case D: Uncertain Commodity Price with Stochastic Interest Rate

In this case we assume that the convenience yield is zero and the interest rate is a stochastic process. Under these assumptions, the price of CLBs is

$$B(.,0) = c\int_0^T P(0,v)dv + FP(0,T) + S(0)N\left(\frac{\ln\frac{S(0)}{P(0,T)K} + \frac{1}{2}\sigma_3^2}{\sigma_3}\right) - KP(0,T)N\left(\frac{\ln\frac{S(0)}{P(0,T)K} - \frac{1}{2}\sigma_3^2}{\sigma_3}\right) \quad (28)$$

This expression is very similar to the result obtained by Schwartz (1982). The key difference is in the appearance of σ_3 which results from different assumptions about interest rates. Note that our result also includes the contribution from coupon rates which is not considered in Schwartz's (1982) formula.

7. Summary and Conclusion

Recent developments in options pricing have led to a plethora of new financial products for agricultural and food firms. For the most part these products have been developed and implemented separately from the firm's capital structure. A new class of financial products called Commodity-Linked Bonds (CLB's) explicitly link the payoff structure of a commodity option to the redemption of a bond, and this offers food and agricultural firms with a new opportunity to balance business and financial risks. With the exception of Jin and Turvey (2002) we are unaware of any previous attempts to consider CLB's in the context of agriculture and food especially at the corporate level. This paper draws on previous models presented in the finance literature. It reveals the complexity of the CLB problem as a means to finance new investment. In this paper, a general model for pricing commodity-linked bonds was developed and a mathematical formula of the bond price based on the HJM framework is provided. The model includes firm risk, commodity price risk, convenience yield risk and interest rate risk. This general model is an extension of models developed by Schwartz (1982) and Atta-Mensah (1992). Specifically, the closed form solutions of commodity-linked bonds obtained by them can be derived as special cases of the general model developed here. For the cases studied by Carr (1987) and Miura and Yamauchi (1998), similar closed form solutions of commodity-linked bonds were derived through the general model. Furthermore, since the model is based upon the HJM framework, many kinds of stochastic processes (at least Wiener process and mean-reverting process), which are satisfied by, interest rates and convenience yields can be transferred to this framework.

REFERENCES

[1] Atta-Mensah, Joseph, “The valuation of commodity-linked bonds.” *Unpublished Ph.D. Thesis, Simon Fraser University*, June 1992.

[2] Black, F., “The pricing of commodity contracts.” *Journal of Financial Economics*, September 1976, pp.167-179.

[3] Black, F. and M. Scholes, “The pricing of options and corporate liabilities.” *Journal of Political Economy*, Vol. 81 1973, pp. 637-659.

[4] Brennan, M. J., “The supply of storage.” *American Economic Review*, 1958.

[5] Carr, P., “A note on the pricing of commodity-linked bonds.” *Journal of Finance*, Vol. 42 1987, pp.1071-1076.

[6] Chen, L., *Interest Rate Dynamics, Derivatives Pricing, and Risk Management*, Springer, 1996.

[7] Cox, J., J. Ingersoll, and S. Ross, “The relation between forward prices and futures prices.”, *Journal of Financial Economics*, Vol. 9 1981, pp.321-46.

[8] Cox, J., J. Ingersoll, and S. Ross, “A theory of the term structure of interest rates.” *Econometrica*, Vol. 53 1985, pp.385-408.

[9] Cox, J., and S. A. Ross, “The valuation of option for alternative stochastic processes.” *Journal of Financial Economics*, Vol. 3 1976

[10] Cox, J., S. A. Ross and M. Rubinstein, “Option pricing: A simplified approach.” *Journal of Financial Economics*, Vol. 7 1979, pp.229-263.

[11] Duffie, D., *Dynamic Asset Pricing Theory*, Princeton University Press, 1992.

[12] Fama, E. and K. French, “Commodity futures prices: Some evidence on forecast power, premiums, and theory of storage.” *Journal of Business*, Vol. 60 1987, pp.50-74.

[13] Frechette, D. L., “The dynamics of convenience and the Brazilian soybean boom.”, *American Journal of Agricultural Economics*, Vol. 79 1997, pp1108-18.

[14] Gibson, R. and E. Schwartz, “Stochastic convenience yield and the pricing of oil contingent claims.” *Journal of Finance*, Vol. 45 July 1990, pp.959-976.

[15] Harrison, J. M. and D. M. Kreps, “Martingales and arbitrage in multi-period securities.” *Journal of Economics Theory*, Vol. 20 1979, pp.381-408.

[16] -------------------- and S. R. Pliska, “Martingales and stochastic integrals in the theory of continuous trading.”, *Stochastic Processes and their Applications*, Vol. 11 1981, pp.215-60.

[17] Heath, D., R. Jarrow and A. Morton, “Bond pricing and the term structure of interest rates: a new methodology.” Econometrica, Vol. 60, 1992, pp.77-105.

[18] Hilliard, J. E. and J. Reis, “Valuation of commodity futures and options under stochastic convenience yields, interest rates, and jump diffusions in the spot.” *Journal of Financial and Quantitative Analysis*, Vol. 33 1998, pp. 61-86.

[19] Hilliard, J. E. and J. Reis, “Jump processes in commodity futures prices and options pricing.” *American Journal of Agricultural Economics*, Vol. 81 1999, pp.273-86.

[20] Hull, John, “Options, Futures and other Derivative Securities.” *Prentice-Hall*, 1989, 1992, 2000.

[21]------------, and A. White, “Pricing interest-rate derivative securities.” *Review of Financial Studies*, Vol. 3 1990, pp.573-92.

[22] Ingersoll, J., “The pricing of commodity-linked bonds, Discussion.” *Journal of Finance*, Vol. 37 1982, pp.540-41.

[23] Kaldor, N., “Speculation and economic stability.” *Review of Economic Studies*, Vol. 7 1939, pp.1-27.

[24] Merton, R. C., “Theory of rational option pricing.” *Bell Journal of Economics and Management*, Vol. 4 1973, pp.141-183.

[25] ----------------, Continuous-time finance, *Blackwell Publishers*, 1990.

[26] Miltersen, K. R. and E. S. Schwartz, “Pricing of options on commodity futures with stochastic term structures of convenience yields and interest rates.”, *Journal of Financial and Quantitative Analysis*, Vol. 33 1998, pp. 33-59.

[27] Miura, R. and H. Yamauchi, “The pricing formula for commodity-linked bonds with stochastic convenience yields and default risk.”, *Asia-Pacific Financial Markets*, Vol. 5 1998, pp.129-58.

[28] O’Hara, M., “Commodity bonds and consumption risks.” *Journal of Finance*, Vol. 39 1984, pp.193-206.

[29] Rajan, R., “Pricing commodity bonds using Binomial option pricing.” *Commodity risk management and finance,* Priovolos, T. and R. C. Duncan (Editors), A World Bank Publication, Oxford University Press, 1991.

[30] Schwartz, E., “The pricing of commodity-linked bonds.” *Journal of Finance*, Vol. 37 1982, pp.525-539.

[31] Schwartz, E., “The stochastic behavior of commodity prices: implications for valuation and hedging.” *The Journal of Finance*, Vol. 52 1997, pp.923-73.

[32] Wilmott, P., *Derivatives, the Theory and Practice of Financial Engineering*, John Wiley & Sons, 1998.

[33] Working, H., “The theory of the price of storage.” *The American Economic Review*, Vol. 39 1948, pp.1254-62.

Appendix A the Price of Commodity-Linked Bonds

The derivation in this appendix was inspired by Miltersen and Schwartz (1998) where they derived the closed-form solutions for the price of options on commodity futures based on the same term structure of interest rate model as well as the future convenience yield model as we have in this paper.

To evaluate the price of CLBs at *t=0*, firstly we write

(A-1) $e^{-\int_0^T f(s,s)ds} = A_1 e^{-X_1}$

with X_1 defined as:

$$\begin{aligned} X_1 &= \int_0^T [\int_0^s \sigma_f(u,s) \bullet dW_u] ds \\ &= \int_0^T [\int_u^T \sigma_f(u,s) ds] \bullet dW_u \end{aligned}$$

and A_1 is residually determined, as

$$A_1 = e^{-\int_0^T [f(0,s) + \int_0^s \mu_f(u,s)du]ds},$$

and A_1 is non-stochastic. In order to evaluate equation (18), we need to derive the expression of the commodity spot price and the value of the firm issuing bonds at maturity date *t=T*. From equation (10), applying Ito's lemma, we obtain

(A-2) $d\ln V(t) = [\mu_V(s) - \frac{1}{2}\sigma_V(s) \bullet \sigma_V(s)]ds + \sigma_V(u) \bullet dW_u$

Integrating (A-Z) from *0* to *T*, the following equation can be obtained

(A-3) $\ln V(T) = \ln V(0) + \int_0^T [\mu_V(s) - \frac{1}{2}\sigma_V(s) \bullet \sigma_V(s)]ds + \int_0^T \sigma_V(u) \bullet dW_u$

Using the expression $\mu_V(s) = r(s) = f(s,s)$ gives us the expression of the value of the firm issuing bonds at maturity date *t=T* as follows

(A-4) $V(T) = A_2 e^{X_2}$

where A_2 is residually determined and defined as:

$$A_2 = V(0)e^{\int_0^T [f(0,s)+\int_0^s \mu_f(u,s)du - \frac{1}{2}\sigma_V(s)\bullet\sigma_V(s)]ds}$$

and X_2 is stochastic and defined as:

$$\begin{aligned} X_2 &= X_1 + \int_0^T \sigma_V(u) \bullet dW_u \\ &= \int_0^T [\int_u^T \sigma_f(u,s)ds + \sigma_V(s)] \bullet dW_u \end{aligned}.$$

A similar approach applied to S, gives

(A-5) $S(T) = A_3 e^{X_3}$

where A_3 is residually determined and defined as:

$$A_3 = S(0)e^{\int_0^T [f(0,s)-\varepsilon(0,s)+\int_0^s (\mu_f(u,s)-\varepsilon_f(u,s))du - \frac{1}{2}\sigma_S(s)\bullet\sigma_S(s)]ds}$$

and X_3 is stochastic and defined as:

$$X_2 = \int_0^T [\int_u^T (\sigma_f(u,s) - \sigma_\varepsilon(u,s))ds + \sigma_S(s)] \bullet dW_u .$$

Let us use the following notations:

$$\sigma_{X_1}(u) = \int_u^T \sigma_f(u,s)ds$$

$$\sigma_{X_2}(u) = \int_u^T \sigma_f(u,s)ds + \sigma_V(u)$$

$$\sigma_{X_3}(u) = \int_u^T (\sigma_f(u,s) - \sigma_\varepsilon(u,s))ds + \sigma_S(u)$$

Now we can price CLBs at *t=0*. Obviously, (X_1, X_2, X_3) is jointly normally distributed with mean zero (Miltersen and Schwartz (1998)). Furthermore, the variances and covariance can be calculated as follows,

$$\sigma_1^2 = \int_0^T \left\| \sigma_{X_1}(u) \right\|^2 du,\ \sigma_2^2 = \int_0^T \left\| \sigma_{X_2}(u) \right\|^2 du,\ \sigma_3^2 = \int_0^T \left\| \sigma_{X_3}(u) \right\|^2 du,$$

$$\sigma_{12} = \sigma_{21} = \int_0^T \sigma_{X_1}(u) \bullet \sigma_{X_2}(u) du,\ \sigma_{13} = \sigma_{31} = \int_0^T \sigma_{X_1}(u) \bullet \sigma_{X_3}(u) du,$$

$$\sigma_{23} = \sigma_{32} = \int_0^T \sigma_{X_2}(u) \bullet \sigma_{X_3}(u) du.$$

The price of CLBs at $t=0$ can now be written as

(A-6)
$$\begin{aligned} B(.,0) = & c\int_0^T [A_1^* \int_{-\infty}^{+\infty} e^{-x_1^*} f_v(x_1^*) dx_1^*] dv \\ & + \iiint [A_1 e^{-x_1} \min[A_2 e^{x_2}, F + \max(0, A_3 e^{x_3} - K)]] f(x_1, x_2, x_3) dx_1 dx_2 dx_3 \end{aligned}.$$

Where,

$$A_1^* = e^{-\int_0^v [f(0,s) + \int_0^s \mu_f(u,s) du] ds},$$

$$X_1^* = \int_0^v [\int_u^v \sigma_f(u,s) ds] \bullet dW_u,$$

$$X_1^* \sim N(0, \sigma_{X_1^*}^2),$$

$$\sigma_{X_1^*}^2 = \int_0^v \left\| \int_u^v \sigma_f(u,s) ds \right\|^2 du,$$

$f_v(x_1^*)$ is the density function of X_1^*

$$f_v(x_1^*) = \frac{1}{\sqrt{2\pi\sigma_{X_1^*}^2}} e^{-\frac{x_1^{*2}}{2\sigma_{x_1^*}^2}}.$$

$f(x_1, x_2, x_3)$ is the density function of (X_1, X_2, X_3), and can be expressed as follows:

$$f(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}} \sqrt{\det \Sigma}} e^{-\frac{1}{2} u^T \Sigma^{-1} u}$$

where

$$u = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

$\det\Sigma$ is the determinant of the matrix Σ, and Σ is variance -covariance matrix, and written as:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_3^3 \end{bmatrix}.$$

Appendix B the Closed-Form Formula without Default Risk

This appendix will deal with the case in which there is no default risk. The closed form formula of the price of CLBs is obtained and several special cases are discussed.

Based on the assumptions in this case (without default risk) and the general formula for pricing CLBs—equation (18), the price of CLBs at *t=0* in this case can be written as follows:

(B-1)
$$\begin{aligned} B(.,0) &= c\int_0^T E[e^{-\int_0^v f(s,s)ds}]dv + FE[A_1 e^{-X_1}] \\ &\quad + A_1 E[e^{-X_1} \max(0, A_3 e^{-X_3} - K)] \\ &= T_1 + T_2 + T_3 \end{aligned} .$$

Since

$$P(t,T) = E[e^{-\int_t^T f(s,s)ds} \Gamma_t]$$

we can obtain the following results as:

(B-2)
$$\begin{aligned} T_1 &= c\int_0^T E[e^{-\int_0^v f(s,s)ds}]dv \\ &= c\int_0^T P(0,v)dv \end{aligned}$$

and

(B-3)
$$\begin{aligned} T_2 &= FE[A_1 e^{-X_1}] \\ &= FP(0,T) \end{aligned} .$$

For the third item, we will employ the same approach as in Miltersen and Schwartz (1998) to deal with it. Using the iterative law, we have

(B-4)
$$\begin{aligned} T_3 &= A_1 E[e^{-X_1}, \max(A_3 e^{X_3} - K, 0)] \\ &= A_1 E[\, E[e^{-X_1} | X_3] \max(A e^{X_3} - K, 0)] \end{aligned}$$

We know that the following equations hold:

$$X_1 \big| X_3 = x_3 \sim N(x_3 \frac{\sigma_{13}}{\sigma_3^2}, \sigma_1^2 (1 - \frac{\sigma_{13}^2}{\sigma_1^2 \sigma_3^2})),$$

$$E[e^{-X_1} \big| X_3 = x_3] = e^{-x_3 \frac{\sigma_{13}}{\sigma_3^2} + \frac{1}{2}\sigma_1^2 (1 - \frac{\sigma_{13}^2}{\sigma_1^2 \sigma_3^2})}.$$

Using the above results, T_3 can be rewritten as

$$\text{(B-5)} \quad \begin{aligned} T_3 &= A_1 e^{\frac{1}{2}\sigma_1^2 (1 - \frac{\sigma_{13}^2}{\sigma_1^2 \sigma_3^2})} E[e^{-X_3 \frac{\sigma_{13}}{\sigma_3^2}} (A_3 e^{X_3} - K)^+] \\ &= A_1 A_3 e^{\frac{1}{2}\sigma_1^2 (1 - \frac{\sigma_{13}^2}{\sigma_1^2 \sigma_3^2})} E[I_{\{X_3 > \frac{K}{A_3}\}} e^{X_3 (1 - \frac{\sigma_{13}}{\sigma_3^2})} - A_1 K e^{\frac{1}{2}\sigma_1^2 (1 - \frac{\sigma_{13}^2}{\sigma_1^2 \sigma_3^2})} E[I_{\{X_3 > \frac{K}{A_3}\}} e^{-X_3 \frac{\sigma_{13}}{\sigma_3^2}}] \end{aligned},$$

here $I_{\{X_3 > \frac{K}{A_3}\}}$ is an indicator function. Computing the expectation directly, we know the following facts:

$$E[I_{\{X_3 > \frac{K}{A_3}\}} e^{X_3 (1 - \frac{\sigma_{13}}{\sigma_3^2})}] = e^{\frac{(\sigma_3^2 - \sigma_{13})^2}{2\sigma_3^2}} N(\frac{\ln \frac{A_3}{K} + \sigma_3^2 - \sigma_{13}}{\sigma_3})$$

and

$$E[I_{\{X_3 > \frac{K}{A_3}\}} e^{-X_3 \frac{\sigma_{13}}{\sigma_3^2}}] = e^{\frac{\sigma_{13}^2}{2\sigma_3^2}} N(\frac{\ln \frac{A_3}{K} - \sigma_{13}}{\sigma_3})$$

where $N(.)$ denotes the standard cumulative normal distribution function. Observe that

$$\begin{aligned} A_1 e^{\frac{1}{2}\sigma_1^2 (1 - \frac{\sigma_{13}^2}{\sigma_1^2 \sigma_3^2})} e^{\frac{\sigma_{13}^2}{\sigma_1^2 \sigma_3^2}} &= A_1 e^{\frac{1}{2}\sigma_1^2} \\ &= A_1 E[e^{-X_1}] \\ &= E[e^{-\int_0^T f(s,s) ds}] \\ &= P(0,T) \end{aligned}$$

and that

$$
\begin{aligned}
A_1 A_3 e^{\frac{1}{2}\sigma_1^2(1-\frac{\sigma_{13}^2}{\sigma_1^2\sigma_3^2})} e^{\frac{(\sigma_3^2-\sigma_{13})^2}{2\sigma_3^2}} &= A_1 A_3 e^{\frac{1}{2}[\sigma_1^2+\sigma_3^2-2\sigma_{13}]} \\
&= A_1 A_3 E[e^{-X_1+X_3}] \\
&= E[A_1 e^{-X_1} A_3 e^{X_3}] \\
&= E[e^{-\int_0^T f(s,s)ds} S(T)]
\end{aligned}
.
$$

Moreover

$$
A_3 e^{\frac{1}{2}\sigma_3^2-\sigma_{13}} = \frac{E[e^{-\int_0^T f(s,s)ds} S(T)]}{P(0,T)}
$$

implying that

$$
\ln A_3 + \frac{1}{2}\sigma_3^2 - \sigma_{13} = \ln \frac{E[e^{-\int_0^T f(s,s)ds} S(T)]}{P(0,T)}.
$$

Defining

$$
G^*(0,T) = E[e^{-\int_0^T f(s,s)ds} S(T)],
$$

using the above results we obtained, the price of CLBs with constant coupon rate c under no default risk at $t=0$ as follows,

$$
\begin{aligned}
B(.,0) &= c\int_0^T P(0,v)dv + FP(0,T) \\
&+ G^*(0,T)N\left(\frac{\ln\frac{G^*(0,T)}{P(0,T)K}+\frac{1}{2}\sigma_3^2}{\sigma_3}\right) - KP(0,T)N\left(\frac{\ln\frac{G^*(0,T)}{P(0,T)K}-\frac{1}{2}\sigma_3^2}{\sigma_3}\right)
\end{aligned}
\tag{B-6}
$$

With the normality assumptions stated, we can compute $G^*(0,T)$ in the following way,

$$
\begin{aligned}
G^*(0,T) &= A_1 A_3 e^{\frac{1}{2}[\sigma_1^2+\sigma_3^2-2\sigma_{13}]} \\
&= A_1 E[e^{-X_1}] A_3 E[e^{X_3}] e^{-\sigma_{13}} \\
&= P(0,T)G(0,T)e^{-\sigma_{13}}
\end{aligned}
$$

With this expression, (B-6) can be simplified as:

$$
\begin{aligned}
B(.,0) &= c\int_0^T P(0,v)dv + FP(0,T) \\
&+ P(0,T)[G(0,T)e^{-\sigma_{13}} N\left(\frac{\ln\frac{G(0,T)}{K} - \sigma_{13} + \frac{1}{2}\sigma_3^2}{\sigma_3}\right) \\
&- KN\left(\frac{\ln\frac{G(0,T)}{K} - \sigma_{13} - \frac{1}{2}\sigma_3^2}{\sigma_3}\right)]
\end{aligned}
\quad . \tag{B-7}
$$