**Multivariate AIM Consumer Demand Model Applied to Dried Fruit, Raisins, and Dried Plums**

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**Abstract:** We estimate a semi-nonparametric demand system based on a multivariate version of the Muntz-Szatz series expansion which is called the *Asymptotically Ideal Model* (AIM). The model is applied to consumer demand for dried fruits, raisins, and dried plums. Results from the first and second order AIM expansions suggest that the second order expansion leads to a more economically consistent model, but the likelihood ratio test indicates the AIM(2) model was not a statistical improvement over the AIM(1) model.

**Keywords:** demand, consumers, AIM

## Introduction

Barnett and Jonas'(1983) *Asymptotically Ideal Model* (AIM) is a multivariate version of the Muntz-Szatz series expansion, which is attractive because it satisfies global flexibility, global regularity, global approximation, and resists overfitting (Barnett and Yue 1988; Havenner and Saha 1999). Global flexibility allows one to estimate the level, first and second-order derivatives at each data point; global regularity ensures theoretically correct demand functions; and global approximation allows elasticity calculations at each data point rather than simply at the mean. Overfitting resistance is especially important to highly flexible functional forms because these functions tend to overfit. Symmetry and homogeneity are easily imposed in the demand system of the AIM model through simple parametric restrictions.

In 1988, Barnett and Yue applied the AIM model to a first- and second-order three-good system using a static indirect utility function. They derived demand functions via the modified Roy's identity. Barnett and Yue imposed linear homogeneity by including exponents that summed to one, imposed symmetry by simple parameter restrictions, and imposed curvature through sufficient nonnegative parameters. The nonnegative parameters is a sufficient, but not necessary condition for imposing curvature. The second order expansion was found to produce the better performing maximum likelihood model. Yue (1991) estimated a money demand model reporting a preference for the second order expansion. Flessig and Swofford (1997) deviated from the previously estimated static indirect utility application to a three-good dynamic (lagged time) estimation that is consistent with Barnett and Yue's estimation procedures. The second order expansion produced results more consistent with economic behavior.

The objectives of this research are to specify and estimate a five good demand system with first and second order AIM expansions. Previous studies used fewer than three equations because of the complexities of estimating the AIM model. The five-good indirect utility function used in this research was based on real-level scanner data for dried plums, raisins, and dried fruits obtained from Infoscan IRI.

The paper proceeds in the following manner. First, this research discusses the background and underlying theory behind the AIM model. Second, estimation issues are discussed. Third, data and descriptive statistics are presented. Fourth, results are reported and discussed. Finally, concluding remarks are drawn.

## AIM Background and Theory

Consider the indirect utility function

$$(1) \qquad u(\mathbf{p}, y) = \max_{\mathbf{x}} \left\{ u(\mathbf{x}) \text{ st } \mathbf{p}'\mathbf{x} = y \right\}$$

where $\mathbf{x}$ is an $(N \times 1)$ vector of goods, $\mathbf{p}$ is an $(N \times 1)$ vector of prices, and $y$ is expenditure. Expenditure normalized prices used in the model specifications below are defined by $\mathbf{v} = \frac{\mathbf{p}}{y}$.

There are multiple motivations for using a Muntz-Szatz series expansion. One motivation is that this expansion converges both pointwise and Sobelev norm to the true utility function and its derivatives, due to spreading the error throughout the function.[1] Other motivations are the attainment of global flexibility, global regularity, and

[1] Convergence in Sobelev norm allows for convergence to the function and its derivatives, which are essential for elasticity estimation (Havenner and Saha 1999).

overfitting resistance of the model. Global flexibility enables the attainment of the true function's level first and second derivatives, which addresses White's concern about estimating the "true" parameters and not the approximation parameters (Havenner and Saha 1999; White 1980). Global regularity is the attainment of monotonicity and curvature, which can be imposed in the AIM model by deleting the diagonal parameters and restricting the remaining parameters to be non-negative. The non-negative parameter constraint is a sufficient, but not necessary condition for obtaining regularity. Thus we choose not to impose global regularity, but test for it. We do not impose global regularity because its imposition forces the substitutability among goods and we want the data to tell us the relationship among the goods. Barnett and Yue indicated global regularity at each expansion makes it difficult to overfit the data or to fit the white noise disturbances (1988). In essence, global regularity in a flexible functional form decreases the potential for overfitting, which is consistent with rational economic behavior (Havenner and Saha 1999). For illustrative purposes and simplicity we present the first order expansion, or AIM(1), of the indirect utility function

$$
\begin{aligned}
(2) \qquad \mathrm{f(v)} = {} & \frac{1}{2}\left(\sum_{i=1}^{n} a_i v_i^{1/2}\right) + \left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right) + 1\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\
& + 2\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\
& + 2\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right)
\end{aligned}
$$

In this first-order expansion, $n$=5, for the five goods discussed in the data section, where $i \neq j$ and $i,j$=1,…,$n$. For the AIM(1) model, this expansion is then the generalized

Leontief functional form that has been used in prior estimation. The second order expansion, or AIM(2), and subsequent derivations are presented in the Appendix.

Under the classical axioms of consumer demand models (Deaton and Muellbauer 1980), necessary demand restrictions include homogeneity, symmetry, adding up requirements, monotonicity, and curvature. The indirect utility function is homogeneous of degree zero in prices and income and is achieved in this modeling structure through expenditure-normalized prices. Symmetry imposes consistent choices of competing bundles of goods and is accomplished by equating the cross effects (i.e. $a_{12}=a_{21}$). The adding-up restriction is a direct result of the implication of a linear budget constraint. Combined with homogeneity, this restriction produces a multiplicative relationship between prices, expenditure, and demanded quantities. Monotonicity implies that the consumer must choose nonnegative quantities of goods. To be theoretically consistent, curvature in a demand system requires that demand functions be negative, and that the second-order determinants of the Hessian matrix are negative semidefinite. The imposition of the restrictions generates results consistent with indirect utility maximization and, hence, rational consumer behavior.

Derivation of the consumer demand equations shown below uses the modified Roy's identity (Barnett and Yue 1988)

$$(3) \qquad x_i = \frac{\frac{\partial f(v)}{\partial v}}{\sum_{i=1}^{n}\left(\frac{\partial f(\mathbf{v})}{\partial v}\right)_i}$$

Share equations are derived as follows:

(4) $$s_i = \frac{v_i\left(\frac{\partial f(v)}{\partial v}\right)}{\sum_{i=1}^{n} v_i\left(\frac{\partial f(v)}{\partial v}\right)}$$

The income elasticity estimation is obtained from the following equations:

(5) $$\eta_i = \frac{\partial x_i}{\partial y}\frac{y}{x_i} = \frac{y}{x_i}\left(\frac{1}{v_i}\frac{\partial s_i}{\partial y} + \frac{s_i}{p_i}\right) = \frac{\partial s_i}{\partial y}\frac{y}{s_i} + 1 \quad \text{where } v_i = \frac{p_i}{y}$$

and

(6) $$\eta_i = \frac{\left(\frac{\partial s_i^t}{\partial y} s_i^b - s_i^t \frac{\partial s_i^b}{\partial y}\right)}{\left(s_i^b\right)^2}\frac{y}{s_i} + 1$$

The estimation of $\sigma_{ij}$ is equation 7 with equation 6 providing the derivation in terms of y.

(7) $$\left[\frac{1}{v_i}\frac{\partial s_i}{\partial p_j} + \frac{s_j}{v_j}\left(\frac{1}{v_i}\frac{\partial s_i}{\partial y} + \frac{s_i}{p_i}\right)\right]\frac{y v_i v_j}{s_i s_j} \qquad \textit{for } i \neq j$$

$$\left[\frac{1}{v_i}\frac{\partial s_i}{\partial p_i} - \frac{s_i}{p_i v_i} + \frac{s_i}{v_i}\left(\frac{1}{v_i}\frac{\partial s_i}{\partial y} + \frac{s_i}{p_i}\right)\right]\frac{y v_i^2}{s_i^2} \quad \text{for } i = j \text{ where } v_i = \frac{p_i}{y}$$

$$\frac{\partial s_i}{\partial p_j} = \frac{\left(\frac{\partial s_i^t}{\partial p_j} s_i^b - s_i^t \frac{\partial s_i^b}{\partial p_j}\right)}{\left(s_i^b\right)^2}$$

The compensated own-price elasticities and cross-price elasticities are calculated using equation 4 for $s_j$ and equation 7 for $\sigma_{ij}$ , respectively. In general, elasticities are calculated using the equation shown below.

(8) $$\varepsilon_{ij}^{*} = s_j \sigma_{ij}$$

## Estimation Issues

Theoretical restrictions can be checked by examining the first and second order conditions of the AIM model. Derivations for the AIM(1) and AIM(2) models are presented in detail in the Appendix. Although curvature can be imposed on the AIM model by restricting the parameters, we choose to estimate the model freely without curvature restrictions. Curvature requirements are satisfied if the Hessian matrix is negative semidefinite (which implies nonpositive own-price demand elasticities) and monotonicity conditions are satisfied by positive share values. To check if the curvature restrictions are met, we calculate eigenvalues of the Hessian matrix.

Theoretical demand requirements are taken into consideration when specifying the models. Since we estimate a share system with five goods, the covariance expression across the five goods will be singular. Hence, we estimate $n$-1=4 share equations. The remaining coefficients are recovered using standard procedures; symmetry restrictions (i.e. $a_{15}=a_{51}$) and adding up restrictions (1-s1-s2-s3-s4=s5). After imposing these restrictions, the parsimonious AIM(1) model has 30 parameters and AIM(2) model has 240 parameters. Both seemingly unrelated AIM expansions are solved using the nonlinear option with the Davidon-Fletcher-Powell algorithm in Shazam Professional Edition version 9. Once convergence is achieved, the derivatives of the shares are calculated using equation 8. The derivatives are used in combination with the shares and expenditure normalized prices for elasticity calculation as shown in equation A5 and A6 in the appendix. Eigenvalues of the Hessian matrix are calculated using Shazam's "eigval" command.

## Data

Data were obtained from Infoscan IRI national, retail scanner data. Quantity and price data are available for raisins, dried fruit, Sunsweet brand dried plums, Del Monte brand dried plums, and Dole brand dried plums. The data were gathered over a fifty-two month retail sale period for fifty-one United States cities. The collection period ran from September 1992 to August 1996 with an observation being a four week period. There were 23 cities with all five demand goods over the duration of the observation period resulting in 1,196 observations for estimation. Quantities were measured in pounds consumed per month while prices were measured in monthly real ($1996) retail prices per pound. Prices were derived from sales and, hence, included store discounts and/or coupons. The three dried plum brands account for roughly 84% of all dried plum sales.

Descriptive statistics for the quantities and prices are shown in table 1. The coefficient of variation indicates the relative dispersion of the data set (Anderson, Sweeney, and Williams 1990). Raisins have the highest quantity consumed, lowest quantity variability, and lowest price. Dole dried plums have the lowest quantity consumed and highest quantity consumed variability. Dried fruit has the highest price. Sunsweet dried plums have the lowest price variability while Del Monte dried plums have the highest price variability. On average, consumers spent $441,340.00 per month in the 23 cities during the observation period. Consumers spent 56% of their dried fruit expenditure on raisins, while only 2% went to Dole dried plums.

## Results

### *AIM(1) Model*

The general demand restrictions are verified by evaluating the eigenvalues of the Hessian matrix, adding up conditions, and share values. Homogeneity and symmetry are imposed and verified for accuracy by elasticity row summation and equal cross Hessian signs, respectively.

The compensated elasticities are shown in table 2. The compensated elasticities are reported because the elasticities are compensated for income, which allows on to focus on the substitution effect. The compensated own price elasticities are negative and inelastic. However, the demand system does not satisfy curvature conditions because the Hessian is not negative semi-definite. Table 3 reports the eigenvalues for the model and shows that curvature is violated because of a single positive eigenvalue.

The cross-price elasticities explain the interactive relationships between the goods. Negative cross-price elasticities are complements while positive cross-price elasticities are substitutes. Raisins are substitutes to the three dried plum brands and complements to dried fruit. Dried fruit is a substitute to all three dried plum brands. Sunsweet, Del Monte, and Dole dried plums are complements to each other.

*AIM(2) Model*

Table 3 shows the eigenvalues for the model and, as in the AIM(1) model, shows curvature violations because of a positive eigenvalue. The own-price elasticities are shown in table 4 to be negative and inelastic. Raisins are substitutes to all three branded dried plums while being a complement for dried fruit. Dried fruit is a substitute to all three branded dried plums. The three brands of dried plums are substitutes for each other.

*Comparison*

In reviewing the results from the first and second order AIM expansion, there is consistency among the models in terms of the relationships of the goods, correct demand negativity, and the number of curvature violations, one eigenvalue violation for each model. However, the degree of violation decreases with the AIM(2) expansion indicating a more economically consistent model. A likelihood ratio test was before due to the models being nested. The value of the likelihood ratio test is 54.044 with 210 degrees of freedom. The value of the tail area is greater than 0.05. AIM(2) is rejected by our test. Thus in this five consumption good estimation, we found it was not beneficial to estimate an expansion past the first order AIM model.

**Conclusions**

In this paper we estimated first and second order AIM expansions using the dried fruit data set. Homogeneity and symmetry were verified. Curvature failed to hold for both expansions although there was a decrease in the degree of violation as the order of expansion increased. Compensated elasticities are calculated and evaluated for raisins, dried fruit, and three brands of dried plums. For both models the elasticities were negative and inelastic.  The relationships among the goods were consistent for both models. Results indicated the unconstrained AIM(2) model was not an improvement over the unconstrained AIM(1) model in a five good case. Thus the AIM model may be reliable for estimation procedures with fewer than five goods and with curvature imposed.

The five good, non-curvature imposed AIM model violated several demand restrictions. Although curvature is not imposed and thus the violation is not a surprise, Bayesian analysis of how often the demand system violates curvature would be beneficial. Also analysis based on consumption per capita instead of total consumption could alleviate some issues surrounding the AIM estimation, especially concerning curvature violations.

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**Table 1. Summary Statistics for Consumption Data Per Month**

| Variable | | Mean | Standard Deviation | Coefficient of Variation | Minimum | Maximum |
|---|---|---|---|---|---|---|
| Quantity (lbs) | | | | | | |
| | Raisins | 127,350.00 | 138,320.00 | 108.62 | 13,307.00 | 1,006,200.00 |
| | Dried Fruit | 23,949.00 | 37,107.00 | 154.95 | 1,926.00 | 380,520.00 |
| | Sunsweet Dried Plums | 34,358.00 | 41,816.00 | 121.71 | 2,626.00 | 326,470.00 |
| | Del Monte Dried Plums | 6,826.00 | 10,393.00 | 152.25 | 7.00 | 72,469.00 |
| | Dole Dried Plums | 4,257.60 | 6,874.40 | 161.46 | 0.00 | 82,781.00 |
| Price ($/lb) | | | | | | |
| | Raisins | 1.93 | 0.17 | 8.92 | 1.48 | 2.47 |
| | Dried Fruit | 4.53 | 0.55 | 12.03 | 3.19 | 5.76 |
| | Sunsweet Dried Plums | 2.16 | 0.17 | 8.07 | 1.54 | 2.92 |
| | Del Monte Dried Plums | 1.95 | 0.25 | 12.96 | 1.32 | 2.9 |
| | Dole Dried Plums | 2.21 | 0.27 | 12.05 | 0 | 2.84 |
| Expenditure ($) | | 441,340.00 | 498,310.00 | 112.91 | 46,663.00 | 3,702,100.00 |
| Share | | | | | | |
| | Raisins | 0.56 | 0.05 | 9.81 | 0.41 | 0.72 |
| | Dried Fruit | 0.22 | 0.07 | 31.55 | 0.09 | 0.46 |
| | Sunsweet Dried Plums | 0.17 | 0.05 | 29.82 | 0.05 | 0.37 |
| | Del Monte Dried Plums | 0.03 | 0.04 | 125.64 | 0 | 0.21 |
| | Dole Dried Plums | 0.02 | 0.01 | 72.47 | 0 | 0.08 |

**Table 2. AIM(1) Compensated Price Elasticities**

| | Raisin | Dried Fruit | Sunsweet | DelMonte | Dole |
|---|---|---|---|---|---|
| Raisin | -0.2292 | 0.0798 | 0.0383 | 0.0823 | 0.0288 |
| Dried Fruit | -2.1754 | -0.6174 | 2.1107 | 0.2525 | 0.4322 |
| Sunsweet | 0.1293 | 0.5561 | -0.5813 | -0.0633 | -0.0407 |
| DelMonte | 0.8562 | -0.0800 | -0.2115 | -0.5548 | -0.0098 |
| Dole | 0.4881 | 0.2748 | -0.2181 | -0.0038 | -0.5409 |

**Table 3. Eigenvalues for Demand AIM**

| | Raisin | Dried Fruit | Sunsweet | DelMonte | Dole |
|---|---|---|---|---|---|
| AIM(1) | 0.3671 | -0.0002 | -0.5020 | -0.6683 | -1.7202 |
| AIM(2) | 0.0089 | -0.5407 | -1.0000 | -1.0000 | -1.3532 |

**Table 4. AIM(2) Compensated Price Elasticities**

| | Raisin | Dried Fruit | Sunsweet | DelMonte | Dole |
|---|---|---|---|---|---|
| Raisin | -0.3195 | -0.0268 | 0.4325 | -0.1227 | 0.0440 |
| Dried Fruit | -0.0756 | -0.7936 | -0.0838 | 0.9301 | 0.0308 |
| Sunsweet | 0.5563 | 0.2269 | -0.8406 | 0.0432 | 0.0254 |
| DelMonte | 0.5563 | 0.2269 | 0.1594 | -0.9568 | 0.0254 |
| Dole | 0.5563 | 0.2269 | 0.1594 | 0.0432 | -0.9746 |

**Appendix**

For the second order expansion, AIM(2), the following equation is estimated.

(*A*1)

$$
\begin{aligned}
f(v)=&\frac{1}{2}\left(\sum_{i=1}^{n}a_i v_i^{1/2}\right)+\left(\sum_{j=1}^{n}\sum_{i=1}^{n}a_{ij}v_i^{1/2}v_j^{1/2}\right)+1\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}a_{ijk}v_i^{1/2}v_j^{1/2}v_k^{1/2}\right)\\
&+2\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}a_{ijkl}v_i^{1/2}v_j^{1/2}v_k^{1/2}v_l^{1/2}\right)\\
&+2\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}a_{ijklm}v_i^{1/2}v_j^{1/2}v_k^{1/2}v_l^{1/2}v_m^{1/2}\right)\\
&+\frac{1}{4}\left(\sum_{i=1}^{n}b_i v_i^{1/4}\right)+\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{i=1}^{n}b_{ij}v_i^{1/4}v_j^{1/4}\right)+\frac{3}{4}\left(\sum_{j=1}^{n}\sum_{i=1}^{n}c_{ij}v_i^{1/2}v_j^{1/4}\right)\\
&+\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}b_{ijk}v_i^{1/2}v_j^{1/4}v_k^{1/4}+1\frac{1}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}c_{ijk}v_i^{1/4}v_j^{1/2}v_k^{1/2}\right)\\
&+\frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}d_{ijk}v_i^{1/4}v_j^{1/4}v_k^{1/4}\right)+\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}e_{ijkl}v_i^{1/4}v_j^{1/4}v_k^{1/4}v_l^{1/4}\\
&+1\frac{1}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}b_{ijkl}v_i^{1/2}v_j^{1/4}v_k^{1/4}v_l^{1/4}\right)+1\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}c_{ijkl}v_i^{1/2}v_j^{1/2}v_k^{1/4}v_l^{1/4}\right)\\
&+1\frac{3}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}d_{ijkl}v_i^{1/2}v_j^{1/2}v_k^{1/2}v_l^{1/4}\right)\\
&+2\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}b_{ijklm}v_i^{1/2}v_j^{1/2}v_k^{1/2}v_l^{1/2}v_m^{1/4}\right)\\
&+1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}c_{ijklm}v_i^{1/4}v_j^{1/4}v_k^{1/4}v_l^{1/4}v_m^{1/4}\right)\\
&+1\frac{3}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}d_{ijklm}v_i^{1/2}v_j^{1/2}v_k^{1/4}v_l^{1/4}v_m^{1/4}\right)\\
&+2\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}f_{ijklm}v_i^{1/2}v_j^{1/2}v_k^{1/2}v_l^{1/4}v_m^{1/4}\right)\\
&+1\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}e_{ijklm}v_i^{1/2}v_j^{1/4}v_k^{1/4}v_l^{1/4}v_m^{1/4}\right)
\end{aligned}
$$

Estimation of the demand equations is found using the modified Roy's identity:

$$(A2) \qquad x_i = \frac{\dfrac{\partial f(v)}{\partial v}}{\sum_{i=1}^{n}\left(\dfrac{\partial f(v)}{\partial v}\right)_i}$$

AIM(1):

$$x_i = \frac{\left[\begin{array}{l} \frac{1}{4}\left(\mathrm{a_i} v_i^{-1/2}\right) + \frac{1}{2}\left(\sum_{j=1}^{n} a_{ij} v_i^{-1/2} v_j^{1/2}\right) \\ + \frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijk} v_i^{-1/2} v_j^{1/2} v_k^{1/2}\right) \\ + \left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijkl} v_i^{-1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\ +1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijklm} v_i^{-1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \end{array}\right]}{\left[\begin{array}{l} \frac{1}{2}\left(\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{a_i} v_i^{1/2}\right) + \left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right) \\ +1\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\ +2\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\ +2\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \end{array}\right]}$$

AIM(2):

$$
x_i = \left[ \begin{aligned}
&\frac{1}{4}\left(a_i v_i^{-1/2}\right) + \frac{1}{2}\left(\sum_{j=1}^{n} a_{ij} v_i^{-1/2} v_j^{1/2}\right) + \frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijk} v_i^{-1/2} v_j^{1/2} v_k^{1/2}\right) \\
&+\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijkl} v_i^{-1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\
&+1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijklm} v_i^{-1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \\
&+\frac{1}{8}\left(b_i v_i^{-1/4}\right) + \frac{1}{8}\left(\sum_{j=1}^{n} b_{ij} v_i^{-1/4} v_j^{1/4}\right) + \frac{3}{8}\left(\sum_{j=1}^{n} c_{ij} v_i^{-1/2} v_j^{1/4}\right) \\
&+\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} b_{ijk} v_i^{-1/2} v_j^{1/4} v_k^{1/4}\right) + \frac{5}{8}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} c_{ijk} v_i^{-1/4} v_j^{1/2} v_k^{1/2}\right) \\
&+\frac{3}{8}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} d_{ijk} v_i^{-1/4} v_j^{1/4} v_k^{1/4}\right) + \frac{1}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} e_{ijkl} v_i^{-1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4}\right) \\
&+\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} b_{ijkl} v_i^{-1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4}\right) + \frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} c_{ijkl} v_i^{-1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4}\right) \\
&+\frac{7}{8}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} d_{ijkl} v_i^{-1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4}\right) \\
&+1\frac{1}{8}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} b_{ijklm} v_i^{-1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/4}\right) \\
&+\frac{5}{16}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} c_{ijklm} v_i^{-1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
&+\frac{7}{8}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} d_{ijklm} v_i^{-1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
&+\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} f_{ijklm} v_i^{-1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4} v_m^{1/4}\right) \\
&+\frac{3}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} e_{ijklm} v_i^{-1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right)
\end{aligned} \right]
$$

$$\left[\begin{array}{l}
\frac{1}{2}\left(\sum_{i=1}^{n} a_i v_i^{1/2}\right)+\left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right)+1\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\
+2\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\
+2\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \\
+\frac{1}{4}\left(\sum_{i=1}^{n} b_i v_i^{1/4}\right)+\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij} v_i^{1/4} v_j^{1/4}\right)+\frac{3}{4}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij} v_i^{1/2} v_j^{1/4}\right) \\
+\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijk} v_i^{1/2} v_j^{1/4} v_k^{1/4}+1\frac{1}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk} v_i^{1/4} v_j^{1/2} v_k^{1/2}\right) \\
+\frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijk} v_i^{1/4} v_j^{1/4} v_k^{1/4}\right)+\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} e_{ijkl} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4} \\
+1\frac{1}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijkl} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4}\right) \\
+1\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4}\right) \\
+1\frac{3}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4}\right) \\
+2\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/4}\right) \\
+1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijklm} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
+1\frac{3}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
+2\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} f_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4} v_m^{1/4}\right) \\
+1\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} e_{ijklm} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right)
\end{array}\right]$$

There are five goods as denoted by n and i=1,…,n. The estimation of the demand equations is a precursor to the estimation of the own-price and the cross-price elasticities for the indirect utility function. To meet the curvature requirements, the own-price demand elasticities must be negative to meet the necessity of a downward sloping demand equation. If the model produces downward sloping demand equations, the model is assumed to meet theoretic curvature restrictions. Share equation estimation is shown below.

$$(A3) \qquad s_i = \frac{v_i\left(\dfrac{\partial f(v)}{\partial v}\right)}{\sum_{i=1}^{n} v_i\left(\dfrac{\partial f(v)}{\partial v}\right)}$$

AIM(1):

$$s_i = \frac{\left[\begin{array}{l} \frac{1}{4}\left(a_i v_i^{1/2}\right) + \frac{1}{2}\left(\sum_{j=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right) + \frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\ \qquad + \left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\ \qquad + 1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \end{array}\right]}{\left[\begin{array}{l} \frac{1}{2}\left(\sum_{i=1}^{n} a_i v_i^{1/2}\right) + \left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right) + 1\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\ \qquad + 2\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\ \qquad + 2\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \end{array}\right]}$$

AIM(2):

$$
s_i = \frac{\left[\begin{aligned}
&\frac{1}{4}\left(a_i v_i^{1/2}\right) + \frac{1}{2}\left(\sum_{j=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right) + \frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\
&+\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\
&+1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \\
&+\frac{1}{8}\left(b_i v_i^{1/4}\right) + \frac{1}{8}\left(\sum_{j=1}^{n} b_{ij} v_i^{1/4} v_j^{1/4}\right) + \frac{3}{8}\left(\sum_{j=1}^{n} c_{ij} v_i^{1/2} v_j^{1/4}\right) \\
&+\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} b_{ijk} v_i^{1/2} v_j^{1/4} v_k^{1/4}\right) + \frac{5}{8}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} c_{ijk} v_i^{1/4} v_j^{1/2} v_k^{1/2}\right) \\
&+\frac{3}{8}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} d_{ijk} v_i^{1/4} v_j^{1/4} v_k^{1/4}\right) + \frac{1}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} e_{ijkl} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4}\right) \\
&+\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} b_{ijkl} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4}\right) + \frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} c_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4}\right) \\
&+\frac{7}{8}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} d_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4}\right) \\
&+1\frac{1}{8}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} b_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/4}\right) \\
&+\frac{5}{16}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} c_{ijklm} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
&+\frac{7}{8}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} d_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
&+\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} f_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4} v_m^{1/4}\right) \\
&+\frac{3}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} e_{ijklm} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right)
\end{aligned}\right]}{\;}
$$

$$
\left[
\begin{aligned}
&\frac{1}{2}\left(\sum_{i=1}^{n} a_i v_i^{1/2}\right)+\left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right)+1\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right)\\
&+2\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right)\\
&+2\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right)\\
&+\frac{1}{4}\left(\sum_{i=1}^{n} b_i v_i^{1/4}\right)+\frac{1}{2}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij} v_i^{1/4} v_j^{1/4}\right)+\frac{3}{4}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij} v_i^{1/2} v_j^{1/4}\right)\\
&+\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijk} v_i^{1/2} v_j^{1/4} v_k^{1/4}+1\frac{1}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk} v_i^{1/4} v_j^{1/2} v_k^{1/2}\right)\\
&+\frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijk} v_i^{1/4} v_j^{1/4} v_k^{1/4}\right)+\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} e_{ijkl} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4}\\
&+1\frac{1}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijkl} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4}\right)\\
&+1\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4}\right)\\
&+1\frac{3}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4}\right)\\
&+2\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/4}\right)\\
&+1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijklm} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right)\\
&+1\frac{3}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right)\\
&+2\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} f_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4} v_m^{1/4}\right)\\
&+1\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} e_{ijklm} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right)
\end{aligned}
\right]
$$

The five commodities are represented by *n* and *i,j,k,l,m*=1,..,*n*. Since this is a share equation system estimation with five goods, there will be *n*-1=4 share equations estimated. The fifth equation can be recovered through symmetry restrictions (i.e. $a_{15}=a_{51}$) and through the adding up restriction (1-s1-s2-s3-s4=s5).

The income elasticity estimation is obtained by:

$$(A4) \qquad \eta_i = \frac{\partial x_i}{\partial y}\frac{y}{x_i} = \frac{y}{x_i}\left(\frac{1}{v_i}\frac{\partial s_i}{\partial y} + \frac{s_i}{p_i}\right) = \frac{\partial s_i}{\partial y}\frac{y}{s_i} + 1$$

where $v_i = \dfrac{p_i}{y}$

$$\eta_i = \frac{\left(\dfrac{\partial s_i^t}{\partial y} * s_i^b - s_i^t * \dfrac{\partial s_i^b}{\partial y}\right)}{\left(s_i^b\right)^2}\frac{y}{s_i} + 1$$

AIM (1):

$$s_i^t = \left[\begin{array}{l} \frac{1}{4}\left(a_i v_i^{1/2}\right) + \frac{1}{2}\left(\sum_{j=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right) \\ + \frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\ + \left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\ + 1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \end{array}\right]$$

$$s_i^b = \left[\begin{array}{l} \frac{1}{2}\left(\sum_{i=1}^{n} a_i v_i^{1/2}\right) + \left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right) \\ + 1\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\ + 2\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\ + 2\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \end{array}\right]$$

AIM(2):

$$
s_i^t = \left[\begin{array}{l}
\frac{1}{4}\left(a_i v_i^{1/2}\right) + \frac{1}{2}\left(\sum_{j=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right) + \frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\
+\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\
+1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \\
+\frac{1}{8}\left(b_i v_i^{1/4}\right) + \frac{1}{8}\left(\sum_{j=1}^{n} b_{ij} v_i^{1/4} v_j^{1/4}\right) + \frac{3}{8}\left(\sum_{j=1}^{n} c_{ij} v_i^{1/2} v_j^{1/4}\right) \\
+\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} b_{ijk} v_i^{1/2} v_j^{1/4} v_k^{1/4}\right) + \frac{5}{8}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} c_{ijk} v_i^{1/4} v_j^{1/2} v_k^{1/2}\right) \\
+\frac{3}{8}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} d_{ijk} v_i^{1/4} v_j^{1/4} v_k^{1/4}\right) + \frac{1}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} e_{ijkl} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4}\right) \\
+\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} b_{ijkl} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4}\right) \\
+\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} c_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4}\right) \\
+\frac{7}{8}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} d_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4}\right) \\
+1\frac{1}{8}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} b_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/4}\right) \\
+\frac{5}{16}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} c_{ijklm} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
+\frac{7}{8}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} d_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
+\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} f_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4} v_m^{1/4}\right) \\
+\frac{3}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} e_{ijklm} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right)
\end{array}\right]
$$

$$
s_i^b = \left[\begin{array}{l}
\frac{1}{2}\left(\sum_{i=1}^{n} a_i v_i^{1/2}\right) + \left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right) + 1\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\
+2\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\
+2\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \\
+\frac{1}{4}\left(\sum_{i=1}^{n} b_i v_i^{1/4}\right) + \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij} v_i^{1/4} v_j^{1/4}\right) + \frac{3}{4}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij} v_i^{1/2} v_j^{1/4}\right) \\
+\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijk} v_i^{1/2} v_j^{1/4} v_k^{1/4} + 1\frac{1}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk} v_i^{1/4} v_j^{1/2} v_k^{1/2}\right) \\
+\frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijk} v_i^{1/4} v_j^{1/4} v_k^{1/4}\right) + \sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} e_{ijkl} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4} \\
+1\frac{1}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijkl} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4}\right) \\
+1\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4}\right) \\
+1\frac{3}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4}\right) \\
+2\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/4}\right) \\
+1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijklm} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
+1\frac{3}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
+2\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} f_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4} v_m^{1/4}\right) \\
+1\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} e_{ijklm} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right)
\end{array}\right]
$$

The compensated price elasticities are calculated using equation A3 for $s_j$ and equation A6 for $\sigma_{ij}$:

$$(A5) \qquad \varepsilon_{ij}^{*} = s_j \sigma_{ij}$$

The estimation of $\sigma_{ij}$ equals equation A6 with equation A4 providing the derivation in terms of y.

$$(A6) \qquad \left[\frac{1}{v_i}\frac{\partial s_i}{\partial p_j} + \frac{s_j}{v_j}\left(\frac{1}{v_i}\frac{\partial s_i}{\partial y} + \frac{s_i}{p_i}\right)\right]\frac{y v_i v_j}{s_i s_j} \qquad for \;\; i \neq j$$

$$\left[\frac{1}{v_i}\frac{\partial s_i}{\partial p_i} - \frac{s_i}{p_i v_i} + \frac{s_i}{v_i}\left(\frac{1}{v_i}\frac{\partial s_i}{\partial y} + \frac{s_i}{p_i}\right)\right]\frac{y v_i^2}{s_i^2} \qquad for \;\; i = j$$

$$\text{where } \mathrm{v}_{\mathrm{i}} = \frac{p_i}{y}$$

$$\frac{\partial s_i}{\partial p_j} = \frac{\left(\frac{\partial s_i^t}{\partial p_j} * s_i^b - s_i^t * \frac{\partial s_i^b}{\partial p_j}\right)}{\left(s_i^b\right)^2}$$

$$\frac{\partial s_i}{\partial p_i} = \frac{\left(\frac{\partial s_i^t}{\partial p_i} * s_i^b - s_i^t * \frac{\partial s_i^b}{\partial p_i}\right)}{\left(s_i^b\right)^2}$$

AIM (1):

$$\mathrm{s}_{\mathrm{i}}^{\mathrm{t}} = \left[\begin{array}{l} \frac{1}{4}\left(\mathrm{a}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}}^{1/2}\right) + \frac{1}{2}\left(\sum_{j=1}^{n} \mathrm{a}_{\mathrm{ij}}\mathrm{v}_{\mathrm{i}}^{1/2}\mathrm{v}_{\mathrm{j}}^{1/2}\right) \\ + \frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n} \mathrm{a}_{\mathrm{ijk}}\mathrm{v}_{\mathrm{i}}^{1/2}\mathrm{v}_{\mathrm{j}}^{1/2}\mathrm{v}_{\mathrm{k}}^{1/2}\right) \\ + \left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} \mathrm{a}_{\mathrm{ijkl}}\mathrm{v}_{\mathrm{i}}^{1/2}\mathrm{v}_{\mathrm{j}}^{1/2}\mathrm{v}_{\mathrm{k}}^{1/2}\mathrm{v}_{\mathrm{l}}^{1/2}\right) \\ + 1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n} \mathrm{a}_{\mathrm{ijklm}}\mathrm{v}_{\mathrm{i}}^{1/2}\mathrm{v}_{\mathrm{j}}^{1/2}\mathrm{v}_{\mathrm{k}}^{1/2}\mathrm{v}_{\mathrm{l}}^{1/2}\mathrm{v}_{\mathrm{m}}^{1/2}\right) \end{array}\right]$$

$$\mathrm{s}_{\mathrm{i}}^{\mathrm{b}} = \left[\begin{array}{l} \frac{1}{2}\left(\sum_{i=1}^{n} \mathrm{a}_{\mathrm{i}}\mathrm{v}_{\mathrm{i}}^{1/2}\right) + \left(\sum_{j=1}^{n}\sum_{i=1}^{n} \mathrm{a}_{\mathrm{ij}}\mathrm{v}_{\mathrm{i}}^{1/2}\mathrm{v}_{\mathrm{j}}^{1/2}\right) \\ + 1\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} \mathrm{a}_{\mathrm{ijk}}\mathrm{v}_{\mathrm{i}}^{1/2}\mathrm{v}_{\mathrm{j}}^{1/2}\mathrm{v}_{\mathrm{k}}^{1/2}\right) \\ + 2\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} \mathrm{a}_{\mathrm{ijkl}}\mathrm{v}_{\mathrm{i}}^{1/2}\mathrm{v}_{\mathrm{j}}^{1/2}\mathrm{v}_{\mathrm{k}}^{1/2}\mathrm{v}_{\mathrm{l}}^{1/2}\right) \\ + 2\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} \mathrm{a}_{\mathrm{ijklm}}\mathrm{v}_{\mathrm{i}}^{1/2}\mathrm{v}_{\mathrm{j}}^{1/2}\mathrm{v}_{\mathrm{k}}^{1/2}\mathrm{v}_{\mathrm{l}}^{1/2}\mathrm{v}_{\mathrm{m}}^{1/2}\right) \end{array}\right]$$

AIM(2):

$$s_i^t=\left[\begin{array}{l}\frac{1}{4}\left(a_i v_i^{1/2}\right)+\frac{1}{2}\left(\sum_{j=1}^{n}a_{ij}v_i^{1/2}v_j^{1/2}\right)+\frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}a_{ijk}v_i^{1/2}v_j^{1/2}v_k^{1/2}\right)\\ +\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}a_{ijkl}v_i^{1/2}v_j^{1/2}v_k^{1/2}v_l^{1/2}\right)\\ +1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}a_{ijklm}v_i^{1/2}v_j^{1/2}v_k^{1/2}v_l^{1/2}v_m^{1/2}\right)\\ +\frac{1}{8}\left(b_i v_i^{1/4}\right)+\frac{1}{8}\left(\sum_{j=1}^{n}b_{ij}v_i^{1/4}v_j^{1/4}\right)+\frac{3}{8}\left(\sum_{j=1}^{n}c_{ij}v_i^{1/2}v_j^{1/4}\right)\\ +\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}b_{ijk}v_i^{1/2}v_j^{1/4}v_k^{1/4}\right)+\frac{5}{8}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}c_{ijk}v_i^{1/4}v_j^{1/2}v_k^{1/2}\right)\\ +\frac{3}{8}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}d_{ijk}v_i^{1/4}v_j^{1/4}v_k^{1/4}\right)+\frac{1}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}e_{ijkl}v_i^{1/4}v_j^{1/4}v_k^{1/4}v_l^{1/4}\right)\\ +\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}b_{ijkl}v_i^{1/2}v_j^{1/4}v_k^{1/4}v_l^{1/4}\right)\\ +\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}c_{ijkl}v_i^{1/2}v_j^{1/2}v_k^{1/4}v_l^{1/4}\right)\\ +\frac{7}{8}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}d_{ijkl}v_i^{1/2}v_j^{1/2}v_k^{1/2}v_l^{1/4}\right)\\ +1\frac{1}{8}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}b_{ijklm}v_i^{1/2}v_j^{1/2}v_k^{1/2}v_l^{1/2}v_m^{1/4}\right)\\ +\frac{5}{16}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}c_{ijklm}v_i^{1/4}v_j^{1/4}v_k^{1/4}v_l^{1/4}v_m^{1/4}\right)\\ +\frac{7}{8}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}d_{ijklm}v_i^{1/2}v_j^{1/2}v_k^{1/4}v_l^{1/4}v_m^{1/4}\right)\\ +\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}f_{ijklm}v_i^{1/2}v_j^{1/2}v_k^{1/2}v_l^{1/4}v_m^{1/4}\right)\\ +\frac{3}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}e_{ijklm}v_i^{1/2}v_j^{1/4}v_k^{1/4}v_l^{1/4}v_m^{1/4}\right)\end{array}\right]$$

$$
s_i^b = \left[\begin{array}{l}
\frac{1}{2}\left(\sum_{i=1}^{n} a_i v_i^{1/2}\right) + \left(\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij} v_i^{1/2} v_j^{1/2}\right) + 1\frac{1}{2}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijk} v_i^{1/2} v_j^{1/2} v_k^{1/2}\right) \\
+2\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2}\right) \\
+2\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/2}\right) \\
+\frac{1}{4}\left(\sum_{i=1}^{n} b_i v_i^{1/4}\right) + \frac{1}{2}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij} v_i^{1/4} v_j^{1/4}\right) + \frac{3}{4}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij} v_i^{1/2} v_j^{1/4}\right) \\
+\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijk} v_i^{1/2} v_j^{1/4} v_k^{1/4} + 1\frac{1}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk} v_i^{1/4} v_j^{1/2} v_k^{1/2}\right) \\
+\frac{3}{4}\left(\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijk} v_i^{1/4} v_j^{1/4} v_k^{1/4}\right) + \sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} e_{ijkl} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4} \\
+1\frac{1}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijkl} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4}\right) \\
+1\frac{1}{2}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4}\right) \\
+1\frac{3}{4}\left(\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijkl} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4}\right) \\
+2\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} b_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/2} v_m^{1/4}\right) \\
+1\frac{1}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijklm} v_i^{1/4} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
+1\frac{3}{4}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right) \\
+2\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} f_{ijklm} v_i^{1/2} v_j^{1/2} v_k^{1/2} v_l^{1/4} v_m^{1/4}\right) \\
+1\frac{1}{2}\left(\sum_{m=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} e_{ijklm} v_i^{1/2} v_j^{1/4} v_k^{1/4} v_l^{1/4} v_m^{1/4}\right)
\end{array}\right]
$$