An option on the average European futures prices for an efficient hog producer risk management

Martial PHÉLIPPÉ-GUINVARC'H, Jean CORDIER***

* **Corresponding author**: Agricultural Insurance Service of GROUPAMA SA, 126 Piazza Mont d'Est, 93199 Noisy Le Grand Cedex et Agrocampus Ouest, UMR1302 SMART, F-35000 Rennes, France
e-mail : martial.phelippe-guinvarc-h@groupama.com

** INRA-Agrocampus Ouest, UMR1302 SMART, F-35000 Rennes, France

Summary *– The volatility of hog prices is high compared to most agricultural commodities. However, European hog producers do not benefit from any agricultural policy support. Through the continuous production process and induced selling activity on spot markets, producers benefit from a natural moving average product pricing. In addition, asymmetric price risk management is able to increase the expected utility of risk adverse hog producers. But, if there is a futures contract at the European Exchange (EUREX), there is no option market and as a consequence no derivative contracts on the European hog market. The article is presenting how financial intermediaries could offer an innovative derivative contract to complement the "naturall " steady price of the French hog producers.*

Keywords: *price risk, margin risk, hog, futures market, replication portfolio, hedging*

Une option sur la moyenne des prix à terme européens pour une gestion efficace du risque revenu de l'éleveur de porc

Résumé – La volatilité des prix du porc est élevée en comparaison des volatilités observées sur les principales matières premières agricoles. Cependant, les éleveurs de porcs ne bénéficient aujourd'hui d'aucun soutien des politiques publiques agricoles. Par leur mode de production en continu, les producteurs parviennent naturellement à obtenir un prix moyen en vendant régulièrement sur le marché physique. Pour autant, une gestion asymétrique des risques de prix serait en mesure d'accroître l'utilité attendue des producteurs de porcs adverse au risque. Mais, s'il existe aujourd'hui un contrat à terme européen (EUREX), il n'y a pas de marché d'option et, en conséquence, pas de contrats dérivés sur le marché du porc européen. L'article décrit comment les intermédiaires financiers pourraient offrir un contrat dérivés novateur en complément de la « naturelle » stabilisation des prix déjà réalisée par les producteurs de porcs français.

Mots-clés : risques agricoles, marge, éleveur de porc, contrat à terme, portefeuille de réplication, couverture de risque

JEL descriptors: G13, Q14, Q18

1. Introduction

Hog prices are more volatile as there is no European public mechanism to limit the natural variability. As a consequence, the hog producer revenue is highly variable from year to year. However, the continuous process of production and selling provides a simple means of stabilizing the producer revenue.

The purpose of the article is to design a derivative contract that could be an add-on to the natural average hog price in bringing asymmetric price risk management. In addition, this contract should participate in increasing market liquidity on the innovative European futures hog contract.

The standardized annual volatility of the hog spot auction market in Brittany (*Marché du porc breton – Plérin, France*) is about 30% on average (Cordier and Debar, 2004). This volatility is managed "naturally" by hog producers through the production process of spreading farrowing sows. The production process is then continuous with sales that can be performed every week for the large producers or every month for small producers. The producers are therefore able to get on semestrial or annual sales prices that are close to the market price average for the same period of time [1]. The short-term market price variability is therefore averaged by the production process. However, the capacity of asymmetric price risk management is questioned by hog producers, directly through financial instruments or indirectly through margin insurance contracts.

The asymmetric price risk management for hog producers is available in the US. Several insurance contracts exist that provide coverage against drops in hog annual prices or hog annual margin on feed costs. Within the European Union, there is no similar means for such asymmetric risk management. Before July 2009, this contract was quoted by RMX Hannover. Today, the EUREX futures market in Frankfort quotes hog futures contract but no option contract. In 2010, there is neither a derivative market on the hog futures price nor any insurance contract. The future of the Common Agricultural Policy may induce changes in such a situation.

In the perspective of financial innovation for managing farm risk, the aim of the article is to assess the feasibility of new offers from financial intermediaries such as banks or insurance companies, to manage price risk level as well as basis risk for producers in the West part of France.

The article demonstrates the technical feasibility of an option on the hog average price when provided to French hog producers. It presents successively:

a) The contract effectiveness, *i.e.* its capacity to limit price risk to the hog producer,

[1] The difference between the reference price from the European futures market (or spot MPB market) and the price paid by the slaughterhouse to the producer is usually called the basis. The basis is first related to location difference between the market delivery place and the effective location of the producer. The basis may also reflect quality differences between the futures contract (or the reference quality of the spot auction market) and the quality of hogs delivered as tested at the slaughterhouse. Basis risk exists and is well documented in the literature. Basis risk is supposed to be marginal as compared to price level risk. *http://www.eurexchange.com/trading/products/COM/AGR/FHOG_en.html*

b) the potential attractiveness of such a contract through its facility of use,
c) the management ability of such a contract by a financial intermediary through an optimized portfolio replication.

2. Context of the contract

2.1. The hog insurance contracts in the United States (US)

The first insurance contract, initially designed in Iowa in 2001 and implemented in nine other States later, the **Livestock Risk Protection Plan** (LRP), is offering a warranty against an average price decrease on livestock [2]. Considering its design, the contract is an option on average price. It does not include any yield or quality risks. The contract benefits from a subsidy from the United States Department of Agriculture (USDA) of 13%.

The second contract initiated in 2001, the **Livestock Gross Margin** (LGM, available in Iowa) provides a warranty to hog producers against a margin loss. The margin is computed over a six months period using lean hog prices, corn and soybean meal prices as observed on futures markets. The insurer who offers this warranty is offsetting margin risk on the futures markets. The innovation of this contract is twofold. First, the insurance contract is dealing with margin and second, the insurer is using futures markets for reinsuring the transferred risk.

2.2. The academic analysis of asymmetric risk management for hog production

Research is exploring a protection against an average cash price for the hog producer. Hart *et al.* (2001) are investigating the use of asiatic options [3] (**average option**) and developing a pricing method. Shao and Roe (2003) designed a contract called "**moving-window contract**", a derivative contract which composes the simultaneous purchase and sale of a basket and asiatic put on prices with different futures. Within their model, the underlying basket is composed of futures contracts on lean hog, corn and soybean meal. Then Shao and Roe are pricing a tunnel option on the average margin of the hog producer.

While the above articles present a theoretical pricing methodology, they do not demonstrate the ability of financial intermediary to offer such contracts.

Numerous institutions are issuing derivative contracts such as warrants, trackers or options. The underlying assets are, for instance, the prices of stocks, stock indices, energy indices or agricultural commodities. These institutions however should not hold these risks. They have to manage their global risk exposition against their own private equity but they should in the meantime hedge these risks by using reinsurance worldwide capacities and/or financial instruments on related futures markets.

[2] *http://www.rma.usda.gov/livestock/*
[3] Asiatic options are options on average price on pre-determined past periods.

The seminal analytical method for hedging risk on derivative instruments was proposed by Black and Scholes in 1973. The method is based upon the first derivative of the option price on the underlying asset price, called delta. It is thus called the neutral delta method. This paper will use the method in order to design innovative derivative instruments to the benefit of hog producers.

3. Definition of an option on the hog average price

3.1. The spot and futures reference prices

The instrument aims to cover the average hog spot price that would correspond to producer risk management horizon. Let us note $\overline{S}_T$ this value. Because we consider this value during a long period $[T_0, T]$ $(0 < T_0 < T)$, we introduce the risk free rate r to actualize this value. Then $\overline{S}_T$ could be defined as:

$$\overline{S}_T = \frac{1}{N_S} \times \sum_{j=0}^{N_S} e^{r.(T-t_j)} \times S(t_j) \tag{1}$$

where Ns is the number of settled spot prices during $[T_0, T]$.

For example, from July to September 2008, the auction spot hog market of Plérin in Brittany elicited 26 prices (Ns = 26). With the risk free rate equal to 3.81% [4], the computed average hog spot price is equal to 1.43 €/kg.

The European futures market provides hog reference prices in Europe on monthly futures with a one-year horizon.

Let's note $(H_i)_{1 \le i \le M}$ the monthly hog future price for month T_i proposed by the European futures market, and noted $T_0 \le T_1 \ldots T_M \le T$. For example, from July to September 2008, the European hog futures market quoted three futures: July (T_1, ended the 24th at 1.760 €/kg), August (T_2, ended the 21st at 1.791€/kg) and September (T_3, ended the 25th at 1.734 €/kg). We have three futures quotes, then M = 3.

3.2. Three possible computed futures prices

The futures contract can compound in different ways to define a contingent claims X that provide a pertinent answer to the three feasibility issues presented in the introduction. They are:

i. **The arithmetic mean of each settlement computed on the last future issue** (ended at T), where N_H is the number of quotation days (equivalent to an Asian futures):

$$X_1 = \frac{1}{N_H} \sum_{1 \le j \le N_H} H_M(t_j) \times e^{r(T-t_i)} \tag{2}$$

[4] From "*Institut des Actuaires*", value at 02/31/2008 for a maturity at 09/31/2008

ii. **The arithmetic mean of last settlement price of each future issue** (the last quotation day is fixed at the 15th of month) [5]:

$$X_2 = \frac{1}{M} \sum_{1 \le i \le M} H_i(T_i) \times e^{r(T-T_i)} \tag{3}$$

iii. **The geometric mean of last settlement price of each future issue**

$$\begin{aligned} X_3 &= GeometricalMean\left(H_i(T_i) \times e^{r(T-T_i)}, 1 \le i \le M\right) \\ &= \sqrt[M]{\prod_{1 \le i \le M} H_i(T_i) \times e^{r(T-T_i)}} \end{aligned} \tag{4}$$

3.3. Discussion for choosing a computed *ex ante* and *ex post* future prices

The theoretical framework of the first answer, X_1, is largely explored in literature. In this case, price and delta of contingent claim would be estimated by the Monte Carlo method. Nevertheless, the use of only one issue is not adequate when the average is computed over a long period of time. First, the strong seasonality of spot hog price would be corrected. Second, even though the hog production process is a continuous process, the supply demand relationship is changing through time. If the average period is "long", several future issues should be integrated within the average spot future price.

The second proposition X_2 is natural relative to producer need. This answer requires the Monte Carlo method to compute price and delta. Lastly, the third answer X_3 does not need to use Monte Carlo method with Samuelson (1965) assumptions (as used by Black and Scholes, 1973). Of course, we note that arithmetical mean of X_2 is the nearest approach to hog producers' needs as opposed to the more theoretical geometrical mean of X_3. Nevertheless, if correlation between X_3 and ST and assumptions are acceptable, X_3 should be chosen for the following reasons:

a) a daily delta management price is feasible. An algebraic answer of X_3 can be implemented in a spreadsheet software,

b) according to the strong variance of X_i, it seems useful to build the model X_3. Indeed, if the variance of X_2 is strong, the variance of (X_2-X_3) is low. Therefore, estimation of X_3 will benefit to strongly reduce the variance of X_2 in Monte Carlo simulation. The same mathematical approach is used in the literature to obtain more precise estimations of Asian options (Musiela and Rutkowski, 1997).

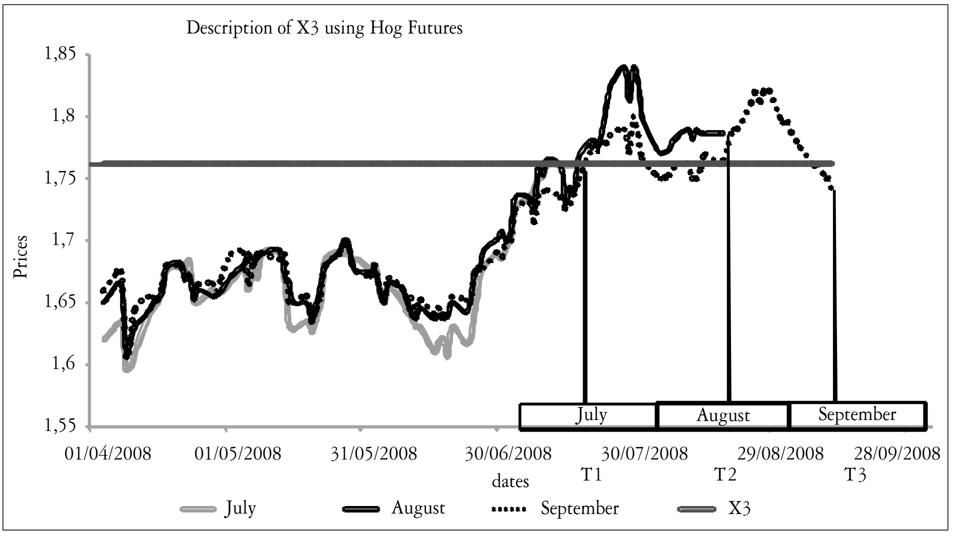
[5] In the last days of maturity, prices are disturbed by agents that have to exit the market or change to the next open issue. Then, calculate the contingent claim using the 15th of each month provides more pertinent result relatively of spot prices.
For example, the three month average price of April, May and June is the mean of the following prices: the April 15th settlement price of the April futures; the May 15th settlement price of the May futures and the June 15th settlement price of the June futures.

From July to September 2007, we observed the following prices (see figure 1):

Table 1.

	Mean Plérin	X_1	X_2	X_3
July to September	1,366 €	1,638 €	1,669 €	1,668 €

Figure 1. **The average hog price calculus illustration**



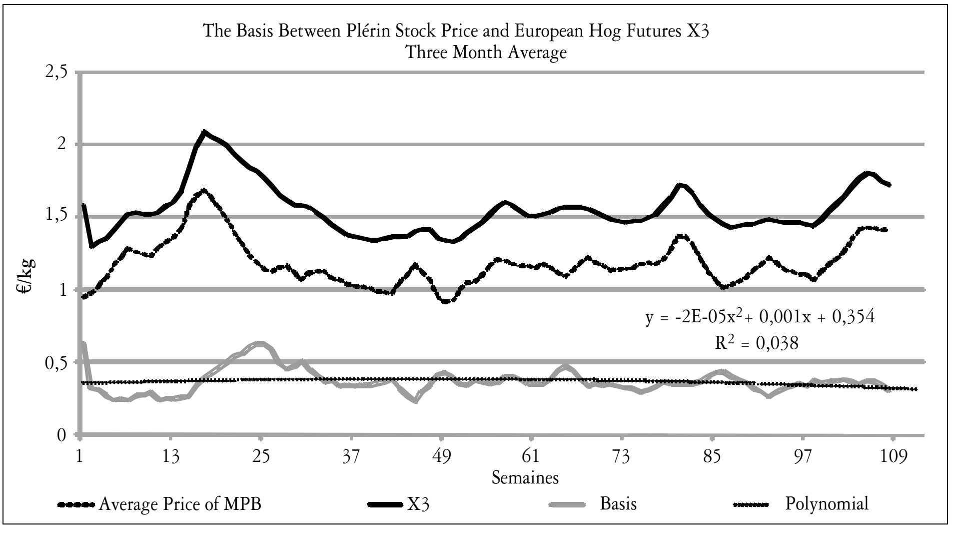
The spread is the basis between the auction spot market in Plérin and the European futures market. The basis is not constant and not even deterministic. Thus, it is required to consider the basis variability or, more generally, the statistical link between X_1, X_2, X_3 and using historical price data (Jan. 2000-Sept. 2008).

Table 2.

	Correlation with Plérin spot prices (%)		
	X_1	X_2	X_3
3 months	92.08	94.24	94.09
6 months	91.33	95.11	94.83

Because X_1 uses only one issue, its correlations with the spot price are lower than correlations using X_2 and X_3. Even if X_2 obtains higher correlations, the spreads with X_3 are quite low. Therefore the X_3 choice seems not only workable but also pertinent for designing and valuing an efficient option in hog farm risk management. Figure 2 illustrates this performance.

Figure 2. **Basis between the geometric average price on the European market and the average price on the MPB spot market**



4. Futures price motion assumptions

Let us note $(F_t^i)_{1\ 1\ N}$ the issues of hog futures price at time t. It is assumed that futures price motion follows geometric Brownian motion with parameters μ_i and σ_i:

$$F_t^i = \exp((\mu_i - \frac{1}{2}\sigma_i^2)\cdot t + \sigma_i W_t^i) \tag{5}$$

for all $t \in [0, T_i]$. We note: $\mu_t^i = \mu_i \times 1_{\{t \le T_i\}}$

$(\mu_t^i)_{1 \le 1 \le N}$ vector is noted μ_t, the diagonal matrix of $(\sigma_i)_{1 \le 1 \le N}$ is noted σ and $(W_t^i)_{1 \le 1 \le N}$ vector is noted W_t. The correlation matrix of Brownian ($\sigma \cdot W_t$) is noted Σ. Let us note Γ superior triangular matrix as $\Sigma = \Gamma\Gamma^*$, where Γ^* is the transposed of Γ. The i^{th} line of Γ is noted Γ_i B_t is a N dimension Brownian motion where Brownian are independent, and where $B_t^i = B_{T_i}^i$ when $t \ge T_i$. It results that: $\sigma \cdot W_t = \Gamma \cdot B_t$. The usual differential equation resolution gives for each $(F^i)_{1 \le 1 \le N}$:

$$F_t^i = \exp((\mu_i^t - \frac{1}{2}\sigma^2 \cdot A_t^*)\cdot t + \Gamma_i \cdot B_t) \quad \forall t \in [0, T_i] \tag{6}$$

where A_t is the line matrix formed by the $1_{\{t \le T_i\}}$.

From 2000-2007 historical hog data on the European futures market, we estimate the following parameters:

Table 3. **In %**

	July	August	September
Sigma_i	15.26	15.62	19.21

Σ is estimated as:

0.0233	0.0154	0.0194
0.0194	0.0244	0.0173
0.0154	0.0173	0.0384

It results that Γ is equal to:

0.10466	0.06902	0.04138
0.00000	0.08904	0.06372
0.00000	0.00000	0.10906

5. The derivative contract valuation

Consider first the average future hog price and its option.

Proposition 1: The average future hog price

At time t, the future price noted H_t is equal to $x_t \prod_{\ell=1}^{N} \sqrt[N]{F_t^\ell}$ *where x_t is defined as:*

$$x_t = \exp\left\{ \sum_{i=1}^{N} \left(-\frac{N-1}{2.N^2} \times \Sigma_{ii} + \frac{1}{N^2} \times \sum_{\substack{j=1 \\ j>i}}^{N} \Sigma_{ij} \right) \times \max(T_i - t, 0) \right\} \tag{7}$$

Proof in appendix A.1.

Proposition 2: an option on the average future hog price

Under previous hypothesis, the put option pure premium on the average future price X at time t is equal to the put option value of Black and Scholes with a strike price E and the equivalent volatility $\sigma_{eq}(t)$ such as:

$$\sigma_{eq}(t) = \frac{1}{N} \times \sqrt{\frac{\Phi_t \Sigma A_t^*}{T-t}} \tag{8}$$

where A_t is the matrix line of the $1_{\{t \leq T_i\}}$ *and where* Φ_t *is the matrix line of the* $\max(T_i - t, 0)$.

Proof in appendix A.1 (Lamberton and Lapeyre, 1997; Nielsen and Sandman, 1998)

Under the hypothesis, the option premium on the average future price can be computed algebraically as it is equivalent to the standard Black and Scholes model.

6. The derivative contract risk management

The derivative should be managed by a financial intermediary. As in the previous section, it is first considered the management of the average hog future price then the option risk management.

Proposition 3: The replication portfolio of the average future hog price

The portfolio replication of H_t is filled with:

$\frac{x_t}{N} \times \frac{g(F,t)}{F_t^i}$ *futures contracts* F_t^i *until the future T_i,* $\forall i = 1, \ldots, N$.

where $g(F,t)=\prod_{\ell=1}^{N}\sqrt[N]{F_t^{\ell}}$.

Proof : The replication portfolio is set from the differential equation of H_t:

$$\begin{aligned} dH_t &= H_t \times \frac{1}{N}\times A_t\cdot\Gamma_i\cdot dB_t^* + r\cdot H_t\cdot dt = \frac{1}{N}\times\left(H_t\cdot A_t\cdot\Gamma_i\cdot dB_t^* + H_t\cdot r\cdot dt\right)\\ &= \frac{1}{N}\times\sum_{i=1}^{N}\frac{d(F,t)}{F_i^t}\times\left(A_t\cdot F_i^t\cdot\Gamma_i\cdot dB_t^* + r\cdot F_i^t\cdot dt_t^i\right)\\ &= \frac{1}{N}\times\sum_{i=1}^{N}\frac{d(F,t)}{F_i^t}\times dF_i^t \end{aligned} \tag{9}$$

This last relation is the proof of proposition 3.

Proposition 4: Risk management of the derivative contract

The replication portfolio of the derivative contract on the average price X at time t is :

- $\Delta\cdot\frac{x_t}{N}\times\frac{g(F,t)}{F_t^i}$ futures contracts F_t^i until the future issue T_i, $\forall i=1,\ldots,N$, where Δ is the delta of the Black et Scholes option on H_t with the volatility $\sigma_{eq}(t)$.
- $X_t-\Delta\cdot H_t$ bond units.

Proof: The result of proposition 4 is the exact application of the replication portfolio of the put option from the Black and Scholes model applied on H_t.

Application of the model to risk management for a financial intermediary:

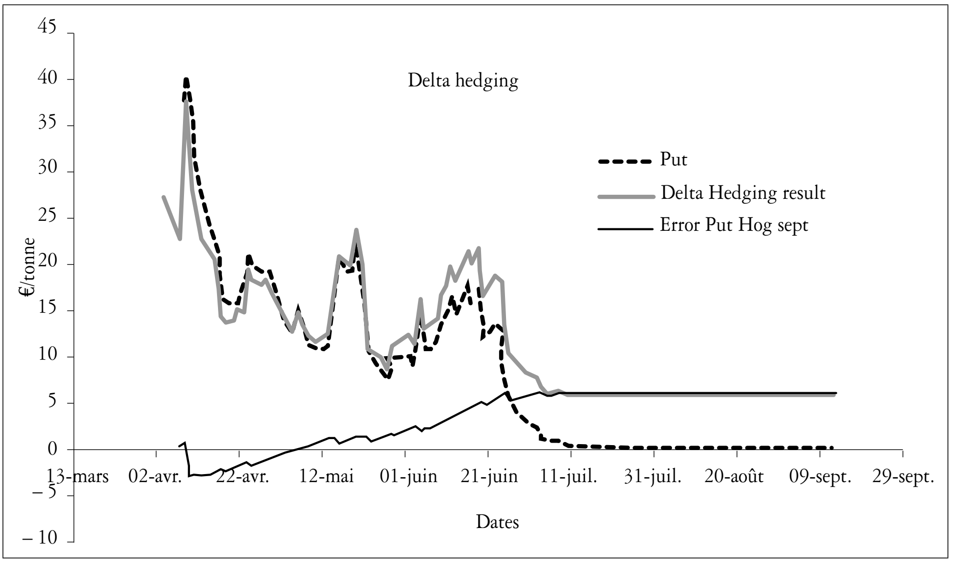
A financial institution likes to offer an option on the three months average of the European futures hog price. The Euribor 6 months rate at April 1st 2008 is 3.81%. This rate is assimilated at the risk free rate.

Appendix A.3 states that assumptions are acceptable using true quotation data statistics.

As stated earlier, the expected value of X_3 at April 1st 2008 is 1,668 €/t. For a strike price equal to 1,600 €/t this option is evaluated at 27.03 €/t.

The test of risk management (proposition 4) is realized using daily quotation data from April 1th 2008 to September 12th 2008. The result is presented in figure 3. Put motion realizes high variations and reaches a maximum of 39 €/t and ends at 0. The result of risk management motion realizes the same high variations and reaches 36 €/t and ends at 5.5 €/t. The error of strategy continuously increases to reach 5.5 €/t. The correlation between the daily error and the daily put variation is no-significant. This error, in favour of financial institution, is related to volatility under-estimation. If the historical volatility of September issue is equal to 19.2%, the 2008 volatility is estimated to 15%.

Figure 3. **Risk Management of the Average Price Put Option**



7. Conclusion and discussion

The original average of quoting futures prices every month on the European futures market is allowing the design of a Put option with an original under-claim on hog averaged futures prices. The derivative contract can be priced and its replication portfolio can be designed under the standard Black and Scholes hypothesis. The tests performed with such derivative contract present efficient results for managing risk of hog producers in the Western part of France selling on the spot auction market. The ability of risk management from the financial intermediary point of view has also been positively tested.

Therefore, the derivative contract is useful for the hog producer, the financial institution, as well as for the futures market. The hog producer benefits from a financial instrument which allows asymmetrical price risk management. The financial institution is allowed to develop a low risk new activity. Finally, the futures market institution benefits from an increased contract liquidity, allowing a better futures price elicitation.

Technically, this study may be improved by shifting from X_3 to X_2 definition of the average structure of futures hog prices. This shift should be able to improve the efficiency of risk management for the hog producer. It requires the use of Monte Carlo methods in addition to the present mathematical results.

Finally, this option may enable full family contracts to manage a large range of farm risk issues, including sales risk management or even farm gross margin risk management.

References

Black F., Scholes M. (1973) The pricing of options and corporate liabilities, *Journal of Political Economy* 81, 637-654.

Cordier J.-E., Debar J.-C. (2004) Gestion des risques agricoles : la voie nord-américaine. Quels enseignements pour l'Union européenne ? *Les Cahiers d'études – Le Demeter* 12, 70 p.

Hart C.E., Badcock B.A. (2001) Ranking of risk management strategies combining crop insurance products and marketing positions, Rapport Technique 01 -WP 267, Iowa State University, *http://www.card.iastate.edu/*.

Hart C.E., Badcock B.A. and Hayes D.J. (2001) Livestock revenue insurance, *Journal of Futures Markets* 21(6), 553-580.

Lamberton D., Lapeyre B. (1997) *Introduction au calcul stochastique appliquée à la finance*, Paris, Ellipses, 174 p.

Musiela M., Rutkowski M. (1997) *Martingale Methods in Financial Modeling. Applications of Mathematics*, Berlin, Springer, 518 p.

Nielsen A.J., Sandman K. (1998) Asian exchange rate options under stochastic interest rates: Pricing as a sum of delayed payment options, Rapport technique, EconPapers.

Samuelson P.A. (1965) Rational theory of warrant price, *Industrial Management Review* 6, 13-33.

Shao R., Roe B. (2003) The design and pricing of fixed- and moving-window contracts: An application of Asian-basket option pricing methods to the hog-finishing sector, *The Journal of Futures Markets 23(11), 1047-1073.*

APPENDICES

A.1. Proof of Proposition 1

From the Girsanov theorem applied to the probability space $(\Omega, \mathcal{H}, P)$, the process B_t^* defined as $B_t^* = B_t - \int_0^t \Gamma^{-1} \cdot (r - \mu_u) \cdot du$ is a Brownian motion under the risk neutral probability P^*. Price of H_t is then defined as:

$$H_t = e^{-r(T-t)} \,\mathrm{E}_{P^*}[H_T \,|\mathcal{H}_t] \tag{A.1}$$

The function g(F,t) is defined as $g(F,t) = \sqrt[N]{\prod_{\ell=1}^{N} F_t^\ell}$. Therefore, $g(F,T) = H_T$ and the partial derivative functions are the following :

$$\begin{aligned} \frac{\partial g(F,t)}{\partial F_t^i} &= \frac{1}{N} \times \frac{g(F,t)}{F_t^i} \\ \frac{\partial^2 g(F,t)}{\partial^2 F_t^i} &= -\frac{N-1}{N^2} \times \frac{g(F,t)}{(F_t^i)^2} \\ \frac{\partial^2 g(F,t)}{\partial F_t^i \partial F_t^j} &= \frac{1}{N^2} \times \frac{g(F,t)}{F_t^i \cdot F_t^j} \end{aligned} \tag{A.2}$$

The Itô formula is then applied:

$$\begin{aligned} dg(F,u) = & \sum_{i,j=1}^{N} 1_{\{(u \le T_i) \cap (u \le T_j)\}} \frac{1}{N} \times \frac{g(F,u)}{F_u^i} \times F_u^i \cdot \Gamma_{ij} \cdot dB_{j,u}^* + \sum_{i=1}^{N} \frac{1}{N} \times \frac{g(F,u)}{F_u^i} \times r F_u^i \cdot du \\ & + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} 1_{\{(u \le T_i) \cap (u \le T_j)\}} \times \frac{1}{N^2} \times \frac{g(F,u)}{F_u^i F_u^j} \times F_u^i \cdot F_u^j \cdot < \Gamma_i, \Gamma_j > du \\ & + \frac{1}{2} \sum_{i=1}^{N} 1_{\{u \le T_i\}} \times \left(-\frac{N-1}{N^2} \right) \times \frac{g(F,u)}{(F_u^i)^2} \times (F_u^i)^2 \cdot < \Gamma_i, \Gamma_i > du \end{aligned} \tag{A.3}$$

where $B_{j,t}^*$ is the j^{th} value at time t of the Brownian B^* with dimension N. As Γ is triangular superior and the respective futures issues are increasingly ordered, it is derived for any *(i, j)* and any $t \le T_i$:

$$\begin{aligned} \sum_{i,j=1}^{N} 1_{\{(u \le T_i) \cap (u \le T_j)\}} \frac{1}{N} \times \frac{g(F,u)}{F_u^i} \times F_u^i \cdot \Gamma_{ij} \cdot dB_{j,u}^* &= \sum_{i,j=1}^{N} \frac{1}{N} \times g(F,u) \cdot \Gamma_{ij} \cdot dB_{j,\min(u,T_j)}^* \\ &= \sum_{i=1}^{N} \frac{1}{N} \times g(F,u) \cdot \Gamma_i \cdot dB_u^* \end{aligned} \tag{A.4}$$

With simplification:

$$dg(F,u)=g(F,u)\times\left\{\frac{1}{N}\times\sum_{i=1}^{N}1_{\{u\le T_i\}}\times\Gamma_i\cdot dB_u^*+r\cdot du+\frac{1}{N^2}\sum_{\substack{i,j=1\\ j>i}}^{N}1_{\{u\le T_i\}}\right.$$
$$\left.\times\Sigma_{ij}\cdot du-\frac{N-1}{2\cdot N^2}\sum_{i=1}^{N}1_{\{u\le T_i\}}\times\Sigma_{ii}\cdot du\right\} \tag{A.5}$$
$$=g(F,u)\times\left\{r\cdot du+\sum_{i=1}^{N}1_{\{u\le T_i\}}\times\left(\frac{1}{N}\times\Gamma_i\cdot dB_u^*-\frac{N-1}{2\cdot N^2}\times\Sigma_{ii}\cdot du\right.\right.$$
$$\left.\left.+\frac{1}{N^2}\times\sum_{\substack{j=1\\ j>i}}^{N}\Sigma_{ij}\cdot du\right)\right\}.$$

The solution of this stochastic differential equation is:

$$g(F,t)=g(F,0)\times\exp\left\{r\cdot t+\sum_{i=1}^{N}\left(\frac{1}{N}\times\Gamma_i\cdot B^*_{\min(t,T_i)}-\left(\frac{1}{2N}\times\Sigma_{ii}+\frac{N-1}{2.N^2}\times\Sigma_{ii}-\frac{1}{N^2}\times\sum_{\substack{j=1\\ j>i}}^{N}\Sigma_{ij}\right)\right.\right.$$
$$\left.\left.\times\min(t,T_i)\right)\right\}$$

As $H_t=e^{-r(T-t)}\mathrm{E}_{p^*}[H_T\,|\,i_t]=e^{-r(T-t)}\mathrm{E}_{p^*}[g(F,T)\,|\,i_{t-}$, then:

$$H_t=g(F,t)\times\exp\left\{\sum_{i=1}^{N}\left(-\frac{N-1}{2.N^2}\times\Sigma_{ii}+\frac{1}{N^2}\times\sum_{\substack{j=1\\ j>i}}^{N}\Sigma_{ij}\right)\times\max(T_i-t,0)\right\} \tag{A.6}$$

This result is proving Proposition 1.

A.2. Proof of Proposition 2

H_t can be written as :

$$dH_t=H_t\times\frac{1}{N}\times\sum_{i=1}^{N}1_{\{t\le T_i\}}\times\Gamma_i\cdot dB_t^*+r\cdot H_t\cdot dt \tag{A.7}$$

When introducing the matrix line A_t composed of the $1_{\{t\le T_i\}}$, it results:

$$dH_t=H_t\times\frac{1}{N}\times A_t\cdot\boldsymbol{\Gamma}_i\cdot dB_t^*+r\cdot H_t\cdot dt \tag{A.8}$$

The instantaneous covariance matrix of B is the identity matrix of dimension N noted $I_{(N)}$. The instantaneous covariance matrix of $A_t\cdot\boldsymbol{\Gamma}_i\cdot B_t^*$ (dimension 1.1) is thus

$A_t \cdot \Gamma \cdot I_{(N)} \left(A_t \Gamma \right)^* = \cdot A_t \Gamma \Gamma^* A_t^* = A_t \Sigma A_t^*$. The instantaneous volatility $\sigma(t)$ is thus $\frac{1}{N} . \sqrt{A_t \Sigma A_t^*} = \frac{1}{N} . \sqrt{\sum_{i,j=1}^{N} 1_{\left\{ \left(t \le T_i \right) \cap \left(t \le T_j \right) \right\}} \times \Sigma_{ij}}$ with a $\Re$ value.

Let's consider H_T at time t under P^*:

$$H_T = H_t \times \exp\left(\frac{1}{N} \times \int_t^T A \cdot \Gamma_i \cdot dB_t^* + r(T-t) \right) \tag{A.9}$$

When introducing W_t a standard Brownian motion of dimension 1, H_T can be written as:

$$\begin{aligned} H_T &= H_t \times \exp\left(\frac{1}{N} \times \int_t^T \sigma(t) \cdot dW_t^* + r(T-t) \right) \\ &= H_t \times \exp\left(\sigma_{eq}(t) \cdot W_{T-t} + \left(r - \frac{1}{2} \sigma_{eq}^2(t) \right) \times (T-t) \right) \end{aligned} \tag{A.10}$$

where

$$\sigma_{eq}^2(t) = \frac{\int_t^T \sigma^2(u) \cdot du}{\left(T-t \right)} = \frac{\int_t^T \sum_{i,j=1}^{N} 1_{\left\{ \left(u \le T_i \right) \cap \left(u \le T_j \right) \right\}} \times \Sigma_{ij} \cdot du}{N^2 \cdot \left(T-t \right)} = \frac{\sum_{i,j=1}^{N} \Sigma_{ij} \times \left(\max\left(T_{\min(i,j)} - t; 0 \right) \right)}{N^2 \cdot \left(T-t \right)} \tag{A.11}$$

Using Equation (A.10) and for t under, P^*, H_T is a geometric Brownian motion with identical constant volatility $\sigma_{eq}(t)$. The Black and Scholes model can then be used.

A.3. Are geometric Brownian motion assumptions acceptable?

To control the acceptability of the geometric Brownian motion assumption, we would test two issues:

1. The log-normality of variations.

2. The no-dependence between two successively variations of price motion.

Then we use the weekly quotation to test the normality of:

$$\Delta = \ln\left(\frac{F_{t+1}}{F_t} \right) \tag{A.12}$$

We obtained for all September issues from January 1st 2000, and a maturity less than three months the following results (121 observations):

Table A1.

Normality tests				
Test	**Statistics**			
Shapiro-Wilk	W	0.986006	Pr < W	0.2233
Kolmogorov-Smirnov	D	0.063185	Pr > D	> 0.1500
Cramer-von Mises	W-Sq	0.063343	Pr > W-Sq	> 0.2500
Anderson-Darling	A-Sq	0.430596	Pr > A-Sq	> 0.2500

Clearly, the normality is acceptable.

Also, if the maturity is between four and six months we obtained (96 observations):

Table A2.

Normality tests				
Test	**Statistics**			
Shapiro-Wilk	W	0.981769	Pr < W	0.2033
Kolmogorov-Smirnov	D	0.090443	Pr > D	> 0.0513
Cramer-von Mises	W-Sq	0.127248	Pr > W-Sq	> 0.0478
Anderson-Darling	A-Sq	0.64747	Pr > A-Sq	> 0.0910

The normality assumption is not refused. Then, if the maturity is not too high, the log-normality of price variations assumption is acceptable.

Now, we have to control the no-dependence between two successively variations of price motion. We obtained for all September issues from 1st January 2001 and a maturity less than three months the following results (121 observations):

Table A3.

Dependance coefficients of Hoeffding, N = 126 Prob > D under H0: D = 0		
	IagLnFt2Ftl_HOG	**LnFt2Ftl_HOG**
LagLnFt2Ftl_HOG	0.99953 < .0001	– 0.00423 0.9896
LnFt2Ftl_HOG	– 0.00423 0.9896	0.99953 < .0001

Clearly, the dependence assumption is refused.

Also, if the maturity is between four and six month we obtain (96 observations):

Table A4.

Dependance coefficients of Hoeffding, Prob > D under H0: D = 0 Nombre d'observations		
	IagLnFt2Ftl_HOG	LnFt2Ftl_HOG
LagLnFt2Ftl_HOG	0.99747 < .0001 95	– 0.00701 0.0841 95
LnFt2Ftl_HOG	– 0.00701 0.0841 95	0.99752 < .0001 96

The dependence assumption is also refused. Then, the no-dependence of price variations assumption is acceptable.

Then, the geometric Brownian motion assumptions are acceptable in regards to true quotations data.