Elliptical Symmetry, Expected Utility, and Mean-Variance Analysis [1]

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Abstract

Mean-variance analysis in the form of risk programming has a long, productive history in agricultural economics research. And risk programming continues to be used despite well known theoretical results that choices based on mean-variance analysis are not consistent with choices based on expected utility maximization. This paper demonstrates that the multivariate distribution of returns used in risk programming must be elliptically symmetric in order for mean-variance analysis to be consistent with expected utility choices. Then a statistical test for elliptical symmetry is developed and demonstrated. This test enables researchers to determine when data will produce significant differences between risk programming choices and expected utility choices.

1 Introduction

Models describing optimal decision making under uncertainty have a long and productive history in agricultural economics research. In recent years, numerical techniques have been introduced that allow the solution of these models under general assumptions. In part, these techniques have been developed because of well-known theoretical results that the expected utility ranking of risky outcomes is consistent with ranking based on mean and variance only under restrictive assumptions of quadratic utility or normality of random returns. Some of the numerical techniques that have been introduced in the agricultural economics literature include: the moment generating function (Collender and Zilberman 1986); separable programming (Lambert and McCarl 1985); and numerical quadrature (Kaylen, Preckel, and Loehman 1987).

Despite the availability of numerical techniques that can produce numerical solutions to expected utility maximization problems, quadratic risk programming continues to be a numerical tool in empirical research (Young and Barry 1987) and mean-variance representations of preferences continue to be employed in the construction of theoretical and econometric models (Coyle 1992). One reason for this is that mean-variance preferences are easier to work with in many analytic derivations. Another reason is that quadratic programming is a more tractable numerical procedure than general nonlinear optimization and numerical integration. Traub, Wasilkowski, and Wozniakowski (1988, pp.177–178) show that, in a worst case, the number of information operations necessary to obtain an ϵ-approximation to a general nonlinear optimization problem is $m(\epsilon) = \theta(\epsilon^{-d/r})$, where d is the dimension of the domain of the objective function, r is the number of continuous derivatives of the objective and constraint functions, and θ is a constant. The same authors point out that it has been proven that an ϵ-approximation to a multivariate integral requires something proportional to $\epsilon^{-d/r}$ information operations, where d is the dimension of the domain of the integrand, and r is the number of continuous derivatives of the integrand. Information-based complexity theory classifies both of these problems as intractable, meaning that the number of required computations increases exponentially in the dimension of the problem. The force of these results for many economic problems is weaker than the general result because of the smooth functions that are commonly employed in economics. Still, these results should be contrasted with the result that minimization of convex functions that satisfy certain regularity conditions can obtain an ϵ-approximate solution in $m(\epsilon) = \theta\left(ln\frac{1}{\epsilon}\right)$ computations (Traub et al.). This is an expression that goes to infinity very slowly as ϵ goes to zero.

Support for the use of mean-variance analysis in models with a univariate source of risk has been provided by recent work of Sinn (1983), and Meyer (1987). They show that neither quadratic utility or normality are necessary for expected utility ranking of alternatives to be consistent with mean-variance ranking. They showed that the necessary condition is that the random variables a decision maker chooses among differ only in location and scale. This result only applies to models where alternatives are represented by univariate random variables, but Meyer and Raasche (1992) were able to apply this result to returns from stock portfolios by using the capital-asset pricing model to express the risk of different portfolios in terms of the non-systematic risk portion of the rate of return. For more general situations that do not allow simplification through the capital-asset pricing model, Chamberlin (1983) showed that elliptical symmetry of multi-variate returns is sufficient for consistency between expected utility and mean-variance solutions. He also showed that elliptical symmetry is necessary for consistent ranking of all portfolios including those with short sales, and those yielding negative wealth.

This paper applies Chamberlin's results on sufficiency of elliptical symmetry by first showing that the distribution of final wealth satisfies Meyer's and Sinn's location and scale condition if the distribution of portfolio returns is elliptically symmetric. The major section of the paper presents the development of a statistical test for elliptical symmetry and demonstrates the test with a sample agricultural portfolio problem. Some comments to generalize the results conclude the paper.

2 Elliptical Symmetry, Expected Utility, and Mean Variance

This section integrates results of Chamberlin (1983), Meyer (1987), and Sinn (1983) to show that mean-variance ranking of agricultural portfolios is consistent with expected utility ranking if the joint distribution of returns is elliptically symmetric. An important implication of this integration is that Meyer's preference representation and comparative statics (Meyer 1987, pp.423–427) can be applied to portfolio choice problems where the distribution of portfolio returns satisfies the condition of elliptical symmetry. Thus, for example, the set of optimal choices for decision makers with varying degrees of risk aversion can be represented by the mean-variance, or mean-standard deviation efficient frontier.

Meyer (1987) and Sinn (1983) independently demonstrated that the location-scale condition is sufficient for consistency of expected utility and mean-variance ranking of risky alternatives. The location-scale condition means that alternatives in the choice set only differ by location and scale parameters. Many economic models with a univariate source of uncertainty satisfy this condition because the interaction between the agent's choice and the source of uncertainty is linear. Sandmo's (Sandmo 1971) model of the competitive firm under output price uncertainty is an example of the type of model that satisfies the location-scale condition. Meyer (1987) also described the representation of preferences and their comparative statics in mean-standard deviation space.

In unrelated work, Chamberlin (1983) characterized the portfolio distributions which cause expected utility to be solely a function of the mean and variance of portfolio returns. Chamberlin resolves confusion that had persisted in earlier literature by showing that multivariate distributions that lead to consistency of expected utility and mean-variance rankings have the common attribute of spherical symmetry. Spherically symmetric random vectors have contours of equal density that are hyper-spheres about the origin.

Chamberlin showed that the portfolio distributions have to be related to spherical symmetry in the following manner. If there is a riskless asset, then the vector of random portfolio returns must be a linear transformation of a spherical random vector. Muirhead (Muirhead 1982, p.34) shows that a linear transformation of a spherical random vector is an elliptically symmetric random vector. The condition is weaker when there is no riskless asset. In this case, one of the random returns can have an arbitrary distribution, and the distribution of the remaining returns conditional on that return must be an elliptically symmetric random vector. Chamberlin's results concentrate on the distributions that make expected utility a function of the mean and variance of returns. His results can be made more useful by combining them with Meyer's (1987) results on the analytics and comparative statics of choice in mean-standard deviation space.

If a random vector, X, follows an elliptically symmetric distribution with mean μ and variance Σ ($X \sim E_m(\mu, \Sigma)$) then a random variable Y formed by taking a linear combination of the elements of X is elliptically symmetric. And, if $Y = \alpha' X$, then $E(Y) = \alpha'\mu$ and $Var(Y) = \alpha'\Sigma\alpha$ (Muirhead 1982, p.34). Further, because Y is symmetric, $Z = (Y - \alpha'\mu) \,/\, (\alpha'\Sigma\alpha)^{1/2}$ is symmetric with mean 0 and variance 1. And, except for location and scale, $\alpha'\mu + (\alpha'\Sigma\alpha)^{1/2} Z$ is equal in distribution to Y for any α. Thus final wealth from a portfolio of assets whose joint distribution is elliptically symmetric, satisfies the location-scale condition.

The analytic tools introduced by Meyer provide a valuable simplification in describing behavior and comparative statics under uncertainty. And the set of choices can be reduced to the mean-variance frontier that can be generated with quadratic programming if the location-scale condition holds. Therefore it would be useful to have a statistical test to determine if a given set of portfolio data satisfies the elliptical symmetry condition demonstrated by Chamberlin. The next section develops and demonstrates the use of such a test.

3 A Statistical Test for Elliptical Symmetry

This section will develop a statistical test for elliptical symmetry that is simple to compute using Monte Carlo techniques to generate the distribution of the test statistic under the null hypothesis. The proposed test is consistent with other computer intensive statistical procedures. In order to motivate the need for a computer intensive generation of the sampling distribution of the test statistic, the literature on tests for elliptical symmetry will first be reviewed.

3.1 Tests for Spherical Symmetry

The literature on testing for symmetry of multivariate distributions has concentrated on spherical symmetry, a special case of elliptical symmetry. In most cases (e.g. finite second moments) elliptically symmetric variables can be transformed to spherical symmetry. This is accomplished by centering and sphering. Namely, given data $X_1, X_2, \ldots, X_n$, it is possible to obtain centered and sphered residuals, Y_i, by the transformation:

$$Y_i = \hat{L}^{-1} (X_i - \hat{\mu}) \tag{1}$$

where $\hat{\mu}$ is the sample mean and $\hat{L}$ is the lower triangular Cholesky factor of the sample covariance matrix $\hat{\Sigma}$. The literature on testing uses an unbiased estimate of the covariance matrix defined as:

$$\hat{\Sigma} = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(X_i - \bar{X}\right)'}{n-1} \tag{2}$$

This estimate of the covariance matrix will be used in the statistical test developed here. However, if elliptical symmetry fails to be rejected and quadratic programming is used to generate an expected value-variance frontier, the covariance matrix in the programming model should be normalized by $(n - m - 1)$, where m is the dimension of the random vector, instead $(n - 1)$ (Chalfant, Collender, and Subramanian 1990). The Y_i's will have a distribution that is approximately spherically symmetric.

Eaton and Kariya (1977) derive a uniformly most powerful test of the null hypothesis of spherical symmetry against the alternative of (nonspherical) elliptical symmetry. This test can not be adapted to test for elliptical symmetry because our alternatives of interest are non-symmetric.

Beran (1979) and Romano (1988) propose tests of spherical symmetry based on the estimated distance of the actual distribution from the family of spherically symmetric distributions. Beran's test statistic is

$$S_n\left(\hat{\mu}, \hat{\Sigma}\right) = \sum_{k=1}^{K_n}\sum_{p=1}^{M_n}\left[n^{-\frac{1}{2}} a_k\left(R_i\left(\hat{\mu}, \hat{\Sigma}\right)\right) b_p\left(\theta_i\left(\hat{\mu}, \hat{\Sigma}\right)\right)\right]^2$$

Here, $\{a_k : K \geq 1\}$ is a family orthonormal with respect to Lebesgue measure on $[0, 1]$ and orthogonal to the constant function, and $\{b_p : p \geq 1\}$ is a family orthonormal with respect to Lebesgue measure on $[0, \pi] \times [0, 2\pi]^{m-2}$ and orthogonal to the constants. The R_i's are the ranks of the distances $\|Y_i\|$, divided by $n+1$, and the θ_i's are angular polar coordinates of the Y_i's. The summation limits K_n and M_n must be chosen. Beran shows that this test is equivalent to

$$\int_0^1 \int_{S_m} \left(\hat{g}_n\left(r, u\right) - 1\right)^2 dM(u)dr$$

where S_m is the surface of the unit sphere in $\mathbb{R}^m$, M is the uniform measure on S_m and $\hat{g}$ is an orthogonal series estimate of the joint density of $r = F_{\|Y\|}^{-1}\left(\|Y\|\right)$ and $U = Y/\|Y\|$. Here, $F_{\|Y\|}$ is the cumulative distribution function for $\|Y\|$, so r is uniformly distributed on S_m and independent of r under the null hypothesis. The summation limits K_n and M_n control the degree of smoothing in the orthogonal series estimator.

Romano's test statistic has the form

$$S_n' = n^{\frac{1}{2}} \underset{V \in \underline{V}}{sup} \| \hat{P}_n\left(V\right) - \tau\left(\hat{P}_n\right)\left(V\right) \|$$

where $\underline{V}$ is a class of subsets of $\mathbb{R}^m$ with certain properties, $\hat{P}_n$ is the empirical measure of the data $Y_1, \ldots, Y_n$, and $\tau\left(\hat{P}_n\right)$ is the spherically symmetric distribution having the same radial distribution as $\hat{P}_n$.

The problem of testing the null hypothesis of spherical symmetry is invariant under monotone transformations of the radial component, so from invariance considerations one could argue that the test should be based on the maximal invariant, $(R_i, Y_i/\|Y_i\|)$, where R_i

is the normalized rank as given above. This has the further advantage that any invariant test has a simple null distribution — its distribution under sampling from the uniform distribution on $S_m \times \{\frac{1}{n}, \ldots, 1\}$. Both Beran's and Romano's test satisfy the invariance requirement.

Both of these tests require certain arbitrary choices — $\underline{V}$ for Romano's test and K_n, M_n, $\{a_k\}$, and $\{b_p\}$ for Beran's test. Since Romano's test has fewer arbitrary choices it is preferable on this count. Even so, it seems difficult to select $\underline{V}$ to achieve invariance under orthogonal transformations. Beran's test will not be rotationally invariant because no selection of b_p's can achieve this.

It is likely that Beran's test will have low power in small samples, This is because one will have to pick K_n and M_n very small — 1 or 2, for example. Since the b_p 's will typically look like the lowest frequency trigonometric functions, only the grossest departures from uniformity will manifest themselves. Typical agricultural economics data sets do not have a large number of observations. The typical sample sizes are too small for reasonable nonparametric density estimation. Romano's test, with appropriate choice of $\underline{V}$, is likely to have power comparable to the test proposed in the next section.

Finally, both of these tests appear to be very difficult to compute. The five dimensional choices in our example would require the solution of a high dimensional nondifferentiable optimization problem to obtain $\underline{V}$ for Romano's test. Furthermore, the significance level is evaluated by Monte Carlo methods, so a practical test should be computable with relatively few operations.

3.2 A Nearest Neighbor Test for Elliptical Symmetry

Let F = the collection of all elliptically symmetric distributions which are absolutely continuous. The hypothesis to be tested is:

$$H_0 : f \in F \quad \text{versus}$$

$$H_A : f \notin F$$

Let $\{X_1, X_2, \ldots, X_n\}$ be a sample of observations on an m-dimensional random vector. Let $\hat{\mu}$, $\hat{\Sigma}$ be estimates of the mean vector and the covariance matrix of the data. If $\hat{\Sigma}$ is positive definite (which will be the case with probability 1 if $n \geq m$) then it has a Cholesky factorization $\hat{\Sigma} = \hat{L}\hat{L}'$, where $\hat{L}$ is lower triangular. Use $\hat{L}$ and $\hat{\mu}$ to transform the sample into standardized deviations from means, Y_i, as in equation 1. If the observed data, X_i, comes from an elliptically symmetric distribution, then Y_i is approximately a sample from a spherically symmetric distribution.

From the properties of spherically symmetric distributions it is known that normalized values of Y_i, $Z_i = Y_i/\|Y_i\|$, under H_0 are approximately uniformly distributed on the unit hypersphere and are independent of the random variables $r_i = \|Y_i\|$ (Muirhead 1982, pages 36–37). The inverse of this transformation provides an easily implemented method for generating samples from a spherically symmetric distribution. First generate n variables, Z_i, uniformly distributed on the unit hypersphere; then, generate a random sample from some univariate distribution with nonnegative support to obtain the radius $r_i, i = 1, 2, \ldots, n$.

Then the spherically symmetric sample is obtained by setting $Y_i = r_i \cdot Z_i, i = 1, 2, \ldots, n$.

Let $R_i = \text{rank}\{\|Y_i\|\}/n$. The nearest neighbor test for elliptical symmetry exploits the property that pairs (R_i, Z_i) are approximately uniformly distributed on the product of $\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ with the unit hypersphere. Uniformity means that the radial distance between nearest neighbors (nodes) that form the shortest path around this space should be similar for all pairs of nodes. This is in contrast to the case of a nonuniform density where the nodes will tend to be clustered in groups that correspond to regions of high density. In this case, distances between nodes in a cluster will be small, but the distance that must be traversed in moving from one cluster to the next cluster will be relatively large. There will also tend to be isolated nodes in low density regions which will have large nearest neighbor distances.

In order to find the shortest path it is necessary to solve the travelling salesman problem. The null distribution of the test statistic will be determined by Monte Carlo methods, and the shortest path must be determined for each Monte Carlo sample. This makes determination of the shortest path by solution of the travelling salesman problem impractical because the solution of hundreds or thousands of travelling salesman problems would be required to calculate one p-value.

Therefore, the shortest path will be approximated by a method that is less computationally intensive than the travelling salesman problem. Consider the n by n symmetric matrix S whose elements are distances between points in the data set. Nearest neighbors will be approximated by choosing the smallest entry from the off-diagonal elements in each column of S. To avoid double counting the same distance, when an element in row i is the smallest entry in column j, $j < i$, row j will be eliminated from consideration when the smallest entry in column i is chosen. This will not produce points that lie on the shortest path around the space, but it will produce points that are nearest neighbors without double-counting the same pairs.

Tests based on nearest neighbor distances have been used in the analysis of spatial point processes to test the hypothesis of complete spatial randomness (Diggle 1983). In particular, Clark and Evans (1954) have proposed a test based on the mean nearest neighbor distance, which is very similar to the test proposed here.

The distances between nearest neighbors should provide information useful for testing approximate spherical symmetry of the Y_i's. There are two problems that need to be addressed: finding an appropriate distance measure; and choosing a good test statistic based on the nearest neighbor distances.

We propose to use Euclidean distance with a suitable weighting of the radial ranks R_i. If Z_1 and Z_2 are independent and uniformly distributed on the unit hypersphere, then:

$$E\|Z_1 - Z_2\|^2 = 2 - 2E\left(Z_1' Z_2\right) = 2. \tag{3}$$

If R_1 and R_2 are independent and uniformly distributed on $\{\frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ then

$$E\left(R_1 - R_2\right)^2 = \left(n^2 - 1\right)/6n^2 \approx \frac{1}{6}. \tag{4}$$

Since the unit hypersphere has dimension $m-1$ and the radius has dimension 1, it is intuitive to have equation 3 contribute $(m-1)$ degrees of freedom and equation 4 contribute 1 degree of freedom. Therefore, we propose:

$$\| (R_1, Z_1) - (R_2, Z_2) \| = \left[6\,(R_1 - R_2)^2 + (m-1)\,(1 - Z_1' Z_2)\right]^{1/2} \tag{5}$$

as the measure of distance.

The test statistic is based on the sample mean and variance of the nearest neighbor distances $D_1, D_2, \ldots, D_n$ derived from a sample $X_1, X_2, \ldots, X_n$. The minimum distances are defined as $D_i = \underset{j}{min} \| (R_i, Z_i) - (R_j, Z_j) \|^2$, where j ranges over observation indices for which (R_i, Z_i) has not already been chosen as the nearest neighbor of (R_j, Z_j). Figure 1 is a plot of 100 mean-variance pairs $\left(\bar{D}, S_D^2\right)$ from the null model with $m = 5$ and $n = 21$, and the alternative of a Burr distribution with parameter $\alpha = 0.25$ – Johnson (Johnson 1987) recommends the Burr distribution as a multivariate non-elliptic distribution. The Burr distribution with parameter α is the distribution of $(1 + \underline{X}/Y)^{-\alpha}$ where $\underline{X}$ is a vector of independent exponentials with mean 1 and Y is an independent gamma random variable with shape parameter α. Figure 2 is a plot of 100 mean-variance pairs from the same null model and an alternative called the inflated normal – this non-elliptic distribution is created by multiplying negative realizations of the first four elements of the random vector by -4. The inflated normal is non-elliptic because four of the five elements of the random vector only take on non-negative values. Inflated normal distributions with multipliers of -2 and -3 were also constructed.

Figures 1 and 2 indicate that a linear discriminant should do a good job of separating the null distribution from each of the two alternatives. The systematic difference in the mean and variance of nearest neighbors is consistent with the preceeding theoretical explanation. The non-elliptic distributions have a smaller mean and larger variance than the normal distribution. Examination of other data sets generated from null and alternative distributions led to a similar conclusion. Table 1 summarizes these results by presenting the best linear discriminant coefficients for various alternative distributions. In each case, samples of size 1000 from the null and alternative were generated, and the discriminant coefficients were computed using the procedure *discr* in Splus (Venables and Ripley 1994).

The test statistic is constructed by assigning weights to the mean and variance of the nearest neighbor distances that are the medians of the values reported in table 1. This produces a simple test statistic of the form:

$$V = 12.83\bar{D} - 5.46S_D^2 \tag{6}$$

where $\bar{D} = \frac{1}{n}\sum_{i=1}^{n} D_i$, $S_D^2 = \frac{1}{n}\sum_{i=1}^{n}\left(D_i - \bar{D}\right)^2$, and $D_1, D_2, \ldots, D_n$ are the nearest neighbor distances obtained from the matrix S with elements $S_{ij} = \|(R_i, Z_i) - (R_j, Z_j)\|^2$. The null hypothesis should be rejected for small values of V, because the discriminant line has a positive slope and places the null distribution to the right of the line. The exact critical values will be obtained from the empirical distribution of the statistic calculated from the Monte Carlo sample of the null distribution. A histogram of the null distribution is presented in Figure 3. The null hypothesis will be rejected for small values of the test statistic, so critical values of the test statistic will be in the region below 17 for 21 observations from a

5 dimensional random vector.

Table 3 presents the power of the test against eight alternatives at three levels of significance. The power function exhibits good performance that is expected of a statistical test. The power is an increasing function of the significance level, rising from .9 to .95 for the Burr distribution with parameter .125 and from .43 to .67 for the inflated normal with an inflation factor of -2. The power also increases as the alternatives get more non-elliptic. The Burr distribution is more non-elliptic when the parameter value is smaller. The power grows monotonically as the Burr parameter decreases. The inflated normal gets more non-elliptic as the inflation factor grows, and the power increases with the inflation factor.

In general, the value of the power function is large, indicating the ability of the test statistic to discriminate alternatives from the null distribution. It should be noted that even the smallest power values of .43 and .53 are relatively good given a sample size of 21 on a 5 dimensional random vector. Other investigations of the test statistic indicate that the power grows as the sample size grows. So applications to larger sample sizes will produce test statistics that are more powerful at distinguishing alternatives. The results for a sample size of 21 are reported in this paper because this is the sample size in the example that will be presented in the following section.

4 Applying the Elliptical Symmetry Test to Agricultural Data

An important risk management tool for agricultural producers is diversification. Portfolio diversification can occur across several dimensions. In some production systems diversification can be accomplished by allocating land to a variety of crops whose yield and/or price risk may be negatively correlated. In addition, most producers have access to financial diversification by allocating investments to various financial assets, whether or not crop diversification is feasible for them. Other producers may diversify income sources by allocating family labor to off-farm income generating activities. These examples highlight the fact that an integrated understanding of the risk management strategies of agricultural producers requires economic models that include a variety of random returns.

Expected utility representation of decision making in such an environment requires a multiple integral objective function that can make the derivation of definite comparative statics difficult or impossible. In contrast, a mean-variance representation of preferences is analytically tractable and can produce clear comparative statics results. Coyle (1992) is an example of a model of multiple output production that uses the mean-variance simplification. When the distribution of multivariate returns is elliptically symmetric, the mean-variance model provides rankings of alternatives that are consistent with expected utility maximization. The test for elliptical symmetry provides researchers a tool to assess the acceptability and potential costs of a mean-variance model. If the test fails to reject the hypothesis of elliptical symmetry support for expected utility, mean-variance consistency is provided. On the other hand, if the test rejects the hypothesis of elliptical symmetry, then there could be some significant asymmetries in the distribution of returns that require further investigation.

For example, Young and Barry (1987) analyzed the benefits of financial diversification for Illinois grain farmers using mean-variance analysis. If the distribution of portfolio returns in this study is elliptically symmetric, then there is support for the argument that mean-variance efficient choices are expected utility maximizing. If elliptical symmetry is rejected then optimal mean-variance portfolios may miss important aspects of utility maximizing portfolio diversification.

A subset of the portfolio data in Young and Barry (1987) consists of 21 annual observations (from 1963 to 1983) on two equity instruments: 1) the Standard and Poors 500 common stock index; 2) a portfolio of small company stocks; returns on three debt instruments: 3) a municipal bonds index; 4) a certificate of deposit rate; 5) a passbook account rate; and 6) returns to grain farming on $500 - 640$ acre farms in northern Illinois. The passbook rate is treated as a risk-free rate. The estimated mean vector and covariance matrix (calculated as in equation 2) for the five risk rates of return are presented in Table 3. The test procedure described above was applied to this portfolio data. This yielded the following values for the estimated mean and variance of the nearest neighbor distances: $\bar{D} = 1.40035$, and $S_D^2 = 0.05916$. The Monte Carlo analysis of data with $n = 21$ and $m = 5$ indicates that these estimates should be weighted by 12.83 and -5.46 yielding a test statistic value of 17.6435, with a p-value of 0.308, which indicates that the null hypothesis of elliptical symmetry fails to be rejected. This result enhances the analysis performed by Young and Barry by showing that the optimal portfolio choices are consistent with expected utility maximization.

5 Conclusion

The analysis of behavior under risk and the development of strategies for managing risk is an important area of agricultural economics research. Expected utility theory continues to be a good behavioral model of decision making under uncertainty, despite experimental evidence on the violation of the independence axiom (Bar-Shira 1992). Mean-variance analysis is consistent with expected utility theory when alternatives only alter the location and/or scale of final wealth. If a set of alternatives does not satisfy the location-scale condition mean-variance ranking can be inconsistent with expected utilty ranking. For example, an increase in variance can make a risk averse decision maker better off if the increase is accompanied by an increase in negative skewness.

Computational and analytical models of behavior in the presence of multiple sources of uncertain returns are simpler and more tractable if preferences can be represented as a function of mean and variance. A sufficient condition for the consistency of mean-variance analysis with expected utility analysis is elliptical symmetry of the distribution of the vector of random returns. This family of multivariate distributions has a regular structure that makes it possible to distinguish them from distributions that are not members of the family.

The structure of the family of elliptically symmetric distributions was used to construct a test statistic for the null hypothesis that an observed set of random vectors comes from an elliptically symmetric distribution. The test rejects the null hypothesis if the normalized values of the random vector are distributed on the unit hypersphere in a manner that is not uniform. The test statistic and its null distribution are easy to compute with Monte Carlo methods. This testing approach is consistent with other computationally intensive testing procedures, such as the bootstrap (Efron and Tibshirani 1993).

The test was shown to have power to distinguish elliptic from non-elliptic distributions. The power of this distinction increases as the alternative gets further away from the elliptic family. And the power increases as the size of the test increases. These properties indicate that the test should reliably distinguish elliptic from non-elliptic distributions.

The properties of the test were demonstrated for a set of 21 observations on a 5 dimensional random vector, in order to apply it to an example of an agricultural portfolio problem. The software for calculating the test statistic is easily adjusted for data sets of other dimensions. And the calculations should be performed in a short time for most agricultural data sets. Investigations with data sets of other dimensions indicate that the power of the test increases as the sample size grows. Further research will attempt to analytically verify the consistency of the test, which is hinted at by these numerical results.

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Table 1

Discriminant Coefficients

Alternative Distribution	Mean Coefficient	Variance Coefficient
Burr-0.125	12.3654	-4.4143
Burr-0.250	11.7803	-6.5138
Burr-0.375	11.3993	-7.9906
Burr-0.500	11.9495	-7.7026
Burr-0.625	13.2892	-6.5606
Inflated-2	17.4581	-4.1218
Inflated-3	16.3713	-3.1802
Inflated-4	15.5079	-3.5152

Table 2

Power of Elliptical Symmetry Test Against Alternatives

Alternative Distribution	5% Pvalue	10% Pvalue	15% Pvalue
Burr-0.125	.9	.93	.95
Burr-0.250	.79	.85	.88
Burr-0.375	.71	.77	.83
Burr-0.500	.61	.69	.75
Burr-0.625	.53	.61	.69
Inflated-2	.43	.58	.67
Inflated-3	.66	.76	.82
Inflated-4	.85	.91	.94

Table 3

Estimated Mean and Covariance Matrix for Portfolio Returns

$$\hat{\mu} = \begin{pmatrix} 15.87 \\ 10.22 \\ 22.49 \\ 4.95 \\ 7.41 \end{pmatrix} = \begin{matrix} \text{farm} \\ \text{S\&P 500} \\ \text{small stock} \\ \text{municipal bond} \\ \text{certificate of deposit} \end{matrix}$$

$$\hat{\Sigma} = \begin{pmatrix} 179.87 & -91.16 & -88.65 & -46,05 & -6.67 \\ -91.16 & 277.33 & 397.42 & 62.39 & -10.42 \\ -88.65 & 397.42 & 931.00 & 88.29 & -8.16 \\ -46.05 & 62.39 & 88.29 & 151.78 & -1.78 \\ -6.67 & -10.42 & -8.16 & -1.78 & 13.44 \end{pmatrix}$$

Figure 1: Mean and Variance for Null and Alternative 1

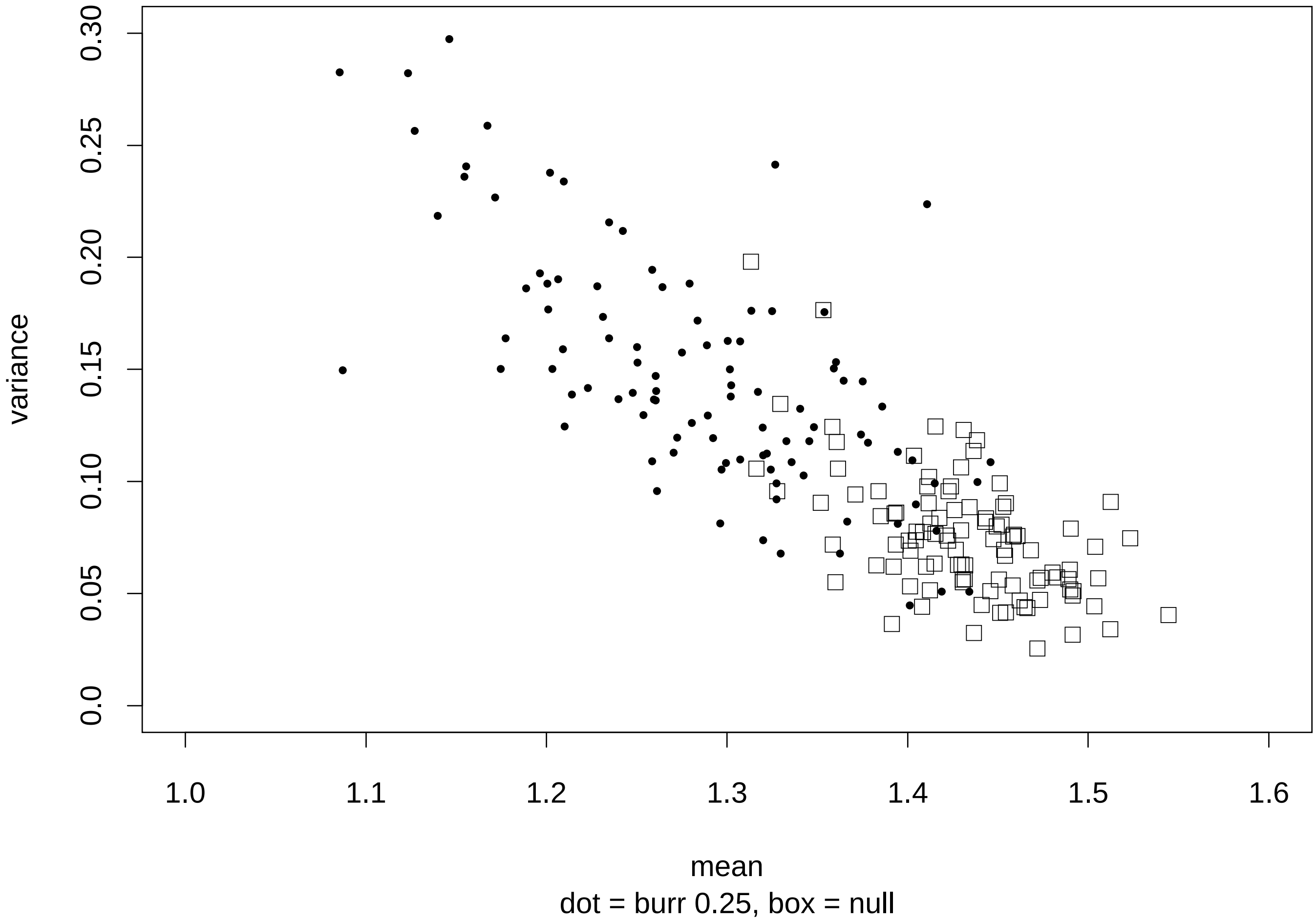

Figure 2: Mean and Variance for Null and Alternative 2

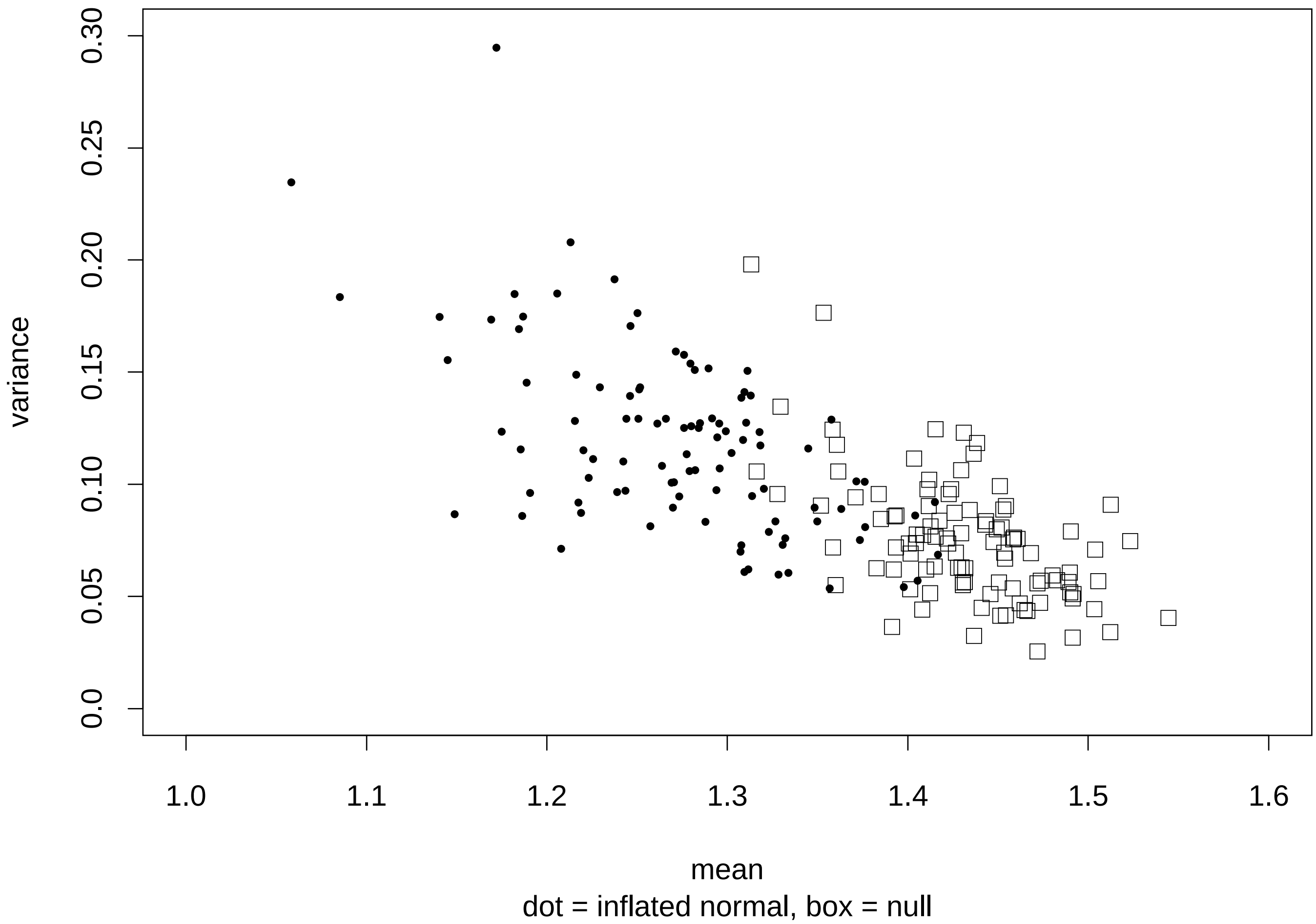

Figure 3: Null Distribution with n=21, m=5