Price Elasticities Implied by Homogeneous Production Functions

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Abstract. *If a production process is characterized by a homogeneous production function, the conditions required for profit maximization imply that the elasticity of demand for each input must be elastic with respect to output price This restriction limits the usefulness of these functions in empirical analysis*

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Chand and Kaul (1986)[1] have demonstrated that using the Cobb-Douglas profit function to characterize a production process imposes a number of restrictions on the price elasticities of input demand However, since every Cobb-Douglas profit function corresponds to a Cobb-Douglas production function, their results are also applicable to the case where the production process is characterized by a Cobb-Douglas production function The purpose of this note is to show that one of the more important restrictions derived by Chand and Kaul also applies to the more general case of a homogeneous production function

Profit Maximization

Assume that the production process requires, at most, n inputs to produce a single output Let f denote the corresponding production function Then

$$y = f(x),$$

where x is an n-dimensional vector of inputs with $x \geqslant 0$, and $y \geqslant 0$ is output [2] Assume that f is strictly quasi-concave and twice continuously differentiable with positive first order derivatives

The profit function is defined as

$$\pi\ (p,\ w,\ x) = p \cdot y - w\ \ x,$$

where w represents the n-dimensional vector of positive input prices and p denotes the positive output price If the producer is a price-taker in all markets, profit maximization involves determining some value of $x^* \geqslant 0$, such that

$$\pi\ (p,\ w,\ x^*) \geqslant \pi\ (p,\ w,\ x)$$

for all $x \geqslant 0$

This constrained optimization problem may be conveniently broken into two parts (Takayama, 1985, p 142) First, minimize the cost function $w \cdot x$ subject to the constraints that $f(x) \geqslant y$ and $x \geqslant 0$ Because f is continuous, the solution to this cost minimization problem gives rise to a minimum cost function, c, defined for all values of w and y, with

$$c\ (w,\ y) \leqslant w \cdot x$$

for all x satisfying the constraints (Diewert, 1982, pp 537-538) Shephard (1981, pp 43-45) has shown that if the production function is homothetic and satisfies the conditions above, then there exist functions, h and g, such that

$$c\ (w,\ y) = h\ (w)\ \ g\ (y) \qquad (1)$$

and

$$x_i\ (w,\ y) = [\partial h\ (w) / \partial w_i] \cdot g\ (y)\ \ (i = 1,\ 2,\ \ \ ,\ n), \qquad (2)$$

where $x_i\ (w,\ y)$ denotes the derived demand for the i^{th} input based on the cost minimization problem The solution to the profit maximization problem is then given by determining that value of $y \geqslant 0$ which maximizes

$$p \cdot y - c\ (w,\ y)$$

Assume now that f is also positively homogeneous of degree k with respect to x As is well known, there does not exist a unique solution to the profit maximization problem, if the production function exhibits either increasing or constant returns to scale ($k \geqslant 1$) Therefore, restricting the production function to having decreasing returns to scale ($k < 1$) is a necessary condition for obtaining a unique

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[1]Sources are listed in the References section at the end of this article

[2]For any vector, z, the notation $z \geqslant 0$ will be used to indicate that each component of the vector is non-negative

input level, x^*, satisfying the profit maximization conditions [3]

Because the production function is homogeneous, f is also homothetic Therefore, Shephard's results are applicable In addition, the function g in equation 1 is given by $g(y) = y^{1/k}$ (Shephard, 1981, p 43) Thus, the profit maximization problem reduces to finding $y \geqslant 0$ which maximizes the expression

$$p \quad y - h(w) \quad y^{1/k}$$

The first order conditions for a maximum imply that

$$p - (1/k) \quad h(w) \quad y^{(1-k)/k} = 0,$$

which, after rearranging terms, yields

$$y(w, p) = [p \quad k \quad h(w)^{-1}]^{k/(1-k)}, \tag{3}$$

where $y(w, p)$ denotes the output level which maximizes profit for each level of w and p Furthermore, substituting equation 3 into equation 2 yields the input demand function corresponding to the profit maximization problem

$$x_i (w, y(w, p)) = [\partial h(w)/\partial w_i] \cdot [p \cdot k \cdot h(w)^{-1}]^{1/(1-k)} \tag{4}$$

Restrictions on the Price Elasticities

Equation 4 implies that the elasticity of the derived demand for input i with respect to output price is $(1 - k)^{-1}$ for $i = 1, 2, \quad , n$ Because profit maximization requires that k be less than one, this implies that $(1 - k)^{-1} > 1$ Hence, the conditions required for profit maximization imply that the elasticity of demand for each input with respect to output price is elastic and that this elasticity is identical for all inputs This restriction is identical to "characteristic five" given by Chand and Kaul (1986) However, the result is now seen to pertain to a much wider class of production functions

Examination of equation 3 yields a result that augments the work of Chand and Kaul (1986) This equation implies that the elasticity of supply with respect to output price is $k \quad (1 - k)^{-1}$ Thus, the elasticity of supply with respect to output price will be inelastic only if $k < 1/2$ This condition may provide a useful check for selecting production functions to characterize a particular industry

[3]If the production function is employed to model aggregate production for a commodity, these remarks are not strictly true Even if the production function for the market exhibits constant returns to scale, price and quantity will be uniquely determined by the interaction of aggregate supply and demand (Samuelson, 1974, pp 78-89) Varian (1984, p 27), however, observes that employing the assumption of decreasing returns is reasonable if we restrict our attention to the short-run Moreover, Chand and Kaul (1986) implicitly employ this assumption in their work

Conclusions

The restrictions derived above should be considered before selecting a homogeneous production function to characterize a particular production process If prior knowledge or empirical evidence suggests that the restrictions implied by the profit maximizing conditions are apt to be violated for a particular industry, alternative methods should be used to model the production process

The preceding results demonstrate that the restrictions needed to ensure profit maximization are inconsistent with inelastic input demand functions Therefore, if there is reason to suspect that input demand is inelastic with respect to changes in output price, homogeneous production functions should not be used to model the production process [4] Moreover, this restriction may be especially pertinent for agricultural commodities Estimates by Ball (1988), for example, indicate that many of the inputs used in agricultural production may be inelastic with respect to output price

The restrictions implied by profit maximization on the elasticity of supply with respect to output price are less serious If supply is believed to be inelastic with respect to output price, only those functions which are homogeneous of degree less than one half are relevant This result only limits the class of homogeneous functions that are appropriate in certain applications It does not preclude their use entirely

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[4]It is noted in passing that, in addition to the Cobb-Douglas function, the Arrow-Chenery-Minhas Solow constant elasticity of substitution function is also homogeneous

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